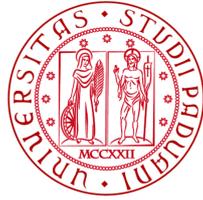


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Tesi di Laurea

Minimizers of the Ginzburg-Landau functional and De Giorgi’s Conjecture

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Introduction

In 1978 De Giorgi formulated the following conjecture [15]:

Conjecture 0.1. (*De Giorgi's Conjecture*) *Let us consider a solution $u \in C^2(\mathbb{R}^n)$ in all \mathbb{R}^n of the partial differential equation*

$$\Delta u = u^3 - u \tag{1}$$

such that

$$|u| \leq 1, \quad \partial_n u > 0 \tag{2}$$

in the whole \mathbb{R}^n . Is it true that all level sets $\{u = \lambda\}$ are hyperplanes, at least if $n \leq 8$?

This conjecture is naturally extended to the case $\Delta u = h'_0(u)$, where h_0 is a “double well” potential. For $n = 2$ the conjecture was proved by N. Ghoussoub and C. Gui in [28] and for $n = 3$ it was proved by L. Ambrosio and X. Cabré in [3]. In [20] we can find a counterexample for $n \geq 9$ by M. Del Pino, M. Kowalczyk and J. Wei. The question remains open for $8 \geq n \geq 4$.

In this thesis we study the following result achieved by O. Savin in [44].

Theorem 0.1. *Let $u \in C^2(\mathbb{R}^n)$ be a solution of the partial differential equation*

$$\Delta u = h'_0(u) \tag{3}$$

in all \mathbb{R}^n such that:

$$|u| \leq 1, \quad \partial_{x_n} u > 0, \quad \lim_{x_n \rightarrow \pm\infty} u(x', x_n) = \pm 1. \tag{4}$$

Then, if $n \leq 8$, the level sets of u are hyperplanes.

This theorem is the solution of a reduced version of De Giorgi's Conjecture and it is strongly related to phase transitions problems. Indeed, we consider u such that $|u| \leq 1$ in \mathbb{R}^n and u is a local minimizer in \mathbb{R}^n of the following energy functional

$$J(u, \Omega) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 + h_0(u) dx, \tag{5}$$

that describes the energy of a fluid in a phase transition regime (see for instance [33], [43], [10]). It is called Ginzburg-Landau functional. Equation (3) is the Euler-Lagrange equation of the functional (5) and, we will see in Section 2.2 that if u satisfies conditions (4) then u is a local minimizer of the Ginzburg-Landau functional.

Using this deep connection between minimizers of the Ginzburg-Landau functional and De Giorgi's Conjecture we can prove that Theorem 0.1 is a consequence of the following result (see Section 2.2)

Theorem 0.2. *Let $u \in H_{loc}^1(\mathbb{R}^n)$ be a local minimizer of J in \mathbb{R}^n and $u(0) = 0$, then the following holds:*

- (i) *If $n \leq 7$ then the level sets of u are hyperplanes.*
- (ii) *If $n = 8$ and $\partial_{x_n} u > 0$ then the level sets of u are hyperplanes.*

In the first part of the thesis, we consider the rescaled minimizer $u_\epsilon = u(\frac{x}{\epsilon})$ and we see how the level sets of u_ϵ converge uniformly, on compact sets, to ∂E when ϵ goes to zero, where E is a set with locally minimal perimeter in \mathbb{R}^n . Namely, L. Modica proved in [40] that there exists a sequence u_{ϵ_k} such that u_{ϵ_k} converge to $\chi_E - \chi_{E^c}$ in $L_{loc}^1(\mathbb{R}^n)$.

One of the main steps for proving Theorem 0.2 is the following density estimate proved in [11]

Theorem 0.3. *Given $\alpha > -1$ and $\beta < 1$, if $u \in H^1(B_R(x))$ is a minimizer of $J(\cdot, B_R(x))$ and $u(x) \geq \alpha$, then there exist a constant c depending only on n and h_0 and a constant $r_0(\alpha, \beta)$ depending on α, β, n , and h_0 such that:*

$$\begin{aligned} \mathcal{L}^n(\{u > \beta\} \cap B_r(x)) &\geq cr^n, \\ \mathcal{L}^n(\{u < \beta\} \cap B_r(x)) &\geq cr^n \end{aligned}$$

for $r \geq r_0(\alpha, \beta)$, provided that $B_{r+2}(x) \subseteq B_R(x)$.

In order to prove this theorem we do not follow the proof given by Caffarelli and Cordoba in [11], but we adapt the techniques used in [52] to our specific functional. Using these techniques we simplify a bit the proof and we correct some minor flaws present in [11].

We see how Theorem 0.3 allows us to pass from the $L_{loc}^1(\mathbb{R}^n)$ convergence to a uniform convergence of $\{u = \lambda\}$, on compact sets, to ∂E . Thanks to Simon's theorem on sets with minimal perimeter in \mathbb{R}^n proved in [51], we prove that the level sets of u are asymptotically flat at ∞ for $n \leq 7$ and, if we also assume $\partial_n u > 0$, this asymptotic behaviour is still true for $n = 8$.

In the second part of the thesis, we show how De Giorgi's conjecture can be proved using this asymptotic behaviour of the level sets. For this purpose we need a more precise estimate of the behaviour at ∞ of the level sets of u . In particular we prove the "Improvement of Flatness" theorem:

Theorem 0.4. *Let $u \in H_{loc}^1(\mathbb{R}^n)$ be a local minimizer of J in $\{|x'| < l\} \times \{|x_n| < l\}$. Assume that $u(0) = 0$ and assume that there exists $\theta \leq l$ such that:*

$$\{u = 0\} \subset \{|x'| < l\} \times \{|x_n| < \theta\}.$$

Then there exist small constants $0 < \eta_1 < \eta_2 < 1$ depending on n and h_0 such that: given $\theta_0 > 0$ there exists $\epsilon_1(\theta_0) > 0$ depending on n, h_0 and θ_0 such that if

$$\frac{\theta}{l} \leq \epsilon_1(\theta_0), \quad \theta_0 \leq \theta,$$

then

$$\{u = 0\} \cap \{|\pi_\xi x| < \eta_2 l\} \times \{|x \cdot \xi| < \eta_2 l\}$$

is included in a flatter cylinder

$$\{|\pi_\xi x| < \eta_2 l\} \times \{|x \cdot \xi| < \eta_1 \theta\},$$

for some unit vector ξ , where $\pi_\xi x = x - (x \cdot \xi)\xi$.

The proof of this result is divided into three steps.

In the first step, we construct two different families of viscosity supersolutions of $\Delta u = h'_0(u)$, and we develop several “sliding methods” that allow us to compare a weak Sobolev solution of $\Delta u = h'_0(u)$ to this two families of viscosity supersolutions. In the second step we use this “sliding methods” to prove that the level sets of the minimizers satisfy the zero mean curvature equation in the viscosity sense. In this two steps we present the results achieved by B. Sciunzi and E. Valdinoci in [50], in particular in the proof of Theorem 3.9 we correct some minor flaws present in Lemma 6.6 of [50].

In the last step, we finally prove the “Improvement of Flatness” Theorem using the Harnack inequality for flat level sets of minimizers and the geometric information on the level sets proved in the second step. In this last step we present the results achieved by O. Savin, B. Sciunzi and E. Valdinoci in [49].

This thesis is structured as follows. In Chapter 1 we introduce De Giorgi’s conjecture, we present the state of art of the conjecture, and we study a link between phase transitions and minimal surfaces. In particular, we prove Theorem 0.3 and an asymptotic flat behaviour of the level sets of phase transitions.

In Chapter 2 we introduce the “Improvement of Flatness” Theorem, we prove De Giorgi’s conjecture for phase transitions and finally we prove the reduced version of De Giorgi’s conjecture.

In Chapter 3 we introduce the notion of hypersurface that satisfies the zero mean curvature equation in the viscosity sense. We construct two different families of viscosity supersolutions of $\Delta u = h'_0(u)$ and we develop several “sliding methods”. Finally, we prove that the level sets of the minimizers satisfy the zero mean curvature equation in the viscosity sense. In particular, in this chapter we present the first two steps of the proof of Theorem 0.4

In Chapter 4, we present the final step of the proof of Theorem 0.4. We introduce the Harnack inequality and, using the geometric information on level sets achieved in Chapter 3, we prove the “Improvement of Flatness” Theorem.

Chapter 1

De Giorgi's Conjecture and Minimal Surfaces

1.1 De Giorgi's Conjecture

De Giorgi's conjecture is related to the study of bounded solution of the semilinear elliptic equation $\Delta u - F'(u) = 0$ in the whole space \mathbb{R}^n , under the assumption that u is monotone in one direction, say $\partial_n u > 0$. In particular the goal is to prove that the solution u is one-dimensional, namely, u only depends on one variable. This question was raised by De Giorgi in 1978, who made the following conjecture (page 175 of [15])

Conjecture 1.1. (*De Giorgi's Conjecture*) *Let us consider a solution $u \in C^2(\mathbb{R}^n)$ in all \mathbb{R}^n of the partial differential equation*

$$\Delta u = u^3 - u \tag{1.1}$$

such that

$$|u| \leq 1, \quad \partial_n u > 0 \tag{1.2}$$

in the whole \mathbb{R}^n . Is it true that all level sets $\{u = \lambda\}$ are hyperplanes, at least if $n \leq 8$?

De Giorgi's conjecture is equivalent to the one-dimensional symmetry property. In fact, if the conjecture is true, then u depends only on the direction orthogonal to the level sets.

The particular elliptic equation (1.1) is called the Allen-Cahn equation, but the results achieved in the past years are dealing with more general elliptic equations of the form:

$$\Delta u(x) - F'(u(x)) = 0, \quad x \in \mathbb{R}^n, \tag{1.3}$$

where $F \in C^2(\mathbb{R})$ and $F(x) > \min\{F(1), F(-1)\}$ for every $x \in (-1, 1)$.

The conjecture remained completely open until 1998 when C. Gui and N. Ghoussoub in [28] proved the result for $n = 2$. Their proof use a Liouville-type theorem for elliptic equations in divergence form, developed by H. Berestycki, L. Caffarelli and L. Nirenberg in [5], applied to the ratio

$$\sigma = \frac{\partial_{x_1} u}{\partial_{x_2} u}.$$

They proved that σ is constant in all \mathbb{R}^2 , and, using this result, they proved the conjecture for $n = 2$. Using similar techniques, L. Ambrosio and X. Cabrè in [3] extended these results to the dimension $n = 3$. We can resume this two works in the following theorem:

Theorem 1.1. *Assume that $F \in C^2(\mathbb{R})$, $F(x) > \min\{F(1), F(-1)\}$ for every $x \in (-1, 1)$ and u is a solution of (1.3) in all \mathbb{R}^n satisfying the conditions (1.2). If $n = 2$ or $n = 3$ then all level sets of u are hyperplanes.*

Another fundamental result was achieved in 2009 by M. Del Pino, M. Kowalczyk and J. Wei in [20]; for $n \geq 9$ they showed examples of solutions u of (1.1), satisfying conditions (1.2), that are not one dimensional. In this way they proved that the upper bound $n \leq 8$ in Conjecture (1.1) is sharp. This is the state of the art on Conjecture (1.1), the problem is still open for dimensions $4 \leq n \leq 8$.

Despite the fact that Conjecture (1.1) is still open, some interesting results were obtained in the past years. For instance N. Ghoussoub and C. Gui showed in [29] that, for $n = 4$ and $n = 5$, the conjecture is true for a special class of solutions that satisfy an anti-symmetry condition.

But the most important result was proved in 2009 by O. Savin that proved in [44] the following theorem:

Theorem 1.2. *Let $u \in C^2(\mathbb{R}^n)$ be a solution of:*

$$\Delta u = u^3 - u$$

in all \mathbb{R}^n , such that

$$|u| \leq 1, \quad \partial_n u > 0, \quad \lim_{x_n \rightarrow \pm\infty} u(x', x_n) = \pm 1. \quad (1.4)$$

If $n \leq 8$ then the level sets of u are hyperplanes.

Savin proved this result not only for the Allen-Cahn equation but for a more general elliptic equation of the form:

$$\Delta u(x) = h'_0(u(x)), \quad x \in \mathbb{R}^n, \quad (1.5)$$

where h_0 is a “double well” potential, we will give the precise definitions in the next section.

The technique used for proving Theorem 1.1 and the technique used for proving Theorem 1.2 are completely different. In Theorem 1.1 the flatness of the level sets is proved using a Liouville-type theorem for elliptic equations and, in particular, the results achieved in [28] and [3] do not use the regularity theory of minimal surfaces. For Theorem 1.2, on the other hand, the regularity theory of minimal surfaces plays a crucial role, and the proof is based on the fact that u , solution of (1.5) with conditions (1.4), is a local minimizer of the following functional:

$$J(u, \Omega) = \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 + h_0(u) \right) dx.$$

Although the two Theorems 1.1 and 1.2 are similar, the ideas behind them are completely different. In this thesis we will study in detail the results achieved by Savin in [44].

1.2 Phase transitions and minimal surfaces

We start by defining the typical phase transition functional. Given a domain $\Omega \subseteq \mathbb{R}^n$, we define the following functional on $H^1(\Omega)$:

$$J(u, \Omega) = \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 + h_0(u) \right) dx. \quad (1.6)$$

From now on we suppose $h_0(1) = h_0(-1) = 0$ and $h_0 \in C^2([-1, 1])$. We assume that, for some $0 < c < 1 < C$ and some $\theta^* \in (0, 1)$,

$$h_0(x) > 0 \quad \text{for any } x \in (-1, 1), \quad (1.7)$$

$$\text{for any } \theta \in [0, 1], \quad c\theta^2 \leq h_0(-1 + \theta) \leq C\theta^2 \text{ and } c\theta^2 \leq h_0(1 - \theta) \leq C\theta^2, \quad (1.8)$$

$$\text{for any } \theta \in [0, \theta^*), \quad c\theta \leq h'_0(-1 + \theta) \text{ and } h'_0(1 - \theta) \leq -c\theta. \quad (1.9)$$

We also assume a convexity property of h_0 near ± 1 , namely that h'_0 is increasing in $(-1, -1 + \theta^*)$ and in $(1 - \theta^*, 1)$.

As a model example for a potential h_0 satisfying the conditions stated above, one may consider:

$$h_0(u) = \frac{1}{4}(1 - u^2)^2. \quad (1.10)$$

In the literature, h_0 is often referred to as a ‘‘double well’’ potential, while its derivative h'_0 is sometimes called a ‘‘bi-stable nonlinearity’’ and the functional (1.6) is called Ginzburg-Landau type functional.

In light of the hypothesis above, with no loss of generality, possibly reducing the size of θ^* , we may assume that

$$h_0(\xi) \geq \max_{[-1, -1 + \theta^*] \cup [1 - \theta^*, 1]} h_0 \quad \text{for any } \xi \in [-1 + \theta^*, 1 - \theta^*]. \quad (1.11)$$

Notice that, if $u \in H^1(\Omega)$, $|u| \leq 1$, is critical for $J(\cdot, \Omega)$, then u satisfies in a weak sense the following elliptic equation:

$$\Delta u(x) = h'_0(u(x)) \quad x \in \Omega,$$

and if we choose the potential (1.10) we obtain the Allen-Cahn equation $\Delta u = u^3 - u$.

Let us briefly explain what is the physical meaning of the functional (1.6). Imagine that we have a two-phase fluid in a domain Ω , and we denote its density at a point x by $u(x)$. Assume its energy is given by a double well potential $h_0(u(x))$ with minima at u_1 and u_2 i.e.

$$h_0(u_1) = h_0(u_2) = 0, \quad h_0(s) > 0 \quad \text{if } s \neq u_1, u_2.$$

The densities u_1 and u_2 correspond to the stable fluid phases, for simplicity in our model we set $u_1 = -1$ and $u_2 = 1$. Then a candidate energy functional of the fluid is given by the integral:

$$\int_{\Omega} h_0(u(x)) \, dx.$$

But this is not a satisfactory physical model since any density function $u(x)$, that takes only the values u_1 and u_2 , minimizes the density energy. In particular the stable phases u_1 and u_2 could coexist along any complicated interface. This problem arises because we ignored the interactions at small scales (such as friction) which penalize the formation of unnecessary interfaces. In order to take into account this kind of interactions we add the term $|\nabla u|^2$ to the functional. This term represents a penalization to the total energy, and keeps under control the formation of interfaces (see). The functional (1.6) represents the energy functional associated to phase transition phenomena, in particular it appears in the Van Der Waals-Allen-Cahn-Hilliard and Ginzburg-Landau theories of phase transition (see, for instance, [43],[10]).

We now discuss the close relation between minimal surfaces and level sets of minimizers of J . We introduce now the definition of local minimizer,

Definition 1.1. A function $u \in H^1(\Omega)$ is a local minimizer of J in Ω if, for every open set $A \subset \Omega$ relatively compact in Ω ,

$$J(u, A) \leq J(u + \phi, A), \quad \forall \phi \in H_0^1(A). \quad (1.12)$$

Minimizers of the energy functional (1.6) are also called “phase transitions”.

Now we want to study the behaviour of u in large domains (recall that De Giorgi's Conjecture is stated for solution in all \mathbb{R}^n), in order to do this we rescale with a parameter ϵ a local minimizer in Ω and we study the behaviour of the rescaled minimizer when ϵ goes to zero. We define $\frac{\Omega}{\epsilon} := \{\frac{x}{\epsilon} \mid x \in \Omega\}$ and we consider u local minimizer of J in the domain $\frac{\Omega}{\epsilon}$, the behaviour of u in large domains is given by the behaviour of the rescaled functions u_ϵ defined as:

$$u_\epsilon(x) := u\left(\frac{x}{\epsilon}\right), \quad x \in \Omega. \quad (1.13)$$

If u is a local minimizer of J in the domain $\frac{\Omega}{\epsilon}$ then, performing a change of variable, we can see that u_ϵ is a local minimizer of the rescaled energy J_ϵ in Ω ,

$$J_\epsilon(v, \Omega) := \int_{\Omega} \left(\frac{\epsilon}{2} |\nabla v|^2 + \frac{1}{\epsilon} h_0(v) \right) dx. \quad (1.14)$$

Now we make an heuristic discussion about minimizers of J_ϵ which highlights a first connection between minimal surfaces and level sets of phase transitions. For a given function v with $|v| \leq 1$, the main contribution in $J_\epsilon(v, \Omega)$, for ϵ small, comes from the potential energy which is minimized when v is equal either to 1 or -1 . Instant jumps from a region where $v = 1$ to a region where $v = -1$ are not allowed since the kinetic energy $\int \frac{\epsilon}{2} |\nabla v|^2$ would become infinite.

From the elementary inequality $a^2 + b^2 \geq 2ab$ we clearly obtain

$$\int_{\Omega} \left(\frac{\epsilon}{2} |\nabla v|^2 + \frac{1}{\epsilon} h_0(v) \right) dx \geq \int_{\Omega} \sqrt{2h_0(v)} |\nabla v| dx, \quad (1.15)$$

now we can use the coarea formula and we get

$$\begin{aligned} \int_{\Omega} \sqrt{2h_0(v(x))} |\nabla v(x)| dx &= \int_{-1}^1 \left(\int_{\Omega \cap \{v(x)=s\}} \sqrt{2h_0(s)} d\mathcal{H}^{n-1}(y) \right) ds \\ &= \int_{-1}^1 \sqrt{2h_0(s)} \mathcal{H}^{n-1}(\{v = s\}) ds. \end{aligned}$$

Finally the inequality (1.15) becomes:

$$J_\epsilon(v, \Omega) \geq \int_{-1}^1 \sqrt{2h_0(s)} \mathcal{H}^{n-1}(\{v = s\}) ds. \quad (1.16)$$

The energy J_ϵ is minimized by functions for which the inequality (1.16) becomes an equality and for which the \mathcal{H}^{n-1} measure of the level sets is as small as possible.

We have an equality in (1.15) and (1.16) if and only if

$$|\nabla v| = \frac{1}{\epsilon} \sqrt{2h_0(v)},$$

this equality gives

$$v(x) = g_0\left(\frac{d_\Gamma(x)}{\epsilon}\right), \quad (1.17)$$

where $d_\Gamma(x)$ represents the sign distance from the 0-level set $\Gamma := \{v = 0\}$ and g_0 is the solution to the ODE

$$\begin{cases} g'_0 = \sqrt{2h_0(g_0)} \\ g_0(0) = 0. \end{cases}$$

We want also to minimize the \mathcal{H}^{n-1} measure of the level sets, but in general the level sets of the function v cannot be all with minimal perimeter. However, if for example the 0-level set Γ is minimal then the s -level sets are essentially minimal as long as s is not too close to ± 1 and ϵ is small. In fact, heuristically, we have that (1.17) is a continuous increasing function and depends only on $\frac{d_\Gamma(x)}{\epsilon}$, so if we consider the s -level set, we have that the distance between these level sets and Γ is small if ϵ is small. On the other hand, when s is close to ± 1 the weight $\sqrt{2h_0(s)}$ becomes negligible. All these heuristic discussions suggest us that the level sets of minimizers of J_ϵ converge to a minimal surface as $\epsilon \rightarrow 0$.

Now we want to make all these arguments rigorous. First of all we define the perimeter of a set:

Definition 1.2. *Given $\Omega \subseteq \mathbb{R}^n$ open, let E be a measurable set, the perimeter of E in Ω is defined as:*

$$P(E, \Omega) = \sup \left\{ \int_E \operatorname{div} \psi \, dx : \psi \in C_c^1(\Omega, \mathbb{R}^n), \|\psi\|_\infty \leq 1 \right\}. \quad (1.18)$$

When Ω is the whole \mathbb{R}^n we use the shorter notation

$$P(E) := P(E, \mathbb{R}^n).$$

We introduce also the concept of minimal surface:

Definition 1.3. *We say that E is a set with minimal perimeter in Ω or, shortly, ∂E is minimal surface in Ω if, for every $A \subset \Omega$ relatively compact in Ω ,*

$$P(E, A) \leq P(F, A) \quad (1.19)$$

whenever E and F coincide outside a compact set included in A

The asymptotic behaviour of u_ϵ was first studied in a rigorous way by L. Modica and S. Mortola in [38] and by L. Modica in [40] within the framework of Γ -convergence.

All the heuristic arguments concerning the convergence of level sets of u_ϵ to minimal surfaces are made rigorous by the results of Modica achieved in [40]. In particular he proved the following Theorem

Theorem 1.3. (Modica) *Given $\Omega \subseteq \mathbb{R}^n$ open, let u_ϵ be local minimizers for the energies $J_\epsilon(\cdot, \Omega)$, then there exists a sequence u_{ϵ_k} such that,*

$$u_{\epsilon_k} \longrightarrow \chi_E - \chi_{E^c} \text{ in } L^1_{loc}(\Omega) \quad (1.20)$$

where E is a set with minimal perimeter in Ω .

This result shows the deep connection between the minimizers of the Ginzburg-Landau functional and minimal surfaces; roughly speaking minimal surfaces and minimizers of J_ϵ should have similar property, at least for small ϵ .

Our goal is to show that the convergence of u_{ϵ_k} in Theorem 1.3 is stronger than $L^1_{loc}(\Omega)$, indeed in Section 1.4 we will show that the level sets of u_{ϵ_k} converge uniformly on compact sets to minimal surfaces.

In order to reach this result we need density estimates for level sets of phase transitions, the following section is devoted to the study of this density estimates.

1.3 Density estimates for level sets of phase transitions

The goal of this section is to prove estimates for the Lebesgue measure of the superlevel sets and sublevel sets of minimizers of (1.6). In the next section, we will see that these estimates are crucial for proving that the L^1_{loc} convergence, given by Theorem 1.3, can be improved to a uniform convergence (in the sense of the Hausdorff distance) on compact sets of the level sets of u_ϵ to minimal surfaces.

The following density estimates are proved by Caffarelli and Cordoba in [11]

Theorem 1.4. (Caffarelli-Cordoba) *Given $\alpha > -1$ and $\beta < 1$, if u is a minimizer of J in $B_R(x)$ and $u(x) \geq \alpha$, then there exist a constant c depending only on n and h_0 and a constant $r_0(\alpha, \beta)$ depending on α, β, n , and h_0 such that:*

$$\mathcal{L}^n(\{u > \beta\} \cap B_r(x)) \geq cr^n, \quad (1.21)$$

$$\mathcal{L}^n(\{u < \beta\} \cap B_r(x)) \geq cr^n \quad (1.22)$$

for $r \geq r_0(\alpha, \beta)$, provided that $B_{r+2}(x) \subseteq B_R(x)$.

Before proving this theorem we highlight another analogy between phase transitions and minimal surfaces: all the ideas behind the density estimates for phase transitions and the improving of the convergence for level sets come from analogous results for minimal surfaces.

We recall the standard compactness theorem for sets with minimal perimeter, a proof can be found in the book of Giusti [32].

Theorem 1.5. *If E_n is a sequence of sets with minimal perimeter in Ω then there exists a subsequence E_{n_k} that converges to a set with minimal perimeter E , i.e.,*

$$\chi_{E_{n_k}} \longrightarrow \chi_E \text{ in } L^1_{loc}(\Omega). \quad (1.23)$$

Now we can pass from this L^1_{loc} convergence to a uniform convergence on compact sets using the following density estimates

Theorem 1.6. *Assume that E has minimal perimeter in B_1 and $0 \in \partial E$. There exists a constant $c > 0$ depending only on the dimension n such that for all $r \in (0, 1)$*

$$\mathcal{L}^n(E \cap B_r) \geq cr^n, \quad \mathcal{L}^n(E^c \cap B_r) \geq cr^n.$$

We see a perfect analogy between Theorems 1.3 and 1.4, about phase transitions, and Theorems 1.5 and 1.6, about minimal surfaces. It is clear that, in order to prove the convergence results for phase transitions, we use the same strategy as in the theory of minimal surface. First of all we prove a compactness result and then, with the density estimates, we improve the convergence.

There is a huge literature regarding density estimates for phase transitions, see for instance [11],[52],[42],[41], and [23]. In this section we present the results achieved in [11] and, in particular, we adapt the techniques used in [52] to our specific functional (1.6).

Theorem 1.4 is a direct consequence of the following result:

Theorem 1.7. *Let u be a local minimizer of J in Ω , then:*

(i) *there exist positive constants c, r_0 (depending only on n and h_0) such that*

$$J(u, B_r(x)) \leq cr^{n-1} \quad (1.24)$$

for any $r \geq r_0$, provided that $B_{r+2}(x) \subset \Omega$;

- (ii) for any $\theta_0 \in [0, 1)$, for any $\theta \in (-\theta_0, \theta_0)$ and for any $\mu_0 > 0$, if there exists $K > 0$ such that $\mathcal{L}^n(\{u > \theta\} \cap B_K(x)) \geq \mu_0$, then there exist positive constants c^* , r_0 (depending only on n and h_0) such that

$$\mathcal{L}^n(\{u > \theta\} \cap B_r(x)) \geq c^* r^n \quad (1.25)$$

for any $r \geq r_0$, provided that $B_{r+2}(x) \subset \Omega$. Analogously if $\mathcal{L}^n(\{u < \theta\} \cap B_K(x)) \geq \mu_0$, then

$$\mathcal{L}^n(\{u < \theta\} \cap B_r(x)) \geq c^* r^n \quad (1.26)$$

for any $r \geq r_0$, provided that $B_{r+2}(x) \subset \Omega$.

Proof. For simplicity of notation we introduce two constants C and c , depending only on n and h_0 , that can change from line to line.

- (i) We start by noticing that, from standard energy inequality, we have $J(u, B_1(x_0)) \leq C$ provided that $B_2(x_0) \subset \Omega$. Indeed, we define the following function w on $B_2(x_0)$ that depends only on $r := |x - x_0|$,

$$w(r) = \begin{cases} -1 & \text{if } r \leq 1 \\ 2r - 3 & \text{if } 1 < r \leq 2. \end{cases}$$

From the fact that $|u| \leq 1$ we obtain the inclusion $B_1(x_0) \subset \{w < u\} \subseteq B_2(x_0)$ and, by comparing w with u on the open set $\{w < u\}$, we obtain

$$J(u, B_1(x_0)) \leq J(u, \{w < u\}) \leq J(w, \{w < u\}) \leq J(w, B_2(x_0)) \leq C.$$

We now fix $x_0 \in \Omega$ and $r > 0$ sufficiently large. Let g be a radial smooth function that is identically equal to -1 on $B_{r-1}(x_0)$ and identically to 1 on $\partial B_r(x_0)$. We define $u^* = \min\{u, g\}$. Clearly, since h_0 is bounded and $h_0(-1) = 0$ we obtain $\int_{B_r(x_0)} h_0(u^*) = \int_{B_r(x_0) \setminus B_{r-1}(x_0)} h_0(u^*) \leq C r^{n-1}$.

Since in the H^1 -sense ∇u^* is equal to ∇u or ∇g almost everywhere, we conclude that $\nabla u^* = \nabla g = 0$ on $B_{r-1}(x_0)$. Therefore we have

$$\begin{aligned} J(u, B_r(x_0)) &\leq J(u^*, B_r(x_0)) \leq C \left(\int_{B_r(x_0) \setminus B_{r-1}(x_0)} |\nabla u^*|^2 dx + r^{n-1} \right) \leq \\ &\leq C \left(\int_{B_r(x_0) \setminus B_{r-1}(x_0)} \{|\nabla u|^2 + |\nabla g|^2\} dx + r^{n-1} \right) \leq \\ &\leq C \left(\int_{B_r(x_0) \setminus B_{r-1}(x_0)} |\nabla u|^2 dx + r^{n-1} \right). \end{aligned}$$

Let us now cover $B_r(x_0) \setminus B_{r-1}(x_0)$ with balls $B_1(z_1), \dots, B_1(z_K)$ with radius 1 and with $K \leq C_1 r^{n-1}$ for some constant C_1 .

Now we have $z_i \in B_r(x_0) \setminus B_{r-1}(x_0)$ for all $i = 1, \dots, K$ and, given the assumption $B_{r+2}(x) \subset \Omega$, we have that $B_2(z_i) \subset \Omega$ for all $i = 1, \dots, K$; hence $J(u, B_1(z_i)) \leq C$ and in particular $\int_{B_1(z_i)} |\nabla u|^2 \leq C$ for all $i = 1, \dots, K$. Then, from the estimates above, we obtain

$$\begin{aligned} J(u, B_r(x_0)) &\leq C \left(\sum_{i=1}^K \int_{B_1(z_i)} |\nabla u|^2 dx + r^{n-1} \right) \leq \\ &\leq C(C_1 r^{n-1} + r^{n-1}) \leq c r^{n-1}, \end{aligned} \quad (1.27)$$

for some constant c . The estimate (1.27) is true for every $x_0 \in \Omega$ such that $B_{r+2}(x_0) \subset \Omega$, and this proves (i).

- (ii) We prove the estimate (1.25), the proof of the estimate (1.26) being analogous. We fix $x \in \Omega$ and for simplicity we define $B_r := B_r(x)$. First of all we notice that it is enough to prove (ii) for θ close to -1 . Indeed, assume the result is true for θ^* close to -1 , and let $\theta \in [-\theta_0, \theta_0]$ with $\theta^* \leq -\theta_0$, then

$$\mu_0 \leq \mathcal{L}^n(\{u > \theta\} \cap B_K) \leq \mathcal{L}^n(\{u > \theta^*\} \cap B_K).$$

In general we have that,

$$\int_{B_r \cap \{\theta^* < u \leq \theta\}} h_0(u) dx \geq \mathcal{L}^n(\{\theta^* < u \leq \theta\} \cap B_K) \inf_{u \in [\theta^*, \theta_0]} h_0(u),$$

and with the assumption (1.7) on h_0 we have $\inf_{u \in [\theta^*, \theta_0]} h_0(u) \neq 0$. We obtain the following estimate:

$$\begin{aligned} cr^n &\leq \mathcal{L}^n(\{u > \theta^*\} \cap B_r) \leq \mathcal{L}^n(\{u > \theta\} \cap B_r) + \mathcal{L}^n(\{\theta^* < u \leq \theta\} \cap B_r) \leq \\ &\leq \mathcal{L}^n(\{u > \theta\} \cap B_r) + \frac{1}{\inf_{u \in [\theta^*, \theta_0]} h_0(u)} \int_{B_r} h_0(u) dx \leq \\ &\leq \mathcal{L}^n(\{u > \theta\} \cap B_r) + cr^{n-1}. \end{aligned} \quad (1.28)$$

In the last inequality we used (1.24) for the estimate $\int_{B_r} h_0(u) dx \leq J(u, B_r) \leq cr^{n-1}$. Then we finally obtain:

$$cr^n - cr^{n-1} \leq \mathcal{L}^n(\{u > \theta\} \cap B_r),$$

and for r sufficiently large, for some constant c , we obtain that $cr^n \leq \mathcal{L}^n(\{u > \theta\} \cap B_r)$. From these considerations, in the rest of the proof, we can assume that θ is close to -1 .

We use suitable positive parameters Θ and T : the idea is that we will fix Θ small enough and then choose T suitably large. Set $k \in \mathbb{N}$, we introduce a barrier function $g = g_k \in C^2(B_{(k+1)T})$ so that $-1 \leq g \leq 1$ in $B_{(k+1)T}$, $g = 1$ on $\partial B_{(k+1)T}$ and also g verifies the following inequalities:

$$g + 1 \leq Ce^{-\Theta T} \quad \text{in } B_{kT} \quad (1.29)$$

$$|\Delta g| \leq C\Theta(g + 1) \quad \text{in } B_{(k+1)T}. \quad (1.30)$$

From the last inequality and from our assumption (1.9) on the potential we have, for Θ small enough:

$$|\Delta g| \leq \sqrt{\Theta} h'_0(g) \quad \text{in } B_{(k+1)T}. \quad (1.31)$$

An explicit construction of g can be found in [45] or in [52].

Define $\theta' = \theta - Ce^{-\Theta T}$, if T is large enough we have that $\theta' > -1$. Define also

$$\sigma = \min\{u, g\} \quad \text{and} \quad \beta = \min\{u - \sigma, 1 + \theta'\}.$$

Since $g = 1$ on $\partial B_{(k+1)T}$ we have that $\beta = 0$ on $\partial B_{(k+1)T}$ so we can apply the Gagliardo-Nirenberg-Sobolev inequality and, using the elementary inequality $Aa^2 + \frac{b^2}{A} \geq 2ab$ for all $a, b \in \mathbb{R}$ and $A > 0$, we obtain

$$\begin{aligned} \left(\int_{B_{(k+1)T}} \beta^{\frac{2n}{n-1}} \right)^{\frac{n-1}{n}} &\leq \int_{B_{(k+1)T}} |\beta| |\nabla \beta| = \int_{B_{(k+1)T} \cap \{u - \sigma \leq 1 + \theta'\}} |\beta| |\nabla \beta| \leq \\ &\leq AC \left(\int_{B_{(k+1)T} \cap \{u - \sigma \leq 1 + \theta'\}} |\nabla u|^2 + |\nabla \sigma|^2 - 2|\nabla u| |\nabla \sigma| \right) + \\ &\quad + \frac{C}{A} \int_{B_{(k+1)T} \cap \{u - \sigma \leq 1 + \theta'\}} (u - \sigma)^2. \end{aligned} \quad (1.32)$$

Hence from (1.32) we obtain

$$\begin{aligned} \left(\int_{B_{(k+1)T}} \beta^{\frac{2n}{n-1}} \right)^{\frac{n-1}{n}} &\leq AC \left(\int_{B_{(k+1)T} \cap \{u-\sigma \leq 1+\theta'\}} |\nabla u|^2 - |\nabla \sigma|^2 - 2|\nabla(u-\sigma)||\nabla \sigma| \right) + \\ &+ \frac{C}{A} \int_{B_{(k+1)T} \cap \{u-\sigma \leq 1+\theta'\}} (u-\sigma)^2. \end{aligned} \quad (1.33)$$

Now from the minimality property of u we obtain

$$\int_{B_{(k+1)T} \cap \{u-\sigma \leq 1+\theta'\}} (|\nabla u|^2 - |\nabla \sigma|^2) \leq \int_{B_{(k+1)T} \cap \{u-\sigma \leq 1+\theta'\}} (h_0(\sigma) - h_0(u)).$$

Using this estimate and an integrating by parts in (1.33) we finally get:

$$\begin{aligned} \left(\int_{B_{(k+1)T}} \beta^{\frac{2n}{n-1}} \right)^{\frac{n-1}{n}} &\leq AC \left(\int_{B_{(k+1)T} \cap \{u-\sigma \leq 1+\theta'\}} h_0(\sigma) - h_0(u) + 2\Delta\sigma(u-\sigma) \right) + \\ &+ \frac{C}{A} \int_{B_{(k+1)T} \cap \{u-\sigma \leq 1+\theta'\}} (u-\sigma)^2, \end{aligned} \quad (1.34)$$

where A is a free parameter, to be chosen suitably large in what follows.

Let us now define

$$\Lambda(r) = \mathcal{L}^n(B_r \cap \{u \geq \theta\}).$$

We now estimate the left hand side of (1.34). From (1.29) we obtain that if T is big enough we get $\theta - g \geq \frac{(1-\theta_0)}{2}$ in B_{kT} , hence we have $\beta \geq \frac{(1-\theta_0)}{2} > 0$ in $B_{kT} \cap \{u \geq \theta\}$. Thus there exists a constant C such that the left hand side of (1.34) is bigger than $C\Lambda(kT)^{\frac{n-1}{n}}$.

Let us now estimate the right hand side of (1.34). First of all, we consider the contribution in $\{u \leq \theta\}$. We observe that, since $-1 \leq \sigma \leq u \leq 1$,

$$\begin{aligned} (u+1)^2 - (\sigma+1)^2 - \frac{1}{2}(u-\sigma)^2 &= \\ &= (u+\sigma)(u-\sigma) + 2(u-\sigma) - \frac{1}{2}(u-\sigma)^2 = \\ &= (u-\sigma)\left(\frac{1}{2}u + \frac{3}{2}\sigma + 2\right) \geq 0; \end{aligned}$$

accordingly, recalling (1.9), in $\{\sigma < u \leq \theta\}$ we have

$$\begin{aligned} h_0(u) - h_0(\sigma) &= \int_{\sigma}^u h'_0(\xi) d\xi \geq \\ &\geq C \int_{\sigma}^u (1+\xi) d\xi = C[(u+1)^2 - (\sigma+1)^2] \geq \\ &\geq C(u-\sigma)^2. \end{aligned} \quad (1.35)$$

Consequently, choosing A suitable large and recalling (1.31), the contribution of the right hand side of (1.34) in $\{u \leq \theta\}$ is controlled by

$$\int_{B_{(k+1)T} \cap \{\sigma < u \leq \theta\}} (h_0(\sigma) - h_0(u) + C\sqrt{\Theta} h'_0(\sigma)(u-\sigma)). \quad (1.36)$$

We now show that this quantity is indeed negative. Since we can choose θ to be close to -1 , we have that h_0 and h'_0 are monotone in $(-1, \theta)$. Hence, in $\{\sigma < u \leq \theta\}$, $h_0(\sigma) - h_0(u)$ is negative and, furthermore,

$$|h'_0(\sigma)(u - \sigma)| \leq |h_0(\sigma) - h_0(u)|.$$

Since we assumed Θ to be small, we have shown that the quantity in (1.36) is negative and clearly we can consider in our estimates only the contributions in $\{u > \theta\}$.

Let us now bound the right hand side of (1.34) in $\{u > \theta\}$. First we notice that this term has no contribution in B_{kT} : indeed, from condition (1.29), we have:

$$u \leq \sigma + 1 + \theta' \leq g + 1 + \theta' \leq Ce^{-\Theta T} + \theta - Ce^{-\Theta T} = \theta,$$

and this means that

$$B_{kT} \cap \{\sigma < u - \sigma \leq +1 + \theta'\} \subseteq B_{kT} \cap \{\sigma < u \leq \theta\}.$$

Thus, from all the estimates above and from condition (1.30), it follows that the right hand side of (1.34) is bounded by

$$\int_{(B_{(k+1)T} \setminus B_{kT}) \cap \{u > \theta\}} (h_0(\sigma) - h_0(u) + (\sigma + 1)(u - \sigma) + (u - \sigma)^2). \quad (1.37)$$

Now the integrand is limited, so this term can be bounded by:

$$C \mathcal{L}^n(\{u > \theta\} \cap (B_{(k+1)T} \setminus B_{kT})). \quad (1.38)$$

Collecting all the estimates, we finally get

$$C(\Lambda(kT))^{\frac{n-1}{n}} \leq \Lambda((k+1)T) - \Lambda(kT). \quad (1.39)$$

Let us define $\alpha_k := \Lambda(kT) - \Lambda((k-1)T)$. Notice that

$$\begin{aligned} \sum_{1 \leq j \leq k} \alpha_j &= \Lambda(kT) - \Lambda((k-1)T) + \Lambda((k-1)T) - \Lambda((k-2)T) + \dots \\ &= \Lambda(kT), \end{aligned}$$

and therefore from inequality (1.39) we get

$$C \left(\sum_{1 \leq j \leq k} \alpha_j \right)^{\frac{n-1}{n}} \leq \alpha_{k+1}. \quad (1.40)$$

Now by induction we prove that there exists a constant c such that $\alpha_k \geq ck^{n-1}$. The first step of the induction is true by hypothesis, indeed if we take $T \geq K$ we have $\alpha_1 = \Lambda(T) \geq \Lambda(K) \geq \mu_0$. Suppose $\alpha_j \geq cj^{n-1}$ for every $j \leq k$. We show the same estimate for $k+1$. Recalling the elementary inequality $\int_0^k x^{n-1} dx \leq \sum_{1 \leq j \leq k} j^{n-1}$, we get

$$\begin{aligned} \alpha_{k+1} &\geq C \left(\sum_{1 \leq j \leq k} \alpha_j \right)^{\frac{n-1}{n}} \geq C \left(\sum_{1 \leq j \leq k} j^{n-1} \right)^{\frac{n-1}{n}} \geq \\ &\geq C \left(\int_0^k x^{n-1} dx \right)^{\frac{n-1}{n}} \geq Ck^{n-1} \geq \\ &\geq \frac{C}{2^{n-1}} (k+1)^{n-1}. \end{aligned}$$

This proves that $\alpha_k \geq ck^{n-1}$ for some constant c .

Now we prove that $\Lambda(kT) \geq ck^n$ for some constant c . From all the above estimates we have

$$\begin{aligned} \Lambda(kT) &= \sum_{1 \leq j \leq k} \alpha_j \geq c \sum_{1 \leq j \leq k} j^{n-1} \geq \\ &\geq c \left(\int_0^k x^{n-1} dx \right) \geq \frac{c}{n} k^n. \end{aligned}$$

From the estimate above we obtain:

$$\mathcal{L}^n(\{u > \theta\} \cap (B_{kT})) \geq ck^n,$$

performing the change of variable $kT = r$ we finally obtain

$$\mathcal{L}^n(\{u > \theta\} \cap (B_r)) \geq \frac{c}{T^n} r^n, \quad (1.41)$$

and this proves (ii). □

Theorem 1.4 follows directly from Theorem 1.7.

Proof of Theorem 1.4. For simplicity we define $B_r := B_r(x)$. Since u is a solution of an elliptic equation, from regularity theory for elliptic partial differential equations (see for instance [2],[4],[30]) we have that u is Hölder continuous, so in particular $\mathcal{L}^n(B_1 \cap \{u > \frac{\alpha}{2}\}) \geq \mu_0 > 0$. Thus by Theorem 1.7

$$\mathcal{L}^n(B_r \cap \{u > \frac{\alpha}{2}\}) \geq cr^n,$$

for r large enough.

Now we have two cases, $\beta \leq \frac{\alpha}{2}$ or $\beta > \frac{\alpha}{2}$. In the first case the theorem follows immediately. In the second case we use the same argument used in the proof of Theorem 1.7, where we were restricted to the case θ near to -1 (estimates (1.28)). We obtain

$$\begin{aligned} cr^n &\leq \mathcal{L}^n(\{u > \frac{\alpha}{2}\} \cap B_r) \leq \mathcal{L}^n(\{u > \beta\} \cap B_r) + \mathcal{L}^n(\{\frac{\alpha}{2} < u \leq \beta\} \cap B_r) \leq \\ &\leq \mathcal{L}^n(\{u > \beta\} \cap B_r) + \frac{1}{\inf_{u \in [\frac{\alpha}{2}, \beta]} h_0(u)} \int_{B_r} h_0(u) dx \leq \\ &\leq \mathcal{L}^n(\{u > \beta\} \cap B_r) + cr^{n-1}, \end{aligned}$$

and, for r large enough, we finally get

$$\mathcal{L}^n(B_r \cap \{u > \beta\}) \geq cr^n. \quad \square$$

In the next section, using Theorem 1.4 we improve the convergence of the level sets of minimizers of J_ϵ . The density estimates allow us to pass from a convergence in measure (L^1_{loc} convergence) to a uniform convergence (in the sense of the Hausdorff distance). Combining this result with the regularity theory of minimal surfaces we will obtain an asymptotic flat behaviour of the level sets of u , at least in low dimension.

1.4 Asymptotic behaviour of level sets

We prove that the convergence in Theorem 1.3 is stronger than L^1_{loc} convergence. In the previous sections we say that the convergence is, actually, a uniform convergence on compact sets. We now define what actually means that a sequence of sets converge uniformly on compact set to another sets.

We introduce the Hausdorff distance in \mathbb{R}^n :

Definition 1.4. *Let X and Y be two subset of \mathbb{R}^n , the Hausdorff distance between X and Y is:*

$$d_H(X, Y) := \inf\{\epsilon > 0 \mid X \subseteq Y_\epsilon \text{ and } Y \subseteq X_\epsilon\}. \quad (1.42)$$

Where X_ϵ is:

$$X_\epsilon := \bigcup_{x \in X} \{z \in \mathbb{R}^n \mid \|x - z\| \leq \epsilon\} \quad (1.43)$$

We now prove, using Theorem 1.4, that the convergence in Theorem 1.3 is a convergence in the Hausdorff distance.

Corollary 1.8. *Given $\Omega \subseteq \mathbb{R}^n$ open, let u_ϵ be local minimizers for the energies $J_\epsilon(\cdot, \Omega)$, then there exists a sequence u_{ϵ_k} such that $\{u_{\epsilon_k} = 0\}$ converge in Hausdorff distance to ∂E , where E is a set with minimal perimeter in Ω .*

Proof. We assume by contradiction that $d_H(\{u_{\epsilon_k} = 0\}, \partial E) > 0$ for every $k \geq \bar{k}$, for some \bar{k} . We have two possible cases:

- (i) There exists $\delta > 0$ such that for $k \geq \bar{k}$ exists $x_k \in \{u_{\epsilon_k} = 0\} \cap B_\delta(z_0)$, with $B_{2\delta}(z_0) \subset E$.
- (ii) There exists $\delta > 0$ such that for $k \geq \bar{k}$ exists $x_k \in \{u_{\epsilon_k} = 0\} \cap B_\delta(z_0)$, with $B_{2\delta}(z_0) \subset E^c$.

We analyze the case (i). From the estimate (1.22) we obtain that:

$$\mathcal{L}^n(\{u_{\epsilon_k} < 0\} \cap B_\delta(z_0)) \geq c\mathcal{L}^n(B_\delta(z_0)).$$

We recall that $B_{2\delta}(z_0) \subset E$, in particular $\chi_E(x) = 1$ for every $x \in B_\delta(z_0)$, from this consideration and the estimate above we obtain:

$$\begin{aligned} 1 &\leq \int_{\{u_{\epsilon_k} < 0\} \cap B_\delta(z_0)} |1 - u_{\epsilon_k}| dx \leq \frac{1}{c\mathcal{L}^n(B_\delta(z_0))} \int_{\{u_{\epsilon_k} < 0\} \cap B_\delta(z_0)} |1 - u_{\epsilon_k}| dx = \\ &= \frac{1}{c\mathcal{L}^n(B_\delta(z_0))} \int_{\{u_{\epsilon_k} < 0\} \cap B_\delta(z_0)} |\chi_E - u_{\epsilon_k}| dx \xrightarrow[k \rightarrow \infty]{} 0, \end{aligned}$$

that is a contradiction. The case (ii) is similar. \square

Our purpose, in this section, is to obtain an asymptotic behaviour of the level sets of phase transitions. We have just proved that the level sets of the rescaled phase transitions converge uniformly on compact sets to a minimal surface.

In order to obtain more precise results we must investigate the geometry of minimal surfaces in \mathbb{R}^n . We summarize in the following theorem some fundamental results about minimal surfaces:

Theorem 1.9. *Let E be a set with minimal perimeter in \mathbb{R}^n , then the following holds:*

- (i) if $n \leq 7$, then ∂E is a hyperplane.
- (ii) The Simons cone $\{x_1^2 + x_2^2 + x_3^2 + x_4^2 < x_5^2 + x_6^2 + x_7^2 + x_8^2\}$ is a set with minimal perimeter in \mathbb{R}^8 .
- (iii) If $n \leq 8$ and if we also assume that ∂E is a graph in some direction, then ∂E is a hyperplane.
- (iv) If $n \geq 9$ there exist non-affine minimal graphs

This Theorem combine several classical results. The main contributions are the papers of De Giorgi [16] and [17], Simons [51] and Bombieri, De Giorgi and Giusti [6]. A detailed proof of the Theorem can be found in the book of Giusti [32] and a short proof of (ii) can be found in the paper of De Philippis and Paolini [18].

We consider u , a local minimizer of the functional (1.6) in \mathbb{R}^n with $n \leq 7$, and we assume that $u(0) = 0$. From Corollary 1.8 and Theorem 1.9 we have that $\{u_{\epsilon_k} = 0\}$ uniformly converge on compact sets to ∂E , where E is a set with minimal perimeter in \mathbb{R}^n . We have that ∂E is an hyperplane, because $n \leq 7$. We also have that $0 \in \partial E$ because $u_{\epsilon_k}(0) = 0$ for every k , so we can assume, without lost of generality, that $\partial E = \{x_n = 0\}$. Indeed, if it is not true, we can rotate the coordinates in such a way that the hyperplane ∂E coincide with the hyperplane $\{x_n = 0\}$.

We finally obtain that there exists a sequence $\delta_k \rightarrow 0$ such that:

$$\{u_{\epsilon_k} = 0\} \cap B_1(0) \subseteq \{|x_n| \leq \delta_k\}. \quad (1.44)$$

If we rescale back the minimizers we obtain:

$$\{u = 0\} \cap B_{\frac{1}{\epsilon_k}}(0) \subseteq \{|x_n| \leq \frac{\delta_k}{\epsilon_k}\}. \quad (1.45)$$

This asymptotic behaviour is still true also if we assume that $n = 8$ and $\partial_{x_n} u > 0$. Indeed, in this case, we have that the level sets of u_{ϵ_k} are rescaling of the level sets of u , that is a graph in the e_n direction. We obtain that ∂E is a minimal graph in \mathbb{R}^8 and, from point (iii) in Theorem 1.9, we conclude that ∂E is a hyperplane.

All the arguments above are true not only for the 0-level set but also for all the s -level sets, with $|s| < 1$. In particular (1.45) is true for $\{u = s\}$ with $|s| < 1$.

We notice that the estimate (1.45) gives us an asymptotically flat behaviour of the level sets: from the limit $k \rightarrow \infty$ we obtain information on the level sets in all \mathbb{R}^n . The level sets are trapped into cylinders, and, if $k \rightarrow \infty$, we obtain that $\epsilon_k \rightarrow 0$, so the basis of these cylinders tends to all \mathbb{R}^n . But we don't know the behaviour of the heights of these cylinders. Indeed if $k \rightarrow \infty$ we know that $\delta_k \rightarrow 0$ and $\epsilon_k \rightarrow 0$, but we don't know the limit of the ratio $\frac{\delta_k}{\epsilon_k}$.

We notice that, if we can prove that $\frac{\delta_k}{\epsilon_k} \rightarrow 0$, we can conclude that the level sets of phase transitions, for which (1.45) holds, are hyperplanes and this proves De Giorgi's Conjecture for this minimizers. Savin in [44] gives precise estimates on the behaviour of the heights of this cylinders, using these estimates he proves De Giorgi's Conjecture for phase transitions. In the next Chapter we present the results achieved by Savin in [44].

Chapter 2

Proof of De Giorgi's Conjecture for Phase Transitions

For simplicity, in this section we consider only the 0-level set of the phase transition u . We prove that the 0-level set is a hyperplane, but all the arguments can be adapted to the s -level sets, with $|s| < 1$. From now on we frequently use the following notation: $x = (x', x_n) = (x_1, x_2, \dots, x_{n-1}, x_n) \in \mathbb{R}^n$.

2.1 Main results

In this section we present the statements of the main theorems proved by Savin in [44], that prove De Giorgi's Conjecture for phase transitions.

We have seen in Section 1.4 that, if in the estimate (1.45) we prove that $\frac{\delta_k}{\epsilon_k} \rightarrow 0$, then we conclude that the 0-level set is a hyperplane. In order to obtain this result Savin proves in [44] the following theorem for level sets of u :

Theorem 2.1. (*Improvement of Flatness*) *Let u be a local minimizer of J in $\{|x'| < l\} \times \{|x_n| < l\}$. Assume that $u(0) = 0$ and assume that there exists $\theta \leq l$ such that:*

$$\{u = 0\} \subset \{|x'| < l\} \times \{|x_n| < \theta\}.$$

Then there exist small constants $0 < \eta_1 < \eta_2 < 1$ depending on n and h_0 such that: given $\theta_0 > 0$ there exists $\epsilon_1(\theta_0) > 0$ depending on n , h_0 and θ_0 such that if

$$\frac{\theta}{l} \leq \epsilon_1(\theta_0), \quad \theta_0 \leq \theta,$$

then

$$\{u = 0\} \cap \{|\pi_\xi x| < \eta_2 l\} \times \{|x \cdot \xi| < \eta_2 l\}$$

is included in a flatter cylinder

$$\{|\pi_\xi x| < \eta_2 l\} \times \{|x \cdot \xi| < \eta_1 \theta\},$$

for some unit vector ξ , where $\pi_\xi x = x - (x \cdot \xi)\xi$.

This theorem is valid for any s -level set, with $|s| < 1$, but it was stated for $s = 0$ for simplicity. We will study in detail this theorem in Chapter 4.

Theorem 2.1 gives us a precise estimate on the decay of the heights of the cylinders in the estimate (1.45). Indeed the theorem heuristically says that, if the 0-level set of u

is included in a flat cylinder, then, up to a rotation of coordinates, in the interior it is included in a flatter cylinder. This result is fundamental because it tells us that the cylinders that trap the 0-level set of u in the estimate (1.45) become flatter if $k \rightarrow \infty$.

This deep geometric interpretation of Theorem 2.1 is crucial in order to prove De Giorgi's Conjecture for phase transitions.

Theorem 2.2. *Let u be a local minimizer of the Ginzburg-Landau functional (1.6) in \mathbb{R}^n and $u(0) = 0$, then the following holds:*

- (i) *If $n \leq 7$ then the level sets of u are hyperplanes.*
- (ii) *If $n = 8$ and $\partial_{x_n} u > 0$ then the level sets of u are hyperplanes.*

Obviously, Theorem 2.2 does not imply the full De Giorgi's Conjecture 1.1. Indeed the conjecture is stated for critical points of the functional J , instead Theorem 2.2 concerns local minimizers.

A direct consequence of Theorem 2.2 is the following theorem that gives us a solution for a weaker version of De Giorgi's Conjecture:

Theorem 2.3. *Let $u \in C^2(\mathbb{R}^n)$ be a solution of the partial differential equation*

$$\Delta u = h'_0(u) \quad (2.1)$$

in all \mathbb{R}^n such that:

$$|u| \leq 1, \quad \partial_{x_n} u > 0, \quad \lim_{x_n \rightarrow \pm\infty} u(x', x_n) = \pm 1. \quad (2.2)$$

Then, if $n \leq 8$, the level sets of u are hyperplanes.

2.2 Proof of Theorems 2.2 and 2.3

In this section we prove De Giorgi's Conjecture for phase transitions. We use Theorem 2.1 to prove the following lemma:

Lemma 2.4. *Let u be a local minimizer of J in \mathbb{R}^n with $u(0) = 0$. Suppose that there exist sequences of positive numbers θ_k, l_k and unit vectors ξ_k with $l_k \rightarrow \infty, \theta_k l_k^{-1} \rightarrow 0$ such that*

$$\{u = 0\} \cap (\{|\pi_{\xi_k} x| < l_k\} \times \{|x \cdot \xi_k| < l_k\}) \subset \{|x \cdot \xi_k| < \theta_k\}. \quad (2.3)$$

Then the 0-level set is a hyperplane.

Proof. We fix $\theta_0 > 0$ and we choose k large such that $\theta_k l_k^{-1} \leq \epsilon \leq \epsilon_1(\theta_0)$, where $\epsilon_1(\theta_0)$ is the quantity involved in Theorem 2.1. If $\theta_k \geq \theta_0$ then we apply Theorem 2.1 and we obtain that $\{u = 0\}$ is trapped in a flatter cylinder with height $\eta_1 \theta_k$. We apply Theorem 2.1 repeatedly until the height θ'_k of the new cylinder becomes less than θ_0 .

In some system of coordinates we obtain

$$\left(\{u = 0\} \cap (\{|y'| < l_k\} \times \{|y_n| < l_k\}) \right) \subset \{|y_n| < \theta'_k\} \subset \{|y_n| < \theta_0\}. \quad (2.4)$$

Let θ''_k be the height of the cylinder before the last application of Theorem 2.1, we have $\theta'_k = \eta_1 \theta''_k$. We notice that, if we apply Theorem 2.1 repeatedly, the lowest value of the height of the cylinder that we can obtain is θ_0 , so $\theta_0 \leq \theta''_k$. From this consideration we obtain

$$\theta'_k = \eta_1 \theta''_k \geq \eta_1 \theta_0. \quad (2.5)$$

Suppose we have applied Theorem 2.1 m times: we have that $l'_k = l_k \eta_2^m$ and $\theta'_k = \theta_k \eta_1^m$. Recalling that $\eta_1 < \eta_2$, we finally obtain

$$\frac{\theta'_k}{l'_k} = \frac{\theta_k}{l_k} \cdot \left(\frac{\eta_1}{\eta_2}\right)^m \leq \frac{\theta_k}{l_k} \leq \epsilon. \quad (2.6)$$

Combining the inequalities (2.5) and (2.6) we obtain:

$$l'_k \geq \frac{\eta_1 \theta_0}{\epsilon}.$$

We let $\epsilon \rightarrow 0$ and then, from (2.4), we conclude that $\{u = 0\}$ is included in a strip of width θ_0 . The lemma is proved since θ_0 is arbitrary. \square

Theorem 2.2 is a direct consequence of Lemma 2.4

Proof of Theorem 2.2. We have that u is a local minimizer of J in \mathbb{R}^n and $u(0) = 0$. If one between the two conditions (i) and (ii) is true then, as we have seen in Section 1.4, the estimate (1.45) is true for the 0-level set.

We define $\theta_k = \frac{\delta_k}{\epsilon_k}$ and $l_k = \frac{1}{\epsilon_k}$, clearly $l_k \rightarrow \infty$ and $\theta_k l_k^{-1} \rightarrow 0$. Now u is a local minimizer of J that satisfies the hypothesis of the Lemma 2.4. We conclude that the 0-level set is a hyperplane. All these arguments can be adapted to a general s -level set, with $|s| < 1$, then the theorem is proved. \square

We now prove Theorem 2.3. The proof consists in showing that a solution in all of \mathbb{R}^n of the equation (2.1) that satisfies conditions (2.2), is a local minimizer of the Ginzburg-Landau functional J . The first proof of this result was given by Alberti, Ambrosio and Cabré in [1]. In this proof they used a calibration method, which is quite involved. Another proof can be found in [34]. In this Thesis we present an easier version of the proof that we can find in [9].

Proof of Theorem 2.3. Without loss of generality we can suppose that $u(0) = 0$. Indeed if it is not true, from conditions (2.2), we can easily see that there exists $y \in \mathbb{R}^n$ such that $u(y) = 0$. We define $\tilde{u}(x) = u(x + y)$ and we see that \tilde{u} satisfies the conditions (2.2) and $\tilde{u}(0) = 0$. If we prove that the level sets of \tilde{u} are hyperplanes, then this result is also true for u , because u is obtained by translating \tilde{u} .

We want to show that u is a local minimizer of J in \mathbb{R}^n . Let us consider the functions:

$$u^t(x) := u(x', x_n + t), \text{ for any } t \in \mathbb{R}.$$

By the monotonicity assumption we have that

$$u^t < u^{t'} \text{ in } \mathbb{R}^n, \text{ if } t < t'. \quad (2.7)$$

Thus by the conditions (2.2) we have that the graphs of $u^t(x)$, $t \in \mathbb{R}$, form a foliation filling all of $\mathbb{R}^n \times (-1, 1)$. Moreover, we have that for every $t \in \mathbb{R}$, u^t are solutions of $\Delta u^t = h'_0(u^t)$ in \mathbb{R}^n .

Given a ball B_R we prove that there exists a minimizer $v : \overline{B_R} \rightarrow (-1, 1)$ of J in B_R , such that $v = u$ on ∂B_R . Let v_h be a minimizing sequence for J in B_R , we have that

$v_h = u$ on ∂B_R for every $h \in \mathbb{N}$. We show that the sequence v_h is uniformly bounded in $H^1(B_R)$. Indeed $|v_h| < 1$ and we obtain

$$\int_{B_R} v_h^2 \leq \mathcal{L}^n(B_R) \leq C, \text{ for every } h \in \mathbb{N}.$$

We have that v_h is a minimizing sequence, so $J(v_h, B_R) \rightarrow L$ with $L \in \mathbb{R}$, in particular $J(v_h, B_R) \leq C < \infty$ for every $h \in \mathbb{N}$. With this estimate, recalling that $h_0(u(x)) \geq 0$ for every $x \in B_R$, we obtain the bound of the norm of the weak derivative

$$\int_{B_R} \frac{1}{2} |\nabla v_h|^2 \leq J(v_h, B_R) \leq C \text{ for every } h \in \mathbb{N}.$$

We have that v_h is bounded in $H^1(B_R)$, in particular $\{v_h\}$ is a precompact set in the weak topology of $H^1(B_R)$. Then there exist a subsequence v_{h_k} and an element $v \in H^1(B_R)$ such that v_{h_k} converge to v in the weak $H^1(B_R)$ topology. The domain B_R has a regular boundary and then by Rellich-Kondrachov theorem we have that $H^1(B_R) \subset\subset L^2(B_R)$. Then there exists a subsequence of v_{h_k} that converges to v strongly in $L^2(B_R)$. If we consider another subsequence we have an almost everywhere convergence. Redefining this subsequence by v_h , we finally obtain the following convergences:

$$\begin{aligned} v_h &\rightharpoonup v \text{ in } L^2(B_R), \\ \nabla v_h &\rightharpoonup \nabla v \text{ in } L^2(B_R), \\ v_h &\rightarrow v \text{ in } L^2(B_R), \\ v_h &\rightarrow v \text{ a.e. in } B_R. \end{aligned}$$

We prove that $v \in H^1(B_R)$ is a minimizer for $J(\cdot, B_R)$. We use the lower-semicontinuity of the L^2 norm with respect to the weak topology in order to estimate the kinetic part of the functional and we use the Fatou Lemma for the potential part of the functional. We obtain

$$\begin{aligned} J(v, B_R) &= \int_{B_R} \frac{1}{2} |\nabla v|^2 + \int_{B_R} \liminf_{h \rightarrow 0} h_0(v_h) \leq \\ &\leq \liminf_{h \rightarrow 0} \int_{B_R} \frac{1}{2} |\nabla v_h|^2 + h_0(v_h) = \\ &= \lim_{h \rightarrow 0} J(v_h, B_R). \end{aligned}$$

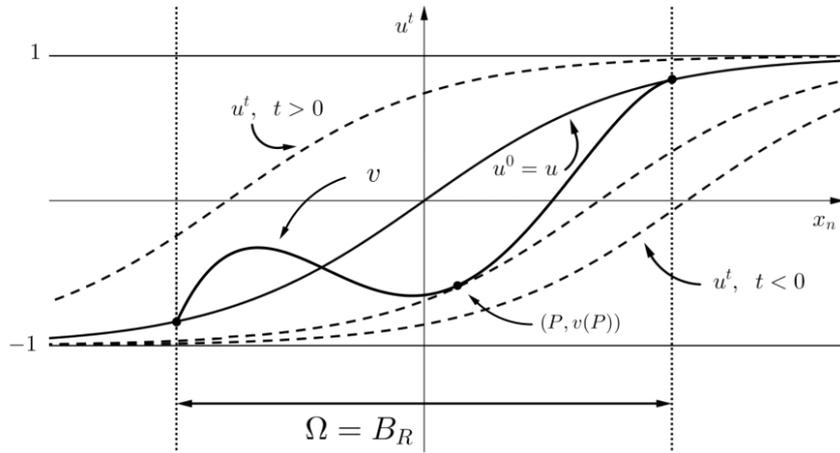
From the fact that v_h is a minimizing sequence we conclude that v is a minimizer of J in B_R and, from the fact that $v_h = u$ on ∂B_R , we have that $v = u$ on ∂B_R .

In particular, v satisfies

$$\begin{cases} \Delta v = v^3 - v & \text{in } B_R \\ |v| < 1 & \text{in } \overline{B_R} \\ v = u & \text{on } \partial B_R. \end{cases} \quad (2.8)$$

We prove that u is the unique solution of (2.8). From this fact follows directly that u is a local minimizer of J in \mathbb{R}^n .

By conditions (2.2) we have that the graph of u^t , in the compact set $\overline{B_R}$, is above the graph of v for t large enough (see Figure 2.1). If $v \neq u$, let us assume that $v < u$ at some point in B_R (the situation $v > u$ somewhere in B_R is done similarly). It follows that,

Figure 2.1: The foliation $\{u^t\}$ and the minimizer v

starting from $t = -\infty$, there will exist a first $t_* < 0$, such that u^{t_*} touches v at a point $P \in \overline{B}_R$. This means that $u^{t_*} \leq v$ in \overline{B}_R and $u^{t_*}(P) = v(P)$.

From (2.7) and from the fact that $v = u = u^0$ on ∂B_R , the point P cannot belong to ∂B_R , because $t_* < 0$.

But then we have that u^{t_*} and v are two solutions of the same semilinear elliptic equation, the graph of u^{t_*} stays below the one of v , and they touch each other at the interior point $(P, v(P))$. This is in contradiction with the strong maximum principle (see Appendix A Corollary A.3).

We have proved that u is a local minimizer of J in \mathbb{R}^n and, by hypothesis $\partial_{x_n} u > 0$, we apply Theorem 2.2 and we obtain that the level sets of u are hyperplanes if $n \leq 8$. \square

Chapter 3

Mean Curvature Properties for Phase Transitions

3.1 Zero mean curvature for phase transitions

To state our results, we need to recall some standard conventions about the sign of the mean curvature of a paraboloid. Let us consider a hyperplane $\pi \subset \mathbb{R}^n$ with normal vector ν . Let S be a hypersurface and P a paraboloid with vertex at some point x , and let us assume that they are tangent to each other and to π at x .

We say that P touches S from below at x in $B_r(x)$ if, for any $y \in S$ and $z \in P$ with $y, z \in B_r(x)$ and $y - z$ in the same direction as ν , we have $(y - z) \cdot \nu \geq 0$. An analogous definition can be given for a paraboloid touching from above.

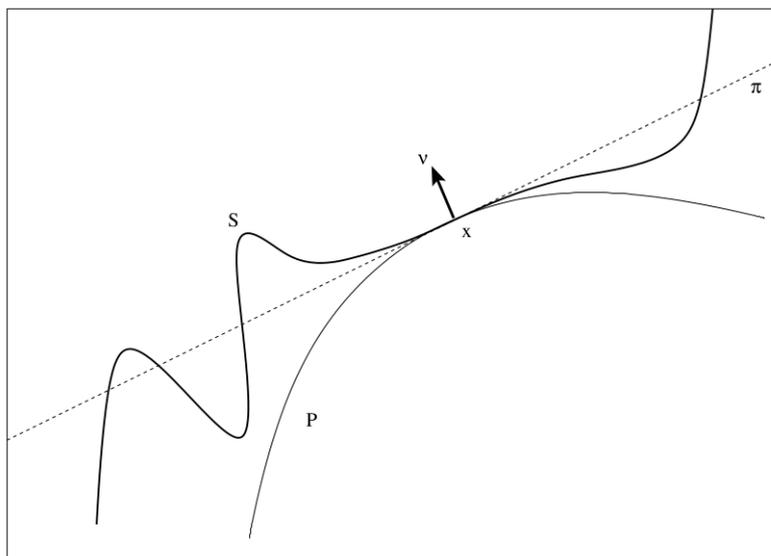


Figure 3.1: P touches S from below at x

Of course, up to a suitable choice of coordinates, one may assume that $x = 0$, $\pi = \{x_n = 0\}$ and $\nu = e_n$. In this set of coordinates, the paraboloid P takes the form

$$\left\{ (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n = \frac{1}{2} x' \cdot M x', M \in \text{Mat}((n-1) \times (n-1)) \right\}.$$

We say that P has non-negative mean curvature if $\text{Tr } M \geq 0$. Analogously, one may define positive, negative, non-positive and zero mean curvature. Obviously, the sign of the mean curvature depends on the orientation of ν , i.e., changing ν to $-\nu$ turns a positive mean curvature into a negative one, and so on. Similarly, changing ν to $-\nu$ turns touching from below into touching from above.

Using the above conventions we define the concept of zero mean curvature in a viscosity sense:

Definition 3.1. *Let $S = \partial E$ be a surface. S satisfies the zero mean curvature equation in the viscosity sense if the following happens:*

let $x^ \in S$ be so that for any $r > 0$,*

$$\mathcal{L}^n((\mathbb{R}^N \setminus E) \cap B_r(x^*)) > 0 \text{ and } \mathcal{L}^n(E \cap B_r(x^*)) > 0$$

assume also that S admits a tangent hyperplane in x^ , then:*

- *if a paraboloid with vertex at x^* touches S from below at x^* , then its mean curvature at x^* must be non-positive;*
- *if a paraboloid with vertex at x^* touches S from above at x^* , then its mean curvature at x^* must be non-negative.*

In particular, if S is C^2 in a neighborhood of x^* , then the mean curvature of S at x^* is zero in the classical sense.

We state the main theorem of this chapter. This theorem is fundamental in order to prove the Improvement of Flatness Theorem.

Theorem 3.1. *Let u be a local minimizer of the Ginzburg-Landau functional (1.6) in \mathbb{R}^n such that $u(0) = 0$ and $|u| \leq 1$. Let $\varrho \in (0, 1)$ and $M \in \text{Mat}((n-1) \times (n-1))$ with*

$$\text{Tr } M > \varrho \|M\| \text{ and } \|M\| \leq \varrho^{-1}.$$

Let

$$\Gamma := \left\{ (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n = \frac{1}{2} x' \cdot M x' \right\}.$$

Then there exist a universal constant $\varrho^ > 0$ and a function $\sigma_0 : (0, 1) \rightarrow (0, 1)$ such that if $\epsilon \in (0, \sigma_0(\varrho))$ and $\varrho \in (0, \varrho^*)$, then Γ cannot touch $\{u_\epsilon = 0\}$ at 0 from below inside the ball $B_{\varrho\sqrt{\epsilon}/\sqrt{\text{Tr } M}}$, where u_ϵ is the rescaled phase transition. More explicitly,*

$$\{u_\epsilon = 0\} \cap \left\{ x_n < \frac{1}{2} x' \cdot M x' \right\} \cap \left\{ |x| < \frac{\varrho\sqrt{\epsilon}}{\sqrt{\text{Tr } M}} \right\} \neq \emptyset.$$

Theorem 3.1 says that $\{u_\epsilon = 0\}$ satisfies the zero mean curvature equation in the viscosity sense, in which we have to specify the size of the neighborhood around the touching point. Indeed Theorem 3.1, roughly speaking, tells us that if we take a paraboloid with non-negative mean curvature, then this paraboloid cannot touch $\{u_\epsilon = 0\}$ from below at 0 in a neighborhood of 0. This fact proves the first point of Definition 3.1.

As we have said above the fact that P has non-negative curvature and the fact that P touches $\{u_\epsilon = 0\}$ from below are matter of conventions. Indeed, if we consider the opposite orientation, i.e., changing ν to $-\nu$, non-negative curvature becomes non-positive curvature and touches $\{u_\epsilon = 0\}$ from below becomes touches $\{u_\epsilon = 0\}$ from above. Using this fact we can reformulate Theorem 3.1 in the following way: if P has non-positive curvature, then P cannot touch $\{u_\epsilon = 0\}$ from above at 0 in a neighborhood of 0. This fact prove

the second point of Definition 3.1. For a rigorous proof that Theorem 3.1 implies that $\{u_\epsilon = 0\}$ satisfies the zero mean curvature equation in the viscosity sense see [50].

We know that $\{u_\epsilon = 0\}$ uniformly converge, on compact sets, to a minimal surface. If we recall that minimal surfaces are surfaces with zero mean curvature, we can say that $\{u_\epsilon = 0\}$ uniformly converge, on compact sets, to a surface with zero mean curvature. Roughly speaking Theorem 3.1 tells us that $\{u_\epsilon = 0\}$ attains a weak version of zero mean curvature property even “before” converging to the limit surface. This fact is crucial, as we will see in Chapter 4, for proving the Improvement of Flatness result.

The main purpose of this chapter is to prove Theorem 3.1 and, in order to do this, in the next section we introduce some useful “barriers” functions.

3.2 Barrier functions

Before going into the details of the argument, we would like to point out some heuristic ideas underlying the construction given below. The crucial idea, which goes back to De Giorgi, is that one dimensional phase transitions are the ones which encode much information on the system. Following this belief, we will construct two barriers, which are suitable modification of one-dimensional solutions.

The first barrier, built in Lemma 3.2, is radially symmetric. More precisely is flat in a ball and then radially increasing. Clearly, since the solution we consider does not has such symmetry, this barrier may provide good bounds in some directions, but poor bounds in other directions. Therefore, in the following section, we will have to slide this barrier to obtain information in all the domain we are interested in.

The second barrier we need is constructed in Lemma 3.3. This is a modification of a one-dimensional solution which takes into account the distance from the level sets. Equation (1.5) will relate the second derivatives of this barrier with the mean curvature of the level sets of our rescaled phase transitions, from this relation we will obtain some useful estimates.

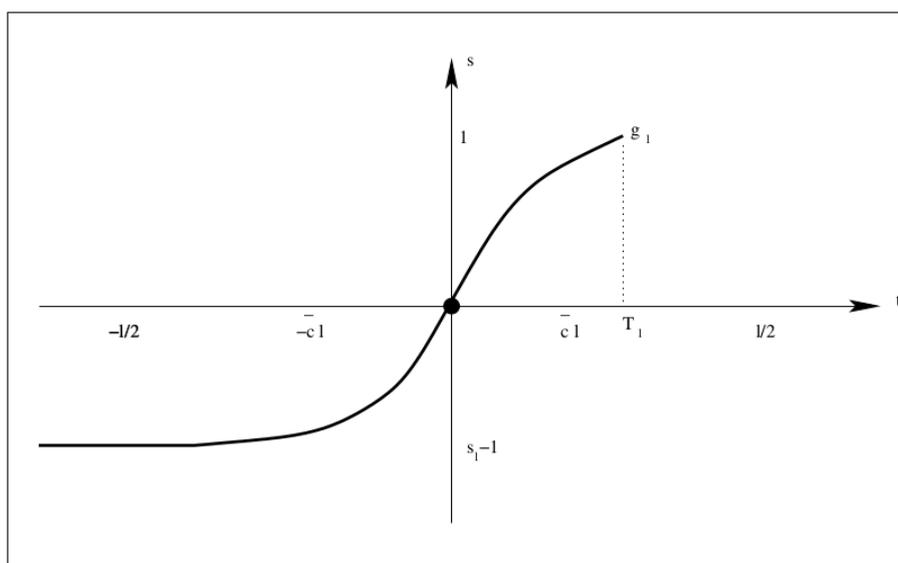


Figure 3.2: The function g_1 introduced in Lemma 3.2

We now construct the first comparison function (sketched in Figure 3.2) that will be

used in the proof of Theorem 3.1. In what follows universal constants are constants that depend only on n and h_0 .

Lemma 3.2. *There exist universal constants $\bar{l} > 1$ and $0 < \bar{c} \leq \frac{1}{2}$ so that, if $l \geq \bar{l}$, we can find $T_l \in [\bar{c}l, \frac{l}{2}]$ and a non-decreasing function*

$$g_l \in C^0(-\infty, T_l) \cap C^{1,1}(-\infty, 0) \cap C^2((-\bar{c}l, T_l) \setminus \{0\})$$

which is constant in an interval I containing $(-\infty, -\frac{l}{2}]$, with $g'_l > 0$ outside I , satisfies $g_l(0) = 0$, $g_l(T_l) = 1$, and if we define

$$\Psi^{y,l}(x) := g_l(|x - y| - l), \quad (3.1)$$

then $\Psi^{y,l}$ is a strict supersolution of 1.5 in the viscosity sense in $B_{T_l+l}(y) \setminus \partial B_l(y)$.

Namely, g_l is constructed as follows. There exist constants $0 < \bar{c}_1 < \bar{C}_1, \bar{C}_2$ so that, if we define

$$\begin{aligned} s_l &:= e^{-\bar{c}_1 l}, \\ h_l(s) &:= \begin{cases} h_0(s) - h_0(s_l - 1) - \frac{\bar{C}_2}{l}((1+s)^2 - s_l^2) & \text{if } s_l - 1 < s < 0 \\ h_0(s) + h_0(1 - s_l) + \frac{\bar{C}_2}{l}((1-s)^2 + s_l(1-s)) & \text{if } 0 \leq s < 1, \end{cases} \\ H_l(s) &:= \int_0^s \frac{1}{\sqrt{2h_l(\xi)}} d\xi, \text{ for any } s \in (-1, 1), \\ H_0(s) &:= \int_0^s \frac{1}{\sqrt{2h_0(\xi)}} d\xi, \text{ for any } s \in (-1, 1), \end{aligned}$$

then the following holds:

- (i) $h_l(s) > 0$ in $s_l - 1 < s < 1$; in particular, H_l is well defined and strictly increasing for $s_l - 1 < s < 1$ and thus we may define $g_l(t) := H_l^{-1}(t)$ for $t \in (s_l - 1, 1)$;
- (ii) $g_l(t)$ is defined to be constantly equal to $s_l - 1$ for $t \leq H_l(s_l - 1)$;
- (iii) the following estimates on H_l hold:

$$H_l(1) \leq \frac{l}{2}; \quad (3.2)$$

$$H_l(s_l - 1) \geq -\frac{l}{2}; \quad (3.3)$$

$$H_0(s) \leq H_l(s) - \frac{\bar{C}_1}{l} \log(1-|s|) \quad \forall |s| \leq 1 - e^{-\bar{c}_1 l/2}; \quad (3.4)$$

$$H_l(1 - e^{-\bar{c}_1 l/2}) \geq \bar{c}l; \quad (3.5)$$

$$H_l(e^{-\bar{c}_1 l/2} - 1) \leq -\bar{c}l. \quad (3.6)$$

Proof. We will focus first on proving (3.2), (3.3), (3.4) and the viscosity supersolution property of $\Psi^{y,l}$.

The proof will consider separately the cases $s_l - 1 < s < 0$ and $0 \leq s < 1$. Let us first consider the case $s_l - 1 < s < 0$. From condition (1.11),

$$h_l(s) \geq h_0(\theta^* - 1)/2 \quad (3.7)$$

if $\theta^* - 1 \leq s < 0$, provided l is suitably large. Also, in light of Lemma B.2, we get

$$h_0(s) - h_0(s_l - 1) \geq c((1 + s)^2 - s_l^2), \quad (3.8)$$

for $s_l - 1 < s < \theta^* - 1$, therefore

$$\text{const}(h_0(s) - h_0(s_l - 1)) \leq h_l(s), \quad (3.9)$$

for $s_l - 1 < s < \theta^* - 1$, provided l is sufficiently large. Now, from the inequality (3.7) and from the conditions on the potential (1.11), we conclude that $h_l(s) > 0$ in $s_l - 1 < s < 0$. This shows that H_l is well defined and strictly increasing in this case. Also, from the definition of H_l and from (3.7), (3.8) and (3.9), we obtain

$$\begin{aligned} -H_l(s_l - 1) &= \text{const} \int_{s_l - 1}^0 \frac{1}{\sqrt{h_l(\xi)}} d\xi \\ &= \text{const} \left(\int_{\theta^* - 1}^0 \frac{1}{\sqrt{h_l(\xi)}} d\xi + \int_{s_l - 1}^{\theta^* - 1} \frac{1}{\sqrt{h_l(\xi)}} d\xi \right) \\ &\leq \text{const} \left(1 + \int_{s_l - 1}^{\theta^* - 1} \frac{1}{\sqrt{h_l(\xi)}} d\xi \right) \\ &\leq \text{const} \left(1 + \int_{s_l - 1}^{\theta^* - 1} \frac{1}{\sqrt{(1 + \xi)^2 - s_l^2}} d\xi \right) \\ &\leq \text{const} \left(1 + \int_{s_l - 1}^0 \frac{1}{\sqrt{(1 + \xi)^2 - s_l^2}} d\xi \right), \end{aligned}$$

hence, from Lemma B.3, we get

$$H_l(s_l - 1) \geq -\frac{l}{2},$$

provided \bar{c}_1 is suitably small. This proves estimate (3.3).

We now show that $\Psi^{y,l}$ is a viscosity supersolution of (1.5) when $|x - y| < l$ (i.e., when $s = g_l(t) < 0$; here and in what follows, we often use the notation $t = |x - y| - l$ and $s = g_l(t) = \Psi^{y,l}(x)$).

Of course, if $|x - y| < \frac{l}{2}$, then $\Psi^{y,l}(x) = s_l - 1$ by (3.3) and the definition of g_l . Therefore, by Lemma B.7,

$$\Delta \Psi^{y,l}(x) \leq 0 < h'_0(s_l - 1) = h'_0(\Psi^{y,l}(x)), \quad (3.10)$$

showing that the viscosity supersolution property of $\Psi^{y,l}$ holds in $\{\Psi^{y,l} = s_l - 1\}$, and, in particular, if $|x - y| < \frac{l}{2}$. Hence, we can now concentrate on the case $\frac{l}{2} \leq |x - y| < l$. In view of Lemma B.6,

$$g'_l(t) = \sqrt{2h_l(g(t))}, \quad g''_l(t) = h'_l(g_l(t)).$$

Thus, by Lemma B.4, we have

$$\begin{aligned} \Delta \Psi^{y,l}(x) &= g''(t) + g'(t) \frac{n-1}{|x-y|} \\ &\leq h'_l(g_l(t)) + K(n-1) \sqrt{h_l(g(t))} \frac{1}{|x-y|} \\ &\leq h'_l(g_l(t)) + \frac{2K(n-1) \sqrt{h_l(g(t))}}{l}, \end{aligned} \quad (3.11)$$

for $|x - y| \geq \frac{l}{2}$, provided K is suitably large.

Hence, by definition of h_l , we get (using again the notation $s = g_l(t)$)

$$h_l(s) \leq h_0(s) - h_0(s_l - 1) \leq h_0(s)$$

and

$$h'_l(s) = h'_0(s) - \frac{2\bar{C}_2}{l}(1 + s)$$

in $s_l - 1 < s < 0$, hence

$$\Delta\Psi^{y,l}(x) < h'_0(s) - \frac{2\bar{C}_2}{l}(1 + s) + \frac{2K(n-1)}{l}\sqrt{h_0(s)}, \quad (3.12)$$

for $s_l - 1 < s < 0$. By condition (1.8), we get, for \bar{C}_2 suitably large,

$$\frac{2\bar{C}_2}{l}(1 + s) \geq \frac{2K(n-1)}{l}\sqrt{h_0(s)} \quad (3.13)$$

and therefore

$$\Delta\Psi^{y,l}(x) < h'_0(\Psi^{y,l}(x)) \quad (3.14)$$

for $s_l - 1 < g_l(t)$ and $|x - y| \geq \frac{l}{2}$.

Estimates (3.10) and (3.14) show that $\Psi^{y,l}$ is a strict viscosity supersolution of (1.5) at any point x so that $|x - y| < l$. This proves that $\Psi^{y,l}$ is a strict supersolution of (1.5) in $s_l - 1 < s < 0$.

Let us now prove (3.4) for $e^{-\bar{c}_1 l/2} - 1 \leq s \leq 0$. Observe that by definition of h_l , recalling condition (1.8),

$$\begin{aligned} h_0(s) - h_l(s) &\leq h_0(s_l - 1) + \frac{\bar{C}_2}{l}((1 + s)^2 - s_l^2) \leq C s_l^2 + \frac{\bar{C}_2}{l}((1 + s)^2 - s_l^2) \\ &\leq \frac{2\bar{C}_2}{l}(1 + s)^2, \end{aligned} \quad (3.15)$$

provided l is sufficiently large. Furthermore, from (3.7), (3.8) and (3.9), it follows that

$$h_l(s) \geq \text{const}(1 + s)^2, \quad (3.16)$$

if $e^{-\bar{c}_1 l/2} - 1 \leq s \leq 0$ and l is large enough. Also, using Lemma B.1 we obtain

$$\begin{aligned} H_0(s) - H_l(s) &= \text{const} \int_s^0 \frac{1}{\sqrt{h_l(\xi)}} - \frac{1}{\sqrt{h_0(\xi)}} d\xi \\ &= \text{const} \int_s^0 \frac{\sqrt{h_0(\xi)} - \sqrt{h_l(\xi)}}{\sqrt{h_0(\xi)h_l(\xi)}} d\xi \\ &\leq \text{const} \int_s^0 \frac{h_0(\xi) - h_l(\xi)}{(\sqrt{h_0(\xi)} + \sqrt{h_l(\xi)})\sqrt{h_0(\xi)h_l(\xi)}} d\xi \\ &\leq \text{const} \int_s^0 \frac{h_0(\xi) - h_l(\xi)}{h_0(\xi)\sqrt{h_l(\xi)}} d\xi. \end{aligned}$$

Consequently, from condition (1.8) and from the inequalities (3.15) and (3.16), we obtain

$$H_0(s) - H_l(s) \leq \frac{\text{const}}{l} \int_s^0 \frac{d\xi}{1 + \xi} \leq -\frac{\text{const}}{l} \log(1 + s)$$

thus proving (3.4) for $e^{-\bar{c}_1 l/2} - 1 \leq s \leq 0$. This completes the proof in the case $s_l - 1 < s < 0$.

Let us now consider the case $0 \leq s < 1$. In this case, $h_l > 0$ by definition, thus H_l is well defined and strictly increasing in $[0, 1]$. Setting $t = |x - y| - l$ and $s = g_l(t) = \Psi^{y,l}(x)$, we notice that $s \geq 0$ corresponds to $|x - y| \geq l$, therefore, arguing as in (3.11), we have,

$$\Delta \Psi^{y,l}(x) \leq h'_l(g_l(t)) + \frac{K(n-1)\sqrt{h_l(g(t))}}{l} \quad (3.17)$$

if $|x - y| \geq l$, provided K is large enough. Since, by definition of h_l and (1.8),

$$h_l(s) \leq \text{const}(h_0(s) + h_0(1 - s_l))$$

for \bar{C}_2 large enough, it follows that

$$\begin{aligned} \Delta \Psi^{y,l}(x) &< h'_0(s) - \frac{\bar{C}_2}{l}(2(1-s) + s_l) + \frac{K(n-1)}{l} \sqrt{(h_0(s) + h_0(1 - s_l))} \\ &\leq h'_0(s); \end{aligned}$$

if \bar{C}_2 is suitably large, where, in the last estimate, (1.8) has been used once more together with the simplest inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$. Thus $\Psi^{y,l}(x)$ is a strict viscosity supersolution of (1.5) for $|x - y| > l$, provided $\Psi^{y,l}(x)$ is well defined.

We now need to prove (3.4) in the case $0 \leq s \leq 1 - e^{-\bar{c}_1 l/2}$. To this end, first notice that, if $0 \leq s \leq 1 - e^{-\bar{c}_1 l/2}$, we have $1 - s \geq \sqrt{s_l}$ and therefore

$$s_l^2 \leq s_l(1-s)^2 \leq \frac{1}{l}(1-s)^2, \quad (3.18)$$

if l is large enough. The definition of h_l , (1.8) and (3.18) imply that

$$h_l(s) - h_0(s) \leq \text{const} \left(s_l^2 + \frac{(1-s)^2}{l} \right) \leq \frac{\text{const}}{l} (1-s)^2, \quad (3.19)$$

for $0 \leq s \leq 1 - e^{-\bar{c}_1 l/2}$. On the other hand, the definition of h_l and (1.8) lead to

$$h_l(s) \geq h_0(s) + h_0(1 - s_l) \geq \text{const}(1 - s)^2, \quad (3.20)$$

for $0 \leq s \leq 1 - e^{-\bar{c}_1 l/2}$. Also we have that

$$\begin{aligned} H_0(s) - H_l(s) &= \text{const} \int_0^s \frac{1}{\sqrt{h_0(\xi)}} - \frac{1}{\sqrt{h_l(\xi)}} d\xi \\ &= \text{const} \int_0^s \frac{\sqrt{h_l(\xi)} - \sqrt{h_0(\xi)}}{\sqrt{h_0(\xi)} h_l(\xi)} d\xi \\ &\leq \text{const} \int_0^s \frac{h_l(\xi) - h_0(\xi)}{(\sqrt{h_0(\xi)} + \sqrt{h_l(\xi)}) \sqrt{h_0(\xi)} h_l(\xi)} d\xi \\ &\leq \text{const} \int_0^s \frac{h_l(\xi) - h_0(\xi)}{h_0(\xi) \sqrt{h_l(\xi)}} d\xi. \end{aligned} \quad (3.21)$$

Then, from estimates (3.18), (3.19), (3.21), condition (1.8) and Lemma B.1, we obtain

$$H_0(s) - H_l(s) \leq \frac{\text{const}}{l} \int_0^s \frac{d\xi}{1-\xi} = -\frac{\text{const}}{l} \log(1-s)$$

if $0 \leq s \leq 1 - e^{-\bar{c}_1 l/2}$. This proves (3.4) in the case $0 \leq s < 1 - e^{-\bar{c}_1 l/2}$.

Let us now prove (3.2). Using the definition of H_l , h_l and (1.8), we get

$$\begin{aligned} H_l(1) &\leq \text{const} \int_0^1 \frac{d\xi}{\sqrt{h_0(\xi) + h_0(1-s_l)}} \leq \text{const} \int_0^1 \frac{d\xi}{\sqrt{(1-\xi)^2 + s_l^2}} \\ &\leq \text{const} \int_0^1 \frac{d\xi}{1-\xi + s_l} \leq \text{const} \log\left(\frac{1}{s_l}\right). \end{aligned}$$

This proves (3.2) provided \bar{c}_1 is chosen to be suitably small.

In particular estimate (3.3) says that, by construction, g_l is constant in $(-\infty, -\frac{l}{2}]$. Also, estimates (3.4) and (1.8) imply that

$$H_l(1 - e^{-\bar{c}_1 l/2}) \geq \text{const} \int_0^{1-e^{-\bar{c}_1 l/2}} \frac{d\xi}{1-\xi} - \text{const} = \text{const} \cdot l - 1 \geq \bar{c}l,$$

provided l is large enough and \bar{c} is small enough, and, analogously,

$$H_l(e^{-\bar{c}_1 l/2} - 1) \leq -\bar{c}l,$$

proving (3.5) and (3.6). These estimates also imply that g_l is strictly increasing at least in $(-\bar{c}l, \bar{c}l)$.

Also, if $T_l := H_l(1)$, by (3.2) and (3.5), we have that $T_l \in [\bar{c}l, l/2]$. We finally notice that the extension in (ii) is $C^{1,1}$, since by Lemma B.6, if $t = H_l(s_l - 1)$,

$$g'_l(t) = \sqrt{2h_l(g_l(t))} = \sqrt{2h_l(s_l - 1)} = 0.$$

This ends the proof of Lemma 3.2 □

We now introduce the second comparison function. This function is an appropriate modification of the comparison function in Lemma 3.2, in order to deal with distance function:

Lemma 3.3. *Let $0 < \epsilon \leq \sigma \leq \delta < 1$, $\xi \in \mathbb{R}^{n-1}$, and $M \in \text{Mat}((n-1) \times (n-1))$. Let Γ be the hypersurface defined as*

$$\Gamma := \left\{ x_n = \frac{\epsilon}{2} x' \cdot M x' + \sigma \xi \cdot x' \right\} \cap \{|x'| < \frac{\sigma}{\epsilon}\}$$

and assume that

$$\text{Tr } M \geq \delta, \quad \|M\| \leq 2/\delta, \quad |\xi| \leq 1/\delta.$$

Define $d_\Gamma(x)$ to be the signed distance of x from Γ , with the assumption that d_Γ is positive above Γ .

Then there exist functions $\sigma_0 : (0, +\infty) \rightarrow (0, 1)$ and $C_0 : (0, +\infty) \rightarrow (0, 1)$ and a number $T_{\epsilon, \delta} \in [0, C_0(\delta) \log(1/\epsilon)]$ such that if $\epsilon \leq \sigma \leq \sigma_0(\delta)$ we can find a function g_Γ with the following properties:

- $g_\Gamma \in C^{1,1}(-\infty, T_{\epsilon, \delta})$;
- g_Γ is constant in $(-\infty, -C_0(\delta) \log(1/\epsilon)]$;
- $g_\Gamma(0) = 0$ and $g_\Gamma(T_{\epsilon, \delta}) = 1$;
- g_Γ is C^2 with g'_Γ non vanishing outside the set where it is constant;

- $g_\Gamma(d_\Gamma(x))$ is a strict viscosity supersolution of (1.5) in its domain of definition (that is, provided $d_\Gamma(x) < T_{\epsilon,\delta}$).

Namely, g_Γ is constructed as follows. Let $c_1 > 0$ be suitably small and let $\rho \in C^1(\mathbb{R})$ be a non-decreasing function so that $\rho(0) = 0$, $\rho(s) = -1$ for $s \leq -1/2$ and $\rho(s) = 1$ for $s \geq 1/2$. For any $s \in (0, 1)$ we define

$$h_\Gamma := \max\{0, h_0(s) + c_1\delta\epsilon\rho(s)\}. \quad (3.22)$$

Let $s_{\delta,\epsilon}$ be the point near -1 for which $h_0(s_{\delta,\epsilon}) = c_1\delta\epsilon$. Define also

$$H_\Gamma(s) := \int_0^s \frac{1}{\sqrt{2h_\Gamma(\xi)}} d\xi.$$

Then:

- (i) There exists a constant $c^\# \in (0, 1)$ so that

$$c^\#\sqrt{\delta\epsilon} \leq 1 + s_{\delta,\epsilon} \leq \frac{1}{c^\#}\sqrt{\delta\epsilon} \quad (3.23)$$

- (ii) for any $s_{\delta,\epsilon} < s \leq 1$,

$$h_\Gamma(s) > 0; \quad (3.24)$$

in particular, H_Γ is well defined and strictly increasing in $[s_{\delta,\epsilon}, 1]$. Thus we may define $g_\Gamma(t) := H_\Gamma^{-1}(t)$ for any $t \in [H_\Gamma(s_{\delta,\epsilon}), H_\Gamma(1)]$ and extend g_Γ to be constantly $s_{\delta,\epsilon}$ for $t \leq H_\Gamma(s_{\delta,\epsilon})$. In particular, if $g_\Gamma > s_{\delta,\epsilon}$, then $g'_\Gamma(t) > 0$.

Proof. First we observe that (3.23) follows from (1.8): indeed, if c and C are as in (1.8), then

$$\left(\frac{c_1}{C}\right)^{1/2}(\sqrt{\delta\epsilon}) \leq 1 + s_{\delta,\epsilon} \leq \left(\frac{c_1}{c}\right)^{1/2}(\sqrt{\delta\epsilon}),$$

and we can choose $c^\# = \min\left\{\left(\frac{c_1}{C}\right)^{1/2}, \left(\frac{c}{c_1}\right)^{1/2}\right\}$.

Without loss of generality we may assume $s_{\delta,\epsilon} < -1 + \theta^*$, in order to use (1.9). Note that since by (1.9), h_0 is increasing in $[s_{\delta,\epsilon}, \theta^*)$, we get $h_0(s) > c_1\delta\epsilon$ in $(s_{\delta,\epsilon}, \theta^*)$. Moreover, from (1.8), if c_1 is small enough we may suppose $h_0(s) > c_1\delta\epsilon$ for $s_{\delta,\epsilon} < s < 0$. From the above discussions, (3.24) follows.

Notice that the constant extension of g_Γ is $C^{1,1}$ since, by Lemma B.6, if $t = H_\Gamma(s_{\delta,\epsilon})$,

$$g'_\Gamma(t) = \sqrt{2h_\Gamma(g_\Gamma(t))} = \sqrt{2h_\Gamma(s_{\delta,\epsilon})} = 0.$$

To estimate the domain in which g_Γ is strictly increasing we have therefore to estimate $H_\Gamma(s_{\delta,\epsilon})$ and $H_\Gamma(1)$. Using Lemma B.2, one obtains

$$h_\Gamma(s) \geq h_0(s) - c_1\delta\epsilon = h_0(s) - h_0(s_{\delta,\epsilon}) \geq \text{const}\sqrt{(1+s)^2 - (1+s_{\delta,\epsilon})^2} \quad (3.25)$$

for any $s \in [s_{\delta,\epsilon}, -1 + \theta^*]$. On the other hand, for any $s \in [-1 + \theta^*, 0]$, (1.11) implies that

$$h_\Gamma(s) \geq h_0(-1 + \theta^*) - c_1\delta\epsilon \geq h_0(-1 + \theta^*)/2. \quad (3.26)$$

Therefore, using the definition of H_Γ , (3.23), (3.25), (3.26) and Lemma B.3, we get

$$\begin{aligned} -H_\Gamma(s_{\delta,\epsilon}) &= \int_{s_{\delta,\epsilon}}^0 \frac{1}{\sqrt{2h_\Gamma(\xi)}} d\xi = \int_{-1+\theta^*}^0 \frac{1}{\sqrt{2h_\Gamma(\xi)}} d\xi + \int_{s_{\delta,\epsilon}}^{-1+\theta^*} \frac{1}{\sqrt{2h_\Gamma(\xi)}} d\xi \\ &\leq \text{const}\left(1 + \int_{s_{\delta,\epsilon}}^{-1+\theta^*} \frac{d\xi}{\sqrt{(1+\xi)^2 - (1+s_{\delta,\epsilon})^2}}\right) \leq C_0(\delta) \log\left(\frac{1}{\epsilon}\right), \end{aligned}$$

or, equivalently,

$$H_\Gamma(s_{\delta,\epsilon}) \geq -C_0(\delta) \log\left(\frac{1}{\epsilon}\right). \quad (3.27)$$

This completes the desired estimate on $H_\Gamma(s_{\delta,\epsilon})$.

Let us now estimate $H_\Gamma(1)$: from the definition of h_Γ and (1.8),

$$\begin{aligned} H_\Gamma(1) &= \int_0^1 \frac{1}{\sqrt{2h_\Gamma(\xi)}} d\xi \leq \text{const} \left(\int_0^{1/2} \frac{d\xi}{1-\xi} + \int_{1/2}^1 \frac{d\xi}{\sqrt{c(1-\xi)^2 + c_1\delta\epsilon}} \right) \\ &\leq \text{const} \left(\int_0^{1/2} \frac{d\xi}{1-\xi} + \int_{1/2}^1 \frac{d\xi}{1-\xi + \sqrt{\delta\epsilon}} \right) \\ &\leq \text{const}(1 - \log(\delta\epsilon)) \leq -C_0(\delta) \log(\epsilon), \end{aligned}$$

or, equivalently,

$$H_\Gamma(1) \leq C_0(\delta) \log\left(\frac{1}{\epsilon}\right). \quad (3.28)$$

The claims on the domain of g_Γ are consequences of (3.27) and (3.28).

Now we deal with the proof of the viscosity supersolution property of g_Γ . First of all notice that in an appropriate coordinate system we have

$$D^2 d_\Gamma = \text{diag}\left(\frac{-k_1}{1-d_\Gamma k_1}, \dots, \frac{-k_{n-1}}{1-d_\Gamma k_{n-1}}, 0\right) \in \text{Mat}(n \times n),$$

where the k_i , with $i = 1, \dots, n-1$, are the principal curvatures of Γ at the point where the distance is realized (see [31] for further details). We also define P as the paraboloid describing Γ , i.e.,

$$P(x') := \frac{\epsilon}{2} x' \cdot M x' + \sigma \xi \cdot x'.$$

Notice that, by hypothesis on M and ξ , $|\nabla P| \leq 1$; thus, by the mean curvature equation (see, for instance, equation (14.103) of [31]), it follows that

$$\begin{aligned} \sum_{i=1}^{n-1} k_i &= \sum_{i=1}^{n-1} \partial_i \left(\frac{\partial_i P}{\sqrt{1+|\nabla P|^2}} \right) = \frac{\Delta P}{\sqrt{1+|\nabla P|^2}} - \frac{(D^2 P \nabla P) \cdot \nabla P}{(1+|\nabla P|^2)^{3/2}} \\ &\geq \frac{1}{2} \Delta P - \text{const} |\nabla P|^2 \|D^2 P\|. \end{aligned}$$

Consequently, if x is so that $|d_\Gamma| \leq C_0(\delta) \log(\frac{1}{\epsilon})$, since, by hypothesis on the paraboloid P , we have that $|k_i| \leq C_1(\delta)\epsilon$, we obtain

$$\begin{aligned} \Delta d_\Gamma = \text{Tr}(D^2 d_\Gamma) &= \sum_{i=1}^{n-1} \frac{-k_i}{1-d_\Gamma k_i} = - \sum_{i=1}^{n-1} k_i - \sum_{i=1}^{n-1} \frac{d_\Gamma k_i^2}{1-d_\Gamma k_i} \\ &\leq - \sum_{i=1}^{n-1} k_i + 2(C_1(\delta)\epsilon)^2 \log\left(\frac{1}{\epsilon}\right) \\ &\leq -\frac{1}{2} \Delta P + \text{const} |\nabla P|^2 \|D^2 P\| + C_1(\delta)\epsilon^{3/2} \\ &\leq -\frac{\epsilon\delta}{2} + C_2(\delta)(\epsilon\sigma^2 + \epsilon^{3/2}) \leq -\frac{\epsilon\delta}{2} + C_3(\delta)\epsilon\sigma^{1/2}, \end{aligned} \quad (3.29)$$

all these estimates are true for ϵ small enough.

Therefore, if $d_\Gamma(x) \in (H_\Gamma(s_{\delta,\epsilon}), H_\Gamma(1))$ (and thus, by (3.27) and (3.27), $|d_\Gamma(x)| \leq C_0(\delta) \log(1/\epsilon)$ and $g'_\Gamma(d_\Gamma(x)) > 0$), by Lemma B.5 we have

$$\begin{aligned} \Delta(g_\Gamma(t)) &= g''_\Gamma(t) + g'_\Gamma(t)\Delta d_\Gamma(t) \\ &\leq g''_\Gamma(t) - \frac{\epsilon}{2}(\delta - C_4(\delta)\sigma^{1/2})g'_\Gamma(t), \end{aligned} \quad (3.30)$$

where we are using the notation $t = d_\Gamma(x)$. Taking into account Lemma B.6, by (3.30) we get

$$\Delta g_\Gamma(t) \leq h'_\Gamma(s) - \frac{\epsilon}{2}(\delta - C_4(\delta)\sigma^{1/2})\sqrt{2h_\Gamma(s)},$$

where we are using the notation $s = g_\Gamma(d_\Gamma(x))$.

Now we choose $\sigma_0(\delta)$ small such that $\delta - C_4(\delta)\sigma^{1/2} \geq \delta/2$ for $\sigma \leq \sigma_0(\delta)$. Thus, if $|d_\Gamma(x)| \leq C_0(\delta) \log(1/\epsilon)$ (and so $s = g_\Gamma(d_\Gamma(x)) > s_{\delta,\epsilon}$), we have (recall also (3.24)) that

$$\begin{aligned} \Delta g_\Gamma(t) &\leq h'_\Gamma(s) - \text{const}\delta\epsilon\sqrt{h_\Gamma(s)} \\ &\leq h'_0(s) + c_1\delta\epsilon\rho'(s) - \text{const}\delta\epsilon\sqrt{h_0(s) + c_1\delta\epsilon\rho(s)}. \end{aligned} \quad (3.31)$$

We now claim that

$$c_1\rho'(s) - \text{const}\sqrt{h_0(s) + c_1\delta\epsilon\rho(s)} < 0 \quad (3.32)$$

for any $s \in (s_{\delta,\epsilon}, 1)$, provided c_1 is small enough. Indeed, if $s \leq -1/2$ or $s \geq 1/2$, then $\rho'(s) = 0$ and therefore the left hand side of (3.32) is under control. On the other hand, if $s \in (-1/2, 1/2)$, then setting $c^* := \inf_{s \in [-1/2, 1/2]} h_0(s)$ (which is strictly positive by condition (1.8)), we bound the left hand side of (3.32) by

$$c_1\|\rho'\|_\infty - \text{const}\sqrt{c^*},$$

which is negative for c_1 small enough. This proves (3.32).

Therefore, by (3.31) and (3.32), if $d_\Gamma(x) \in (H_\Gamma(s_{\delta,\epsilon}), H_\Gamma(1))$, we get

$$\Delta g_\Gamma(t) < h'_0(g_\Gamma(t)).$$

If else $d_\Gamma(x) \leq H_\Gamma(s_{\delta,\epsilon})$, we have

$$\Delta g_\Gamma(t) = 0 < h'_0(s_{\delta,\epsilon}) = h'_0(g_\Gamma(t)),$$

thanks to Lemma B.7. □

To sum up we have introduced in Lemma 3.2 and Lemma 3.3 two different families of strict viscosity supersolution of (1.5) and we have investigated in details their geometric properties. Now the goal is to use the geometric information that we have about these functions in order to understand the geometry of the level sets of phase transitions.

In the next section we introduce the techniques that allow us to compare these barriers functions to phase transitions.

3.3 Sliding techniques

We now develop some slide techniques that allow us to compare phase transitions with the barriers functions introduced in the previous section. The results below are quite general, indeed the theorems are valid for weak Sobolev solutions of (1.5) and not only for local minimizers of the Ginzburg-Landau functional.

Lemma 3.4. *Let u be a weak Sobolev subsolution of (1.5) in some domain Ω . Then u and $\Psi^{y,l}$ cannot coincide in any open domain.*

Proof. To simplify the notation, we set $\Psi := \Psi^{y,l}$, $B := B_{T_l+l}(y)$ and $B' := B_l(y)$. Let $\tau(l) \in [l/2, (1-\bar{c})l]$ be so that Ψ is flat in $B'' := B_{\tau(l)}(y)$. Then $B'' \subset B' \subset B$, the domain of definition of Ψ is B and Ψ is C^2 outside $\partial B' \cup \partial B''$. Suppose by contradiction that $u = \Psi$ in some ball $\mathfrak{B} \subseteq B$. Possibly taking a smaller ball, we may assume that

$$\mathfrak{B} \subset (\Omega \cap B) \setminus (\partial B' \cup \partial B'').$$

Hence $u = \Psi$ is C^2 in \mathfrak{B} ; therefore, for any non-negative smooth function φ supported in \mathfrak{B} we have that

$$\begin{aligned} \int_{\Omega} h_0(\Psi)\varphi &> - \int_{\Omega} \nabla \Psi \cdot \nabla \varphi = - \int_{\Omega} \nabla u \cdot \nabla \varphi \geq \int_{\Omega} h_0(u)\varphi \\ &= \int_{\Omega} h_0(\Psi)\varphi, \end{aligned}$$

which is a contradiction. \square

Lemma 3.5. *Fix $y \in \mathbb{R}^n$ and let $l > 0$ be suitably large. Let u be a weak Sobolev subsolution of (1.5) in some domain Ω . Suppose that $u \in C^1(\Omega)$ and that $|u| \leq 1$. Suppose that $\Psi^{y,l}$ touches u from above at x^* , i.e., $\Psi^{y,l} \geq u$ in their common domain of definition $\Omega \cap B_{T_l+l}(y)$, and $\Psi^{y,l}(x^*) = u(x^*)$, with x^* in the closure of $\Omega \cap B_{T_l+l}(y)$. Then either $x^* \in \partial\Omega$ or $\Psi^{y,l}(x^*) = u(x^*) = 0$.*

Proof. To simplify the notation, we set $\Psi := \Psi^{y,l}$, $B := B_{T_l+l}(y)$ and $B' := B_l(y)$. Assume that

$$x^* \notin \partial\Omega. \tag{3.33}$$

We will show that then $\Psi(x^*) = 0$. First we prove that

$$x^* \notin \partial B. \tag{3.34}$$

Indeed, suppose the contrary. Let us consider the radial direction

$$w := \frac{x^* - y}{|x^* - y|}.$$

Then, by the construction in Lemma 3.2, $\Psi(x^*) = 1$ and $\nabla \Psi(x^*) \cdot w > 0$. On the other hand, $u \leq 1$ and, since $u(x^*) = \Psi(x^*) = 1$, we have $\nabla u(x^*) = 0$. Let $\hat{u} := u - \Psi$. From the above discussion, $\hat{u} \leq 0$ in $B \cap \Omega$ and $\hat{u}(x^*) = 0$, therefore

$$\nabla \hat{u}(x^*) \cdot w \geq 0.$$

But then

$$0 \leq \nabla(u - \Psi)(x^*) \cdot w = -\nabla \Psi(x^*) \cdot w < 0,$$

which is a contradiction. This proves (3.34).

Due to (3.33) and (3.34), x^* is in the interior of $\Omega \cap B$. Also, by Lemma 3.4, u and Ψ cannot agree in any open domain. Then from this fact and Corollary A.3, x^* may only lie on $\partial B'$ where $\Psi = 0$ and it fails to be a supersolution. \square

We have proved two results that allow us to investigate the contact points between u subsolution of (1.5) and the barriers $\Psi^{y,l}$. Now we introduce results which allow us to bound subsolutions of (1.5) by the barriers $\Psi^{y,l}$.

Proposition 3.6. *Let u be a weak Sobolev subsolution of (1.5) in some domain Ω . Suppose that $u \in C^1(\Omega)$ and that $|u| \leq 1$. Let $y \in \mathbb{R}^n$ and $l > 0$ be such that*

$$B_{l+T_l}(y) \subset \{x \in \Omega : u(x) \leq -1 + \theta^*\}. \quad (3.35)$$

Then $u(x) \leq \Psi^{y,l}(x)$ for any $x \in B_{l+T_l}(y)$, provided l is sufficiently large.

Proof. We use the notation $\Psi := \Psi^{y,l}$. Notice that Ψ is defined on $B_{l+T_l}(y)$ and that, if $x \in B_{l+T_l}(y) \setminus B_l(y)$, then

$$\Psi(x) \geq 0 > -1 + \theta^* \geq u(x).$$

Therefore, by (3.35), if the claim of Proposition 3.6 were false, there would be an open set \mathfrak{U} such that

$$\mathfrak{U} \subset B_l(y) \subset \Omega \cap \{u < -1 + \theta^*\}, \quad (3.36)$$

and so that $\Psi < u$ in \mathfrak{U} , and $\Psi = u$ on $\partial\mathfrak{U}$.

Consequently, there exists $k > 0$ so that $v := u - k \leq \Psi$ in \mathfrak{U} , $v < \Psi$ in $\partial\mathfrak{U}$ and $v(x^*) = \Psi(x^*)$ for some $x^* \in \mathfrak{U}$. Note also that

$$v(x^*) = \Psi(x^*) \in (-1, 0), \quad (3.37)$$

since $x^* \in \mathfrak{U} \subset B_l(y)$, and therefore

$$x^* \in \mathfrak{B} := \mathfrak{U} \cap \{|v| < 1\}. \quad (3.38)$$

Since h'_0 is increasing in $B_{l+T_l}(y)$ (thanks to (3.35) and the assumptions on the potential),

$$\Delta v = \Delta u \geq h'_0(u) = h'_0(u + k) \geq h'_0(v), \quad (3.39)$$

weakly in \mathfrak{B} .

Consequently, from Lemma 3.5 we deduce that either $x^* \in \partial\mathfrak{B}$ or $v(x^*) = 0$. The first assertion would contradict (3.38) and the second contradict (3.37). This provides the contradiction which proves the desired result. \square

Proposition 3.6 can be easily sharpened, giving a strict inequality, in the following way:

Corollary 3.7. *Let u be a weak Sobolev subsolution of (1.5) in some domain Ω . Suppose that $u \in C^1(\Omega)$ and that $|u| \leq 1$. Let $y \in \mathbb{R}^n$ and $l > 0$ be such that*

$$B_{l+T_l}(y) \subset \{x \in \Omega : u(x) \leq -1 + \theta^*\}. \quad (3.40)$$

Then $u(x) < \Psi^{y,l}(x)$ for any $x \in B_{l+T_l}(y)$, provided l is sufficiently large.

Proof. By Proposition 3.6, we know that $u \leq \Psi^{y,l}$. If there exists x^* for which the equality holds, then (3.40) and Lemma (3.5) would imply that $u(x^*) = \Psi^{y,l}(x^*) = 0$, which is a contradiction to (3.40). \square

A result analogous to Lemma 3.5 holds for the barrier $g_\Gamma(d_\Gamma)$ constructed in Lemma 3.3. We state the result and we sketch the proof, a more detailed proof can be found in [50].

Lemma 3.8. *Let u be a weak Sobolev subsolution of (1.5) in some domain Ω . Suppose that $u \in C^1(\Omega)$ and that $|u| \leq 1$. Suppose that $g_\Gamma(d_\Gamma)$ touches u from above at x^* . Then $x^* \in \partial\Omega$*

Sketch of the proof. First notice that u and $g_\Gamma(d_\Gamma)$ cannot be identically equal in any open set: this can be proved by an easy modification of the argument in Lemma 3.4. By Corollary A.3, we infer that interior contact points may only lie in the region where $g_\Gamma(d_\Gamma)$ is flat, but this is not possible, see for instance [50]. Thus, x^* cannot be an interior point. This proves that either $x^* \in \partial\Omega$ or lies in the boundary of the domain of $g_\Gamma(d_\Gamma)$. We now show that the latter possibility cannot hold. Indeed, on the boundary of the domain of $g_\Gamma(d_\Gamma)$ we have $g_\Gamma(d_\Gamma) = 1$. On the other hand, if x^* lies on that boundary, but in the interior of Ω , then

$$u \leq 1 = u(x^*) = g_\Gamma(d_\Gamma(x^*))$$

would give $\nabla u(x^*) = 0$. Let now e be any direction pointing from x^* outside the domain of $g_\Gamma(d_\Gamma)$ and let $\hat{u} := u - g_\Gamma(d_\Gamma)$. Then from the hypothesis that $g_\Gamma(d_\Gamma)$ touches u from above at x^* we have that $\partial_e \hat{u}(x^*) \geq 0$ for any outer derivative. If e is taken to be outer normal, however, then

$$\partial_e(g_\Gamma \circ d_\Gamma)(x^*) = g'_\Gamma(d_\Gamma(x^*))\partial_e d_\Gamma(x^*) = g'_\Gamma(d_\Gamma(x^*)) > 0.$$

Collecting the above estimates, we have

$$0 > -\partial_e(g_\Gamma \circ d_\Gamma)(x^*) = \partial_e \hat{u}(x^*) - \partial_e u(x^*) = \partial_e \hat{u}(x^*) \geq 0,$$

and this contradiction shows that the contact point may only lie on $\partial\Omega$. \square

The assumptions on subsolution u in Lemma 3.5, Lemma 3.8, Proposition 3.6 and Corollary 3.7 are, in particular, fulfilled in the case where u is a weak Sobolev solution of (1.5) satisfying $|u| \leq 1$. Indeed the C^1 -regularity is given by the results in [21].

3.4 Proof of Theorem 3.1

In this section we prove the main result of this chapter. In order to prove this result we use the barriers introduced in Section 3.2 and we slide these barriers according to the results achieved in Section 3.3 and we finally obtain estimates on the trace of the touching paraboloid.

Namely we will see that Theorem 3.1 is a direct consequence of the following theorem.

Theorem 3.9. *Let $l, \theta, \delta > 0$ and $M_1 \in \text{Mat}((n-1) \times (n-1))$. Let u be a local minimizer of the Ginzburg-Landau functional (1.6) in $[-l, l]^n$. Assume that $|u| \leq 1$ in $[-l, l]^n$, $u(0) = 0$ and $u(x) < 0$ for any $x = (x', x_n) \in [-l, l]^n$ such that*

$$x_n < \frac{\theta}{2l^2} x' \cdot M_1 x' + \frac{\theta}{l} \xi \cdot x'.$$

Then there exist a universal constant $\delta_0 > 0$ and a function $\sigma : (0, 1) \rightarrow (0, 1)$ so that, if

$$\delta \in (0, \delta_0), \quad \delta \leq \theta, \quad \theta/l \in (0, \sigma(\delta)], \quad \|M_1\| \leq 1/\delta, \quad |\xi| \leq 1/\delta,$$

then $\text{Tr } M_1 \leq \delta$.

Proof. We remark that, by our assumptions, $l \geq \delta/\sigma(\delta)$ and we will assume l to be a large quantity. Let g_l and $\Psi^{y,l}$ be the functions defined in Lemma 3.2. We recall that $T_l \in [\bar{c}l, \frac{l}{2}]$ and it increases if l increases, so we choose $c_1 > 0$ (independent of l) be such that

$$T_{l/4} \geq c_1 l. \tag{3.41}$$

Define also

$$\Gamma_1 := \left\{ (x', x_n) \in [-l, l]^n : x_n = \frac{\theta}{2l^2} x' \cdot M_1 x' + \frac{\theta}{l} \xi \cdot x' \right\}.$$

Let us make some elementary observations on the above paraboloid. First of all, by construction, u is negative below Γ_1 in $[-l, l]^n$. Now we introduce a constant $c_2 > 0$ that we will specify later in the proof, by assumption (if $\sigma(\delta)l/\delta$ is small enough),

$$\Gamma_1 \subseteq \{|x_n| \leq \text{const } \theta/\delta\} \subseteq \{|x_n| \leq \text{const } \sigma(\delta)l/\delta\} \subseteq \{|x_n| \leq c_2 l/8\}. \quad (3.42)$$

Therefore,

$$x_n - c_2 \frac{l}{8} \leq d_{\Gamma_1}(x) \leq x_n + c_2 \frac{l}{8}, \quad (3.43)$$

for any $x \in [-l, l]^n$.

Given $X \in \Gamma_1$ let ν_X be the normal direction to Γ_1 at X pointing downwards and we define a constant $\alpha > 0$ such that

$$\alpha + \frac{c_2}{8} = c_1. \quad (3.44)$$

Let also

$$\mathfrak{C} := \{|x'| \leq \frac{l}{4}\} \times \{x_n \in [-\frac{l}{2}, \alpha l]\}.$$

We claim that

$$\mathfrak{C} \subseteq \bigcup_{X \in \Gamma_1} B_{(c_1 + \frac{1}{4})l}(X + \left(\frac{l}{4}\right)\nu_X). \quad (3.45)$$

To prove this, take any $\xi \in \mathfrak{C}$ and let $X = X(\xi) \in \Gamma_1$ the point that realizes the distance $d_{\Gamma_1}(\xi)$. By (3.43) we have

$$d_{\Gamma_1} \in \left[-\frac{l}{2} - c_2 \frac{l}{8}, \alpha l + c_2 \frac{l}{8}\right]. \quad (3.46)$$

This says, in particular, that $|d_{\Gamma_1}| < 3l/4$. Then the definition of \mathfrak{C} implies that X lies in the interior of $[-l, l]^n$ and therefore $\xi - X$ is orthogonal to Γ_1 at X , that is,

$$\xi = X + \tau \nu_X,$$

for a suitable $\tau \in \mathbb{R}$. Hence,

$$d_{\Gamma_1}(\xi) = -\tau l \quad (3.47)$$

and

$$\left| \xi - \left(X + \left(\frac{l}{4}\right)\nu_X \right) \right| = \left| \tau - \frac{1}{4} \right| l. \quad (3.48)$$

Then by (3.47) and (3.46), we have

$$\tau \in \left[-\alpha - c_2 \frac{1}{8}, \frac{1}{2} + c_2 \frac{1}{8} \right],$$

and so, recalling (3.43), we obtain

$$\tau - \frac{1}{4} \in \left[-\left(\alpha + \frac{c_2}{8}\right) - \frac{1}{4}, \frac{1}{4} + \left(\alpha + \frac{c_2}{8}\right) \right] = \left[-c_1 - \frac{1}{4}, \frac{1}{4} + c_1 \right].$$

This and (3.48) imply that $\xi \in B_{(c_1 + 1/4)l}(X + (l/4)\nu_X)$. This proves the claim (3.45).

We now observe that

$$\frac{\theta \|M_1\|}{l^2} \leq \frac{\theta}{l^2 \delta} \leq \frac{\sigma(\delta)}{l \delta}. \quad (3.49)$$

The bound on the curvature of Γ_1 given in (3.49) implies that, if $\sigma(\delta)/\delta$ is sufficiently small, then, given $X \in \Gamma_1$, there exists a ball of radius $l/4$ which touches Γ_1 from below at X .

The following is the decisive step towards the proof of the desired result. We claim that

$$u(x) \leq g_{\frac{l}{4}}(d_{\Gamma_1}(x)), \quad \forall x \in \mathfrak{C}. \quad (3.50)$$

To prove (3.50), first notice that, from Lemma B.9 and Corollary 3.7, we infer that

$$u(x) < \Psi^{(0, \dots, 0, -l/2), l/4}(x), \quad \forall x \in B_{l/4+T_{l/4}}(0, \dots, 0, -l/2).$$

Then, for a given $X \in \Gamma_1$ we define

$$X' = X'(X) := X + \frac{l}{4}\nu_X,$$

where, as above, we denote by ν_X the normal direction to Γ_1 at X pointing downwards. In particular, from the above observation, $B_{l/4}(X')$ touches Γ_1 from below at X . We now slide the surface $\Psi^{(0, \dots, 0, -l/2), l/4}$ in the direction of the vector

$$v = v(X) = X' - (0, \dots, 0, -l/2),$$

i.e., we consider the surface

$$\Psi^t := \Psi^{(0, \dots, 0, -l/2)+tv, l/4}, \quad \text{for } t > 0.$$

We will show that

$$\Psi^t(\xi) > u(\xi) \quad \forall t \in [0, 1), \quad \forall \xi \in B_{l/4+T_{l/4}}((0, \dots, 0, -l/2) + tv). \quad (3.51)$$

Indeed, let $t \in [0, 1)$ be the first time at which Ψ^t touches u . First of all, note that, since $t < 1$, we have $u < 0$ on $\partial B_{l/4}((0, \dots, 0, -l/2) + tv)$, while $\Psi^t = 0$ there. Therefore, u cannot be equal to Ψ^t , and no touching points occur on $\partial B_{l/4}((0, \dots, 0, -l/2) + tv)$. On the other hand, Lemma 3.5 says that touching points cannot occur anywhere else. This proves (3.51).

We now prove (3.50). We deduce from (3.51) that $\Psi^1(\xi) \geq u(\xi)$ for any $\xi \in B_{l/4+T_{l/4}}(X')$ and, thanks to (3.41), $\xi \in B_{(1/4+c_1)l}(X')$. Therefore, taking now any $\xi \in \mathfrak{C}$ and letting X' be so that $x \in B_{(1/4+c_1)l}(X')$ (recall (3.45)), we have

$$g_{\frac{l}{4}}(d_{\Gamma_1}(x)) = g_{\frac{l}{4}}(|x - X'| - l/4) = \Psi^{X', l/4}(x) = \Psi^1(x) \geq u(x).$$

This proves (3.50).

We now complete the proof by supposing that $\text{Tr } M_1 > \delta$; under this assumption, by Lemma 3.5, $g_{\Gamma_2}(d_{\Gamma_2})$ is a strict supersolution of (1.5), where

$$\begin{aligned} \Gamma_2 &:= \left\{ (x', x_n) \in [-l, l]^n : x_n = \epsilon x' \cdot M_1 x' + \frac{\theta}{l} \xi \cdot x' - \frac{\epsilon \delta}{2(n-1)} |x'| \right\}, \\ \epsilon &:= \frac{\theta}{2l^2}. \end{aligned}$$

Note that

$$\Gamma_2 \subseteq \{|x_n| \leq \sigma(\delta)(\delta + 1/\delta)l\} \subseteq \{|x_n| \leq c_2 l/8\}. \quad (3.52)$$

Now we claim that, if θ/l and δ are small enough, then the following estimates hold

$$h_{\Gamma_2}(s) \leq h_{l/4}(s) \quad \text{if } s_{\delta,\epsilon} \leq s \leq -1 + \sqrt{\delta\theta l^{-2}}, \quad (3.53)$$

$$h_{\Gamma_2}(s) \geq h_{l/4}(s) \quad \text{if } 1 - \sqrt{\delta\theta l^{-2}} \leq s \leq 1. \quad (3.54)$$

If $h_{\Gamma_2} = 0$, then (3.53) follows from (i) in Lemma 3.2. If, on the contrary, $h_{\Gamma_2} > 0$ and $s \in [s_{\delta,\epsilon}, -1 + \sqrt{\delta\theta l^{-2}}]$, then, by definitions of h_{Γ_2} and $h_{l/4}$, by conditions (1.8) and (3.23),

$$\begin{aligned} h_{\Gamma_2}(s) - h_{l/4}(s) &\leq \text{const} \left(-\delta\epsilon + \frac{(1+s)^2}{l} - \frac{s_{l/4}^2}{l} + h_0(-1 + s_{l/4}) \right) \\ &\leq \text{const} \left(-\frac{\delta\theta}{l^2} + \frac{(1+s)^2}{l} + s_{l/4}^2 \right) \\ &\leq \text{const} \left(-\frac{\delta\theta}{l^2} + \frac{\delta\epsilon}{l} + s_{l/4}^2 \right) \\ &= \text{const} \left(-\frac{\delta\theta}{l^2} + \frac{\delta\epsilon}{l} + e^{-\text{const}l} \right), \end{aligned}$$

which is negative for sufficiently large l , completing the proof of (3.53). To prove (3.54), we use condition (1.8), (3.23) and the definitions of h_{Γ_2} and $h_{l/4}$ to deduce that, if $1 - \sqrt{\delta\theta l^{-2}} \leq s \leq 1$, we have

$$\begin{aligned} h_{\Gamma_2}(s) - h_{l/4}(s) &\geq h_0(s) + \text{const} \delta\epsilon - h_{l/4}(s) \\ &\geq \text{const} \delta\epsilon - h_0(1 - s_{l/4}) - \text{const} \frac{(1-s)^2 + s_{l/4}^2}{l} \\ &\geq \text{const} \left(\delta\epsilon - s_{l/4}^2 - \frac{(1-s)^2 + s_{l/4}^2}{l} \right) \\ &\geq \text{const} \left(\delta\epsilon - \frac{\delta\theta}{l^3} - e^{-\text{const}l} \right) \\ &= \text{const} \left(\frac{\delta\theta}{l^2} - \frac{\delta\theta}{l^3} - e^{-\text{const}l} \right), \end{aligned}$$

which is positive if l is large enough, completing the proof of (3.54).

According to (3.53) and (3.54), the function $s \rightarrow H_{\Gamma_2}(s) - H_{l/4}(s)$ is increasing for $s \leq -1 + \sqrt{\delta\theta l^{-2}}$ and decreasing for $s \geq 1 - \sqrt{\delta\theta l^{-2}}$, therefore its maximum occurs in $[-1 + \sqrt{\delta\theta l^{-2}}, 1 - \sqrt{\delta\theta l^{-2}}]$, i. e.,

$$\max_{s \in [s_{\delta,\epsilon}, 1]} (H_{\Gamma_2}(s) - H_{l/4}(s)) = \max_{s \in [-1 + \sqrt{\delta\theta l^{-2}}, 1 - \sqrt{\delta\theta l^{-2}}]} (H_{\Gamma_2}(s) - H_{l/4}(s)). \quad (3.55)$$

Also, recalling the definition of H_0 in Lemma 3.2, if $s \in [0, 1 - \sqrt{\delta\theta l^{-2}}]$,

$$H_{\Gamma_2}(s) = \int_0^s \frac{1}{\sqrt{2h_{\Gamma_2}(\xi)}} d\xi \leq \int_0^s \frac{1}{\sqrt{2h_0(\xi)}} d\xi = H_0(s), \quad (3.56)$$

and analogously, if $s \in [-1 + \sqrt{\delta\theta l^{-2}}, 0]$,

$$-H_{\Gamma_2}(s) = \int_s^0 \frac{1}{\sqrt{2h_{\Gamma_2}(\xi)}} d\xi \geq \int_s^0 \frac{1}{\sqrt{2h_0(\xi)}} d\xi = -H_0(s). \quad (3.57)$$

Hence from (3.56) and (3.57),

$$H_{\Gamma_2} \leq H_0(s), \quad \forall s \in [-1 + \sqrt{\delta\theta l^{-2}}, 1 - \sqrt{\delta\theta l^{-2}}].$$

Consequently, from (3.4), if $s \in [-1 + \sqrt{\delta\theta l^{-2}}, 1 - \sqrt{\delta\theta l^{-2}}]$, then

$$H_{\Gamma_2}(s) \leq H_{l/4}(s) + \frac{\text{const}}{l} \log \frac{l^2}{\delta\theta}.$$

Therefore, by (3.57),

$$H_{\Gamma_2}(s) \leq H_{l/4}(s) + \frac{\text{const}}{l} \log \frac{l^2}{\delta\theta}, \quad \forall s \in [s_{\delta,\epsilon}, 1]. \quad (3.58)$$

Furthermore, by definitions of Γ_1 and Γ_2 , if $|x'| = l/4$, then

$$d_{\Gamma_2}(x) \geq d_{\Gamma_1}(x) + c(\delta),$$

for a suitable $c(\delta) \in (0, 1)$. Hence, using (3.58) and taking l appropriately large, with $s = g_{l/4}(d_{\Gamma_1}(x))$,

$$\begin{aligned} H_{\Gamma_2}(g_{l/4}(d_{\Gamma_1}(x))) &< H_{l/4}(g_{l/4}(d_{\Gamma_1}(x))) + \frac{\text{const}}{l} \log \frac{l^2}{\delta\theta} \\ &= d_{\Gamma_1}(x) + \frac{\text{const}}{l} \log \frac{l^2}{\delta\theta} \leq d_{\Gamma_2}(x), \end{aligned}$$

provided $g_{l/4}(d_{\Gamma_1}(x)) \geq s_{\delta,\epsilon}$ and $|x'| = l/4$. We apply $H_{\Gamma_2}^{-1}$ at the inequality above and, since H_{Γ_2} is increasing in $[s_{\delta,\epsilon}, 1]$,

$$g_{l/4}(d_{\Gamma_1}(x)) < g_{\Gamma_2}(d_{\Gamma_2}(x)), \quad (3.59)$$

for any x so that $g_{l/4}(d_{\Gamma_1}(x)) \geq s_{\delta,\epsilon}$ and $|x'| = l/4$. Of course, if $g_{l/4}(d_{\Gamma_1}(x)) < s_{\delta,\epsilon}$ then (3.59) hold since $g_{\Gamma_2}(d_{\Gamma_2}(x)) \geq s_{\delta,\epsilon}$ by construction (recall (i) of Lemma 3.3). Thus,

$$g_{l/4}(d_{\Gamma_1}(x)) < g_{\Gamma_2}(d_{\Gamma_2}(x)), \quad \forall x \text{ such that } |x'| = l/4, \quad (3.60)$$

provided that $d_{\Gamma_1}(x)$ is in the domain of $g_{l/4}$ and $d_{\Gamma_2}(x)$ is in the domain of g_{Γ_2} . Notice that the first of these conditions is implied by the second:

$$\text{if } d_{\Gamma_2}(x) \text{ is in the domain of } g_{\Gamma_2}, \text{ then } d_{\Gamma_1}(x) \text{ is in the domain of } g_{l/4}. \quad (3.61)$$

To prove this, take x so that $d_{\Gamma_2}(x)$ is in the domain of g_{Γ_2} . Then, by Lemma (3.3) and our choice of parameters,

$$d_{\Gamma_2}(x) \leq C_0(\delta) \log \frac{l^2}{\theta},$$

and thus by (3.41), (3.44) and (3.52) we deduce that

$$d_{\Gamma_1}(x) \leq d_{\Gamma_2}(x) + \frac{c_2 l}{4} \leq C_0(\delta) \log \frac{l^2}{\theta} + \frac{c_2 l}{4} \leq \frac{c_2 l}{2} \leq c_1 l \leq T_{l/4},$$

which says that $d_{\Gamma_1}(x)$ is in the domain of $g_{l/4}$.

Now (3.60), (3.50) and (3.61) imply that

$$u(x) < g_{\Gamma_2}(d_{\Gamma_2}(x)) \quad (3.62)$$

for any x so that $|x'| = l/4$ and $d_{\Gamma_2}(x)$ is in the domain of g_{Γ_2} .

With these estimates we are now ready to deduce the contradiction that will finish the proof. To this end, we slide $g_{\Gamma_2}(d_{\Gamma_2})$ in the e_n -direction till we touch u in \mathfrak{C} . Namely, we consider, for $t \in \mathbb{R}$,

$$g^t(x) := g_{\Gamma_2}(d_{\Gamma_2}(x - te_n)), \quad (3.63)$$

and, first of all, we want to show that there exists a time $t \leq 0$ such that g^t touches u from above. If we denote \mathcal{D}_0 the domain of $g_{\Gamma_2}(d_{\Gamma_2})$, then Lemma 3.3 shows that \mathcal{D}_0 is the subgraph of a paraboloid, namely

$$\mathcal{D}_0 = \{d_{\Gamma_2}(x) \leq T_{\epsilon, \delta}\} \subseteq \left\{x_n \leq C_0(\delta) \log \frac{l^2}{\theta}\right\}. \quad (3.64)$$

and also by construction if $d_{\Gamma_2}(x) = T_{\epsilon, \delta}$ then $g_{\Gamma_2}(d_{\Gamma_2}(x)) = 1$. Notice that, with this notation, g^t is defined in $\mathcal{D}_t := \mathcal{D}_0 + te_n$ and $g^t = 1$ on the top of \mathcal{D}_t . Thus, if $t \ll 0$, then $g^t > u$ in $\mathcal{D}_t \cap \mathfrak{C}$, since $u < 0$ below Γ_1 . On the other hand,

$$g^0(0) = g_{\Gamma_2}(d_{\Gamma_2}(0)) = g_{\Gamma_2}(0) = 0 = u(0),$$

therefore, there is a time $t \leq 0$ of first touch of g^t and u in $\mathcal{D}_t \cap \mathfrak{C}$. Hence, in view of Lemma 3.8, contact points may only happen either on the lateral side of the cylinder \mathfrak{C} (i.e. $|x'| = l/4$) or in its two basis (i.e. $x_n = -l/2$ or $x_n = \alpha l$).

Now the touching points cannot occur in $x_n = \alpha l$, because $x_n = \alpha l$ is the upper face of the cylinder \mathfrak{C} and $t \leq 0$, hence, if l is large enough, \mathcal{D}_t lies below $x_n = \alpha l$, due to (3.64).

We exclude the possibility of touching at $x_n = -l/2$. By applying (3.50), (3.43) and the fact that $g_{l/4}$ is constant in $(-\infty, -l/8]$, we deduce that, if $x_n = -l/2$, then

$$\begin{aligned} u(x) &\leq g_{l/4}(d_{\Gamma_1}(x)) \leq g_{l/4}\left(x_n + c_2 \frac{l}{8}\right) = g_{l/4}\left(\frac{l}{2} + c_2 \frac{l}{8}\right) \leq g_{l/4}\left(-\frac{l}{8}\right) \\ &= -1 + e^{-\text{const}l} < s_{\delta, \epsilon} \leq g^t(x), \end{aligned}$$

which rules out the possibility of touching at $x_n = -l/2$.

Therefore, a contact point $x^* \in \mathcal{D}_t \cap \mathfrak{C}$ between u and g^t does occur when $|x'| = l/4$. Notice now that, from Lemma B.8

$$d_{\Gamma_2}(x^* - te_n) \geq d_{\Gamma_2}(x^*).$$

But then, since g_{Γ_2} is non-decreasing, we deduce from (3.62) that

$$g_{\Gamma_2}(d_{\Gamma_2}(x^* - te_n)) = g^t(x^*) = u(x^*) < g_{\Gamma_2}(d_{\Gamma_2}(x^*)) \leq g_{\Gamma_2}(d_{\Gamma_2}(x^* - te_n)).$$

This contradiction concludes the proof. \square

Now, using Theorem 3.9, we can prove Theorem 3.1, that is the main result of this chapter.

Proof of Theorem 3.1. We apply Theorem 3.9 with the following choice of parameters:

$$l := \frac{\varrho}{\sqrt{\epsilon \text{Tr } M}}, \quad \delta := \theta := \varrho^2, \quad M_1 := \frac{1}{\text{Tr } M} M, \quad \xi := 0.$$

By contradiction, if the claim of Theorem 3.1 were false, by scaling back the phase transition u_ϵ and using the above parameters, we obtain that Γ_1 touches from below the zero level set of u inside $[-l, l]^n$, where

$$\Gamma_1 = \left\{ (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \frac{\theta}{2l^2} x' \cdot M_1 x' + \frac{\theta}{l} \xi \cdot x' \right\}.$$

By Theorem 3.9 we conclude that $1 > \delta \geq \text{Tr } M_1 = 1$, which is the contradiction that proves Theorem 3.1 \square

Chapter 4

Improvement of Flatness

4.1 Improvement of Flatness and Harnack Inequality

In this Section we study the Improvement of Flatness Theorem, that is the key result that allow us to prove De Giorgi's Conjecture for phase transitions.

We now recall the Improvement of Flatness Theorem:

Theorem 4.1. (*Improvement of Flatness*) *Let u be a local minimizer of J in $\{|x'| < l\} \times \{|x_n| < l\}$. Assume that $u(0) = 0$ and assume also that there exists $\theta \leq l$ such that:*

$$\{u = 0\} \subset \{|x'| < l\} \times \{|x_n| < \theta\}.$$

*Then there exist small constants $0 < \eta_1 < \eta_2 < 1$ depending on n and h_0 such that:
given $\theta_0 > 0$ there exists $\epsilon_1(\theta_0) > 0$ depending on n , h_0 and θ_0 such that if*

$$\frac{\theta}{l} \leq \epsilon_1(\theta_0), \quad \theta_0 \leq \theta,$$

then

$$\{u = 0\} \cap \{|\pi_\xi x| < \eta_2 l\} \times \{|x \cdot \xi| < \eta_2 l\}$$

is included in a flatter cylinder

$$\{|\pi_\xi x| < \eta_2 l\} \times \{|x \cdot \xi| < \eta_1 \theta\},$$

for some unit vector ξ , where $\pi_\xi x = x - (x \cdot \xi)\xi$.

This Theorem is a consequence of the Harnack inequality that is a weaker version of the Improvement of Flatness Theorem:

Theorem 4.2. (*Harnack Inequality*) *Let u be a local minimizer of J in $\{|x'| < l\} \times \{|x_n| < l\}$. Assume that $u(0) = 0$ and assume also that there exists $\theta \leq l$ such that:*

$$\{u = 0\} \subset \{|x'| < l\} \times \{|x_n| < \theta\}.$$

Then there exists small constant η_0 depending on n and h_0 such that:

given $\theta_0 > 0$ there exists $\epsilon_1(\theta_0) > 0$ depending on n , h_0 and θ_0 such that if

$$\frac{\theta}{l} \leq \epsilon_1(\theta_0), \quad \theta_0 \leq \theta,$$

then

$$\{u = 0\} \cap \{|\pi_\xi x| < \eta_0 l\} \subset \{|x \cdot \xi| < (1 - \eta_0)l\}$$

for some unit vector ξ , where $\pi_\xi x = x - (x \cdot \xi)\xi$.

We will not prove this result, a proof of this Theorem can be found in [49].

We now present the main ideas of the proof of the Harnack inequality. First of all we introduce suitable barrier functions constructed from the one dimensional phase transitions, and we introduce slide techniques in order to compare these barriers with the minimizer u . Then we need some very precise estimate on the measure of the contact points between the minimizer u and the barriers. Very roughly, we can say that the final target of the proof consists in deducing measure estimates in the above mentioned contact points, which, if the statement of Theorem 4.2 were false, would contradict the minimality of u .

Before proving the Improvement of Flatness Theorem we highlight another analogy between phase transitions and minimal surfaces. A result similar to the Improvement of Flatness Theorem holds for minimal surfaces. Indeed we have the following Harnack inequality for minimal surfaces:

Theorem 4.3. *Assume E is a set with minimal perimeter in B_1 and*

$$\partial E \cap B_1 \subset \{|x_n| \leq \epsilon\}.$$

Then there exist two constants ϵ_1 and $0 < \eta < 1$ such that if $\epsilon \leq \epsilon_1$ we have

$$\partial E \cap B_{\frac{1}{2}} \subset \{|x_n| \leq \epsilon(1 - \eta)\}.$$

Now from this theorem we can prove the Improvement of Flatness Theorem for minimal surfaces

Theorem 4.4. *Assume E is a set with minimal perimeter in B_1 , $0 \in \partial E$ and*

$$\partial E \cap B_1 \subset \{|x_n| \leq \epsilon\}.$$

Then there exist two constants ϵ_1 and r_0 and a unit vector ν_1 such that if $\epsilon \leq \epsilon_1$ we have

$$\partial E \cap B_{r_0} \subset \{|x \cdot \nu_1| \leq \frac{\epsilon}{2} r_0\}.$$

We notice that the geometric interpretation of Theorem 4.4 is similar to the geometric interpretation of Theorem 4.1. Indeed Theorem 4.4 says that if a minimal surface is included in a cylinder then, in its interior, it is included in a flatter cylinder. Theorem 4.4 is deeply used in order to prove smoothness and analytic regularity of minimal surfaces (see for instance [32])

4.2 Proof of the Improvement of Flatness

We assume by contradiction that Theorem 4.1 does not hold. This implies that, if we fix $\theta_0 > 0$, there exist u_k , θ_k and l_k for which:

C1 u_k is a local minimizer of the Ginzburg-Landau functional in $\{|x'| < l_k\} \times \{|x_n| < l_k\}$ with $u_k(0) = 0$.

C2 $\{u_k = 0\} \subseteq \{|x'| < l_k\} \times \{|x_n| < \theta_k\}$, with $\theta_k \geq \theta_0$ and $\frac{\theta_k}{l_k} \rightarrow 0$ when $k \rightarrow \infty$,

but the thesis of Theorem 4.1 does not hold. Let us consider the following rescaling:

$$y' = \frac{x'}{l_k}, \quad y_n = \frac{x_n}{\theta_k}. \quad (4.1)$$

We define $T(x', x_n) = (y', y_n)$. We also define

$$A_k := \{(y', y_n) \text{ s.t. } T^{-1}(y', y_n) \in \{u_k = 0\}\} = T(\{u_k = 0\}).$$

STEP 1: There exists a Hölder continuous function $w : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ such that if we define

$$A_\infty := \{(y', w(y')), |y'| \leq \frac{1}{2}\}$$

then, for any $\epsilon > 0$, $A_k \cap \{|y'| \leq \frac{1}{2}\}$ lies in a ϵ -neighborhood of A_∞ for k sufficiently large.

Proof of Step 1. Let us suppose that

$$y_0 = (y'_0, y_{0n}) \in A_k, \text{ with } |y'_0| \leq \frac{1}{2}.$$

Then $u_k(l_k y'_0, \theta_k y_{0n}) = 0$, and so, by means of **(C2)**, $|\theta_k y_{0n}| < \theta_k$; therefore, using again **(C2)**, we infer that

$$\{u_k = 0\} \subseteq \{|x_n - \theta_k y_{0n}| < 2\theta_k\}.$$

We can exploit Theorem 4.2 in the cylinder

$$\begin{aligned} \{|x' - l_k y'_0| < \frac{l_k}{2}\} \times \{|x_n - \theta_k y_{0n}| < 2\theta_k\} &\subseteq \\ \subseteq \{|x'| < l_k\} \times \{|x_n| < l_k\} \end{aligned} \quad (4.2)$$

and get that there exists a universal constant $\eta_0 > 0$ such that

$$\{u_k = 0\} \cap \{|x' - l_k y'_0| < \eta_0 \frac{l_k}{2}\} \subseteq \{|x_n - \theta_k y_{0n}| < 2(1 - \eta_0)\theta_k\},$$

provided

$$\frac{4\theta_k}{l_k} \leq \epsilon_0(2\theta_0),$$

where $\epsilon_0(\cdot)$ is the one given by Theorem 4.2. Rescaling back, we get

$$A_k \cap \{|y' - y'_0| < \frac{\eta_0}{2}\} \subseteq \{|y_n - y_{0n}| < 2(1 - \eta_0)\}.$$

By iterating, we get

$$A_k \cap \{|y' - y'_0| < \frac{\eta_0^m}{2}\} \subseteq \{|y_n - y_{0n}| < 2(1 - \eta_0)^m\}, \quad (4.3)$$

provided

$$\frac{4\theta_k}{l_k} \leq \eta_0^{m-1} \epsilon_0(2(1 - \eta_0)^{m-1} \theta_0). \quad (4.4)$$

We now fix $m_0 \in \mathbb{N}$ and consider $m \leq m_0$ (later on, during a limit procedure performed later, we let $m_0 \rightarrow \infty$). Notice that in this setting, (4.3) (and therefore (4.4)) is fulfilled for k suitably large, say $k \geq k^*(m_0)$. We claim that $A_k \cap \{|y'| \leq 1/2\}$ is above the graph of

$$\Psi_{y_0, k}(y') := y_{0n} - 2(1 - \eta_0)^{m_0} - \alpha |y' - y'_0|^\beta, \quad (4.5)$$

where α and β depend only on η_0 .

In order to prove this, let $(y', y_n) \in A_k \cap \{|y'| \leq 1/2\}$. Since $|y'_0| \leq 1/2$ we have that $|y' - y'_0| \leq 1$. Now, we consider three different cases: the case $|y' - y'_0| \leq \frac{\eta_0^{m_0}}{2}$, the case $\frac{\eta_0^{m_0}}{2} < |y' - y'_0| < \frac{1}{2}$, and the case $\frac{1}{2} \leq |y' - y'_0| \leq 1$.

In case $|y' - y'_0| \leq \frac{\eta_0^{m_0}}{2}$, (4.5) follows immediately from (4.3), with $m = m_0$. If, on the other hand, $\frac{\eta_0^{m_0}}{2} < |y' - y'_0| < \frac{1}{2}$, then we argue as follows. We first note that, in this case, there exist m with $0 \leq m \leq m_0$ and p with $0 \leq p \leq 1$, such that

$$\frac{\eta_0^{m+1}}{2} \leq |y' - y'_0| \leq \frac{\eta_0^p}{2}. \quad (4.6)$$

Consequently, from (4.3), we have that

$$2(1 - \eta_0)^m \geq |y_n - y_{0n}|. \quad (4.7)$$

By (4.6) and the fact that $0 < \eta_0 < 1$, we also get

$$p \leq \frac{-\ln(2|y' - y'_0|)}{\ln(\frac{1}{\eta_0})} \leq m + 1.$$

In particular, it follows that

$$\begin{aligned} (1 - \eta_0)^m &\leq (1 - \eta_0)^{\frac{-\ln(2|y' - y'_0|) - 1}{\ln(\frac{1}{\eta_0})}} = \\ &= \frac{1}{(1 - \eta_0)} e^{\beta \ln(2|y' - y'_0|)} = \frac{(2|y' - y'_0|)^\beta}{(1 - \eta_0)}, \end{aligned}$$

where $\beta := \frac{-\ln(2|y' - y'_0|)}{\ln(\frac{1}{\eta_0})}$.

Therefore, recalling (4.7), it follows

$$|y_n - y_{0n}| \leq \frac{2^{\beta+1}}{(1 - \eta_0)} |y' - y'_0|^\beta,$$

which is the desired result, with $\alpha := \frac{2^{\beta+1}}{(1 - \eta_0)}$. Finally, adding a constant to α , the result also follows for the case $|y' - y'_0| \in [1/2, 1]$.

Note now that, as y_0 varies, $\Psi_{y_0, k}$ are Hölder continuous functions with Hölder modulus of continuity bounded by the function αt^β (recall that m_0 is fixed for the moment, and that α and β depending only on η_0). Therefore, if we set

$$\psi_k(y') := \sup_{|y'_0| \leq \frac{1}{2}, y_0 \in A_k} \Psi_{y_0, k}(y'),$$

then ψ_k is a Hölder continuous function (with Hölder modulus of continuity bounded via the function αt^β), and $A_k \cap \{|y'| \leq 1/2\}$ is above the graph of ψ_k .

Arguing in the same way, possibly taking α and β larger (depending only on η_0), we also get that, if we define

$$\Phi_{y_0, k}(y') := y_{0n} + 2(1 - \eta_0)^{m_0} + \alpha |y' - y'_0|^\beta,$$

then $A_k \cap \{|y'| \leq 1/2\}$ is below the graph of $\Phi_{y_0, k}$. Arguing as above we define

$$\phi_k(y') := \sup_{|y'_0| \leq \frac{1}{2}, y_0 \in A_k} \Phi_{y_0, k}(y'),$$

so that ϕ_k is Hölder continuous function (with Hölder modulus of continuity bounded via the function αt^β), and $A_k \cap \{|y'| \leq 1/2\}$ is below the graph of ϕ_k .

In particular $A_k \cap \{|y'| \leq 1/2\}$ lies between $\psi_k(y')$ and $\phi_k(y')$ for any $k \geq k^*(m_0)$ and, by construction,

$$0 \leq \phi_k(y') - \psi_k(y') \leq 4(1 - \eta_0)^{m_0}. \quad (4.8)$$

Also, for m_0 fixed, by Ascoli-Arzelà Theorem, letting $k \rightarrow \infty$, it follows that $\psi_k(y')$ uniformly converges in $|y'| \leq 1/2$ to a Hölder continuous function which depends only on m_0 , say

$$\psi_k(y') \xrightarrow[k \rightarrow \infty]{} w_{m_0}^-(y')$$

Analogously, we find an Hölder continuous function $w_{m_0}^+$, such that

$$\phi_k(y') \xrightarrow[k \rightarrow \infty]{} w_{m_0}^+(y')$$

uniformly.

Also, by construction, we have that $w_{m_0}^- \leq w_{m_0}^+$ and that $A_k \cap \{|y'| \leq 1/2\}$ lies between the graphs of $w_{m_0}^- - \frac{\epsilon}{2}$ and $w_{m_0}^+ + \frac{\epsilon}{2}$ for k large.

Let now $m_0 \rightarrow \infty$. In this case, by Ascoli-Arzelà Theorem, (we remark that, by construction of α and β above, the Hölder constants of $w_{m_0}^\pm$ depend on η_0 , but are independent of m_0) we get that there exists a Hölder continuous function w such that $w_{m_0}^-$ uniformly converges to w . By (4.8), also $w_{m_0}^+$ uniformly converges to w . This concludes the proof. \square

STEP 2: The function w constructed in the first step is harmonic.

Proof of Step 2. We prove that w is harmonic in the viscosity sense. Then it follows that it is harmonic in the classical sense (see for instance [8]).

Let P be the quadratic polynomial

$$P(y') := \frac{1}{2} y'^T M y' + \xi \cdot y'.$$

Assume, by contradiction, that $\Delta P > 0$, that P touches the graph of w , say at 0, and that P stay below it in $|y'| < 2r$, for some $r \in (0, 1)$. Let now $\delta_0 > 0$ be the universal constant of Theorem 3.9 and let us define

$$\delta := \min \left\{ \left(\frac{\Delta P}{2\theta_0} \right)^{\frac{1}{2}}, \frac{1}{2\theta_0 \|M\|}, \frac{1}{2\theta_0 |\xi|}, \left(\frac{\delta_0}{2\theta_0} \right)^{\frac{1}{2}}, r \right\}.$$

Thus, δ is such that

$$\begin{aligned} \Delta P > 2\delta^2 \theta_0, \quad \|M\| \leq \frac{1}{2\delta \theta_0}, \quad |\xi| \leq \frac{1}{2\delta \theta_0}, \\ \delta^2 \theta_0 \leq \frac{\delta_0}{2}. \end{aligned} \quad (4.9)$$

Note that, eventually replacing δ with 2δ and $P(y')$ with $P(y') - \delta|y'|^2$, we may assume, with no loss of generality, that P touches the graph of w at 0 and stays strictly below it in $|y'| < 2\delta < 2$. Therefore, since $A_k \cap \{|y'| \leq 1/2\}$ uniformly converge to the graph of w , it follows that, for k large, we find points $y_k = (y'_k, y_{k_n})$ close to 0, such that $P(y') - K_k$ touches A_k at (y'_k, y_{k_n}) and stays below it in $|y' - y'_k| \leq \delta$, for an appropriate $K_k \in \mathbb{R}$. In particular, we have

$$y_{k_n} + K_k = \frac{1}{2} y'^T M y' + \xi \cdot y'. \quad (4.10)$$

Let us now consider the following translation

$$z' = y' - y'_k, \quad z_n = y_n - (y_{k_n} + K_k).$$

Exploiting (4.10) we find a surface

$$z_n = \frac{1}{2} z'^T M z' + \xi_k \cdot z',$$

with

$$\xi_k := M y'_k + \xi$$

that touches A_k at the origin and stays below it in $|z'| < \delta$. Notice also that, by construction,

$$|\xi_k| \leq \frac{1}{\delta \theta_0}. \quad (4.11)$$

Rescaling back, we get that the surface

$$x_n = \frac{\theta_k}{l_k^2} \frac{1}{2} x'^T M x' + \frac{\theta_k}{l_k} \xi_k \cdot x'$$

touches $\{u_k = 0\}$ at the origin and stays below it, if $|x'| < \delta l_k$. We write now the above surface in the form

$$x_n = \frac{\delta^2 \theta_k}{(\delta l_k)^2} \frac{1}{2} x'^T M x' + \frac{\delta^2 \theta_k}{\delta l_k} \frac{1}{\delta} \xi_k \cdot x'$$

and we exploit Theorem 3.9, we obtain that

$$\Delta P \leq \delta^2 \theta_0$$

against the assumption. This contradiction shows that $\Delta P \leq 0$. By arguing in the same way, one may prove that $\Delta P \geq 0$ if P touches w by above, so Step 2 is proved. \square

CONCLUSION: Since w is harmonic, by standard elliptic estimates (see for instance [31]) we have that $\|D^2 w\|$ is bounded on compact sets. Therefore, since by construction $w(0) = 0$, by Taylor's formula, it follows that

$$|w(y') - \nabla w(0) \cdot y'| < C' \eta_2^2,$$

for $|y'| < 2\eta_2$. In particular, for η_2 sufficiently small, setting

$$\xi' := \nabla w(0),$$

we get that there exist positive constants $0 < \eta_1 < \eta_2 < 1$, for which

$$|w(y') - \xi' \cdot y'| < \frac{\eta_1}{2}, \quad (4.12)$$

for $|y'| < 2\eta_2$.

Now let us consider

$$\xi_k := \frac{\left(\frac{\theta_k}{l_k} \xi', -1\right)}{\sqrt{\frac{\theta_k^2}{l_k^2} |\xi'|^2 + 1}}. \quad (4.13)$$

Consider the rescaling given by (4.1) we obtain

$$\{\pi_{\xi_k} x < \eta_2 l_k\} \times \{|\xi_k \cdot x| < \eta_2 l_k\} \subset \{|x'| < 2\eta_2 l_k\} \subset \{|x'| < l_k/2\}. \quad (4.14)$$

Since $A_k \cap \{|y'| \leq 1/2\}$ uniformly converges to the graph of w , for k sufficiently large (thanks to Step 1), we may suppose that $A_k \cap \{|y'| \leq 1/2\}$ is in a $\frac{\eta_1}{4}$ -neighborhood of the graph of w . Consequently, by (4.12), taking into account the rescaling, it follows that

$$\{u_k = 0\} \cap \{|x'| \leq l_k/2\} \subset \{|x_n - \frac{\theta_k}{l_k} \xi' \cdot x'| < \frac{3}{4} \theta_k \eta_1\}.$$

From (4.13), we have that

$$\{u_k = 0\} \cap \{|x'| \leq l_k/2\} \subset \{|x \cdot \xi_k| < \frac{3}{4} \theta_k \eta_1\},$$

which, together with (4.14), is a contradiction with the fact that u_k does not satisfies the statement of Theorem 4.1. This end the proof of Theorem 4.1.

Appendix A

PDE Tools

In this appendix we recall the basic concepts we need about the theory of viscosity solutions for partial differential equations and we present some classical comparison results. The references for a more detailed presentation of the theory of viscosity solutions for second order partial differential equations are for instance [35] or [13].

We will first consider a general degenerate elliptic second order partial differential equation, and then we will focus on the particular equation (1.5).

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \rightarrow \mathbb{R}$ be a continuous function such that for any two symmetric matrices X and Y such that $Y - X$ is positive definite and any values $x \in \Omega$, $u \in \mathbb{R}$ and $p \in \mathbb{R}^n$ we have the inequality $F(x, u, p, X) \geq F(x, u, p, Y)$. We consider the following partial differential equation

$$F(x, u, \nabla u, D^2 u) = 0, \quad \text{in } \Omega. \quad (\text{A.1})$$

This equation is called degenerate elliptic. We now introduce the definition of viscosity solution for a degenerate elliptic equation

Definition A.1. *Let $u \in C^0(\Omega)$, we say that u is a viscosity supersolution of (A.1) if, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that $u(x_0) = \phi(x_0)$ and $u(x) \geq \phi(x)$ in Ω , we have*

$$F(x_0, \phi(x_0), \nabla \phi(x_0), D^2 \phi(x_0)) \geq 0. \quad (\text{A.2})$$

Analogously, we say that u is a viscosity subsolution of (A.1) if, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that $u(x_0) = \phi(x_0)$ and $u(x) \leq \phi(x)$ in Ω , we have

$$F(x_0, \phi(x_0), \nabla \phi(x_0), D^2 \phi(x_0)) \leq 0. \quad (\text{A.3})$$

u is called viscosity solution if it is both viscosity subsolution and viscosity supersolution.

During Chapter 3, in order to show that the barriers functions are strict supersolution of the equation (1.5), we have not used Definition A.1. We have used another characterization of viscosity supersolution for second order partial differential equations. Now we want to explain the characterization that we used in order to prove that the barriers functions are viscosity supersolutions. First of all we introduce the superjets and the subjects of a function u .

Definition A.2. *Let $u : \Omega \rightarrow \mathbb{R}$ be a continuous function, we define*

- *the superjet of u at $x \in \Omega$ is denoted by $J^{2,+}u(x) \subseteq \mathbb{R}^n \times \text{Sym}(n)$ and it is defined in the following way*

$$(p, X) \in J^{2,+}u(x) \iff u(y) \leq u(x) + p \cdot (y - x) + \frac{1}{2}X(y - x) \cdot (y - x) + o(|y - x|^2)$$

- the subject of u at $x \in \Omega$ is denoted by $J^{2,-}u(x) \subseteq \mathbb{R}^n \times \text{Sym}(n)$ and it is defined in the following way

$$(p, X) \in J^{2,-}u(x) \iff u(y) \geq u(x) + p \cdot (y - x) + \frac{1}{2}X(y - x) \cdot (y - x) + o(|y - x|^2)$$

We now present the result that gives us an equivalent definition of viscosity subsolution and viscosity supersolution:

Proposition A.1. *Let $u : \Omega \rightarrow \mathbb{R}$ be a continuous function, then the following are equivalent:*

- (i) u is a viscosity subsolution (resp., supersolution) of (A.1)
- (ii) for every $x \in \Omega$ and $(p, X) \in J^{2,+}u(x)$ (resp., $J^{2,-}u(x)$) we have $F(x, u(x), p, X) \leq 0$ (resp., ≥ 0)

Proof. The proof of this theorem can be found in [13]. □

In the Lemmas 3.2 and 3.3 we have used the characterization of viscosity supersolution given by the Proposition A.1. In our case the function F has the following form:

$$F(x, u, \nabla u, D^2u) = -\Delta u + h'_0(u),$$

and, during the proof of Lemmas 3.2 and 3.3, we have proved that

$$-\Delta u(x) + h'_0(u(x)) \geq 0, \tag{A.4}$$

when x was in a region where u was C^2 .

This proves that u is a viscosity supersolution because, in the region where u is C^2 , we have that $J^{2,-}u(x) = \{(\nabla u, D^2u)\}$ and the estimate (A.4) proves (ii) of the Proposition A.1. If the inequality in (A.4) is strict we call u strict viscosity supersolution.

Now we state two comparison principle that are useful during the proof of the main results.

Theorem A.2. (Strong Comparison Principle 1) *Let Ω be an open subset of \mathbb{R}^n , let $\Lambda \in \mathbb{R}$ and let $u, v \in C^1(\Omega)$ satisfy (in a weak sense) the following inequalities*

$$-\Delta u + \Lambda u \leq -\Delta v + \Lambda v, \quad u \leq v \text{ in } \Omega.$$

If there exists $x_0 \in \Omega$ such that $u(x_0) = v(x_0)$ then $u = v$ in the connected component of Ω containing x_0

Proof. We can find a proof of a more general result in [21]. This theorem is a particular case of Theorem 1.4 in [21] □

An easy consequence of the above result is the following one, which is very useful for our applications

Corollary A.3. (Strong Comparison Principle 2) *Let Ω be an open subset of \mathbb{R}^n , and let $u, v \in C^1(\Omega)$ satisfy (in a weak sense) the following inequalities*

$$-\Delta u + f(u) \leq -\Delta v + f(v), \quad u \leq v \text{ in } \Omega,$$

with f locally Lipschitz continuous. If there exists $x_0 \in \Omega$ such that $u(x_0) = v(x_0)$ then $u = v$ in the connected component of Ω containing x_0 .

Proof. Let $\epsilon > 0$ be so that $B_\epsilon(x_0) \subset \Omega$ and let

$$M_{u,v} := \max\{\|u\|_{L^\infty(B_\epsilon(x_0))}, \|v\|_{L^\infty(B_\epsilon(x_0))}\},$$

$$\Lambda := \sup_{\{|U|, |V| \leq M_{u,v}, U \neq V\}} \frac{|f(U) - f(V)|}{|U - V|}.$$

Then

$$\begin{aligned} -\Delta u + \Lambda u &\leq -\Delta v + f(v) - f(u) + \Lambda u \leq -\Delta v + \Lambda|v - u| + \Lambda u \\ &= -\Delta v + \Lambda(v - u) + \Lambda u = -\Delta v + \Lambda v, \end{aligned}$$

hence the result follows from Theorem A.2. □

Appendix B

Technical Lemmas

In this appendix we collect some elementary lemmas and technical lemmas that are useful during the proofs of the main results. The proofs of this lemmas can be found in [50].

Lemma B.1. *There exists a positive constant C such that*

$$\sqrt{a} - \sqrt{b} \leq C \frac{a - b}{\sqrt{a} - \sqrt{b}}$$

for any $a \geq b \geq 0$, $a \neq 0$.

Lemma B.2. *For any $0 \leq s \leq t \leq \theta^*$,*

$$h_0(-1 + t) - h_0(-1 + s) \geq c(t^2 - s^2)$$

for a suitable constant $c > 0$

Lemma B.3. *There exists a positive constant \tilde{C} , so that*

$$\int_{-1+b}^0 \frac{d\xi}{\sqrt{(1+\xi)^2 - a^2}} \leq \tilde{C} \left(1 + \log\left(\frac{1}{b}\right)\right)$$

for any $0 < a \leq b \leq 1$

Lemma B.4. *Let U be an open subset of \mathbb{R} . Let $g \in C^2(U)$ and assume that g has no critical points. Define*

$$\Psi^{y,l}(x) = g(|x - y| - l).$$

Then, for $t = |x - y| - l \in U$ and $x \neq y$, we have

$$\Delta \Psi^{y,l}(x) = g''(t) + g'(t) \frac{n-1}{|x-y|}.$$

Lemma B.5. *Let U be an open subset of \mathbb{R} . Let $g \in C^2(U)$ and assume that g has no critical points. Let Γ be a smooth hypersurface in \mathbb{R}^n and let $d_\Gamma(x)$ be the distance function to Γ . Suppose that if $x \in \Omega$, then $d_\Gamma(x) \in U$. Then*

$$\Delta g(d_\Gamma(x)) = g''(d_\Gamma(x)) + g'(d_\Gamma(x)) \Delta d_\Gamma(x)$$

Lemma B.6. *Let $I \ni 0$ be an interval of \mathbb{R} and let $h \in C^1(I)$ satisfy $h(s) > 0$ for any $s \in I$. Let*

$$H(s) := \int_0^s \frac{1}{\sqrt{2h(\xi)}} d\xi, \text{ for any } s \in I.$$

Define also g as the inverse of H , that is, $g(t) = H^{-1}(t)$, for any $t \in H(I)$. Then $g \in C^2(H(I))$ and

$$g'(t) = \sqrt{2h(g(t))}, \quad g''(t) = h'(g(t)),$$

for any $t \in H(I)$.

Lemma B.7. Let Ω be an open domain in \mathbb{R}^n and let $x_0 \in \Omega$. Let $w \in C^1(\Omega)$ and $v := \nabla w(x_0)$. Assume that there exists $\bar{\omega} \in \mathbb{R}^n \setminus \{0\}$ such that

$$w(x_0 + x) \leq v \cdot x + w(x_0)$$

for any $x \in \mathbb{R}^n$ so that $x + x_0 \in \Omega$ and $\bar{\omega} \cdot x \geq 0$. If $P \in C^2(\Omega)$ is a quadratic function touching w from below at x_0 , then $\Delta P \leq 0$ in the viscosity sense. Analogously, if

$$w(x_0 + x) \geq v \cdot x + w(x_0)$$

for any $x \in \mathbb{R}^n$ so that $x + x_0 \in \Omega$ and $\bar{\omega} \cdot x \geq 0$. If $P \in C^2(\Omega)$ is a quadratic function touching w from above at x_0 , then $\Delta P \geq 0$ in the viscosity sense.

Lemma B.8. Let $M \in \text{Mat}((n-1) \times (n-1))$ and $V \in \mathbb{R}^{n-1}$. Define the paraboloid

$$\Gamma := \left\{ (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n = \frac{1}{2} x' \cdot M x' + V \cdot x' \right\}.$$

Let d_Γ be the signed distance to Γ . then for any $\tau \geq 0$

$$d_\Gamma(x + \tau e_n) \geq d_\Gamma(x).$$

Lemma B.9. Let u be a local minimizer of the Ginzburg-Landau functional in $[-l, l]^n$. Then, if l is large enough, the following happen: given $\omega \in S^{n-1}$, there exist a universal constant L so that, for any $k \geq L$,

- if $\{u = 0\} \cap \{|x - (\omega \cdot x)\omega|_\infty \leq k\} \subset \{\omega \cdot x \geq -\frac{k}{10}\}$, then $u < -1 + \theta^*$ for any $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ satisfying

$$\omega \cdot x \leq -\frac{k}{5} \quad \text{and} \quad |x - (\omega \cdot x)\omega|_\infty \leq \frac{k}{2};$$

- if $\{u = 0\} \cap \{|x - (\omega \cdot x)\omega|_\infty \leq k\} \subset \{\omega \cdot x \leq \frac{k}{10}\}$, then $u > -1 + \theta^*$ for any $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ satisfying

$$\omega \cdot x \geq \frac{k}{5} \quad \text{and} \quad |x - (\omega \cdot x)\omega|_\infty \leq \frac{k}{2}.$$

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