Università degli Studi di Padova

DIPARTIMENTO DI MATEMATICA "TULLIO LEVI-CIVITA"
Corso di Laurea Magistrale in Matematica

Tesi di Laurea Magistrale

Non-Minimality of the Double Logarithm Spiral

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Sessione di Laurea del 20 luglio 2018
Anno Accademico 2017-2018

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Introduction

The problem considered in this thesis is set in a particolar framework, the subRiemannian one. The word 'subRiemannian' has been introduced by Strichartz in [Str86], and the choice of the prefix 'sub' is motivated by the fact that in our manifold M we are fixing a subbundle of the tangent bundle. Roughly speaking this subbundle can be seen as a constraint on the velocities of the curves that live in M: we consider Lipschitz curves whose velocity lies for almost every t in the subbundle. These curves are called horizontal. Since we want to assign a length to horizontal curves we endow our subbundle with a smooth metric on it (that is a smooth 2-covariant, strictly positive and symmetric tensor). It is clear that the Riemannian manifold is a special case of a subRiemannian manifold, the one obtained taking the whole TM as subbundle.
With this notion of length we can define the distance between two points as in the Riemannian case, that is, as the infimum of the lengths of the horizontal curves joining the two points. We are interested in length minimizing curves, i.e. the curves whose length realizes the distance, and in particular to their regularity.
It is known, see for instance [ABB15], that in the Riemannian case length minimizers are smooth ($\mathscr{C}^\infty$). What makes the subRiemannian problem interesting is that in this framework we have two kind of horizontal curves that satisfy the first order necessary conditions for minimality: the normal and the abnormal ones. This is not the case in Riemannian geometry, where there are not abnormal curves. In [Str86] Strichartz proved that normal length minimizers are smooth ($\mathscr{C}^\infty$) and, not taking into account the existence of abnormal length minimizers, he thought the problem of regularity was solved. Nevertheless, the situation turned out to be not so simple, indeed Montgomery exhibited in [Mon94] the first example of a length minimizer which is abnormal, and this curve turned out to be $\mathscr{C}^\infty$. Indeed, up to this day all the known examples of abnormal length minimizers are $\mathscr{C}^\infty$ but nobody has succeded in proving a general regularity result. We don't even know if abnormal length minimizers are $\mathscr{C}^1$.
We present some partial result in the positive. In [LM08] Leonardi and Monti proved that for a large class of subRiemannian manifolds, length minimizers do not have corner-like singularities. Their result has been extended to all subRiemannian manifolds in [HLD15], by Hakavouri and LeDonne. In [MPV18] Monti, Pigati and Vittone proved a necessary condition satisfied by length minimizers and related to the existence of a tangent line in the tangent cone. In this last paper is also detailed the cut and correction technique for Carnot groups, which was already introduced (in a different formulation) in [LM08]. Carnot groups are a special type of subRiemannian manifold.
The purpose of this thesis is to understand to what extent the cut and correction technique can be used to prove the non minimality of curves satisfying the necessary condition for minimality. To be more specific, in [LLMV] is presented a kind of spiral with the following properties:

- the curve is rectifiable;
- the curve satisfies the necessary condition found in [MPV18], that is the tangent

cone at the origin contains a line. Actually, in this case the tangent cone at the origin contains all the lines.

Moreover this curve is not $\mathscr{C}^1$ at the origin and in the article arises as an abnormal extremal satisfying also the Goh conditions for minimality. Of course, due to the spiral-like behaviour we would like to say that this curve is not a minimizer. The problem is that to this day there are no tools to deal with curves with this kind of singularity. At this point we have to admit that in a Carnot group of step 3 it is already known that length minimizers are smooth, thus a curve like the one in [LLMV] cannot be a minimizer in this framework.
In the first chapter we set up the framework and the techniques that will be used in the second chapter to investigate our case study: the double logarithm spiral. We will focus on the positive branch of this spiral. The second chapter has been devoted to use the new ideas and methods to prove the following

Theorem. Let G be a Carnot group of step 3 and rank 2 and $\gamma : [0, 1[\to \mathbb{G}$ a horizontal curve with $\gamma(0) = e$,

$$\pi(\gamma(t)) = t\cos(\phi(t))X_1 + t\sin(\phi(t))X_2,$$

where $\phi : (0, 1) \to (0, \infty)$

$$\phi(t) := \log(-\log(t)).$$

Then, for $T < 1$, $\gamma|_{[0,T]}$ is not a length minimizer between $\gamma(0)$ and $\gamma(T)$.

We outline briefly the ideas behind the proof. We want to find an admissible competitor joining the two points e and $\gamma(T)$. We start by modifying the curve in some subinterval $[a, b] \subset [0, T]$ in such a way that the new curve is still admissible and has shorter length. The subinterval is one of the unknowns of the problem. Moreover this operation creates a shorter curve, but its endpoint is no more $\gamma(T)$. In the first two paragaphs of the second chapter we estimate how much the new curve is shorter and what is the error on the final point. To "restore the final point with a gain of length" we use an iterative procedure (with a finite number of steps), which creates at each step a new curve:

1) that is obtained modifying the curve of the previous step on a subinterval of $[0, T]$ (to be found);

2) that is longer than the one of the previous step;

3) that is strictly shorter than the initial curve;

4) with endpoint "nearer" to $\gamma(T)$ than the endpoint of the previous curve.

Moreover at each step we will find an existence condition for the subintervals used for the modifications. If all these conditions are compatible, then the final curve is the shorter competitor joining e and $\gamma(T)$ that we were looking for. Unfortunately we had to impose the assumption on the step, due to problems on the propagation of the error on the final point. The removal of this assumption will be object of further investigations.

1 Carnot groups

1.1 Basic facts about subRiemannian manifolds

In this chapter we introduce some basic facts about Carnot groups and the techniques that will be used in the next chapter to prove that the double logarithm spiral is not a length minimizer. To give a flavour of the subRiemannian framework we devote this first paragraph to introduce the Hörmander condition and the notion of subRiemannian manifold. From the following paragraph we will focus on Carnot groups. We will show how a Carnot group is naturally endowed with a subRiemannian structure. Actually, Carnot groups have more structure than a general subRiemannian manifold and due to this fact enjoys nicer properties. We will describe some of them in this chapter. In the following the manifolds and the vector fields will always assumed to be smooth. We denote by $\Gamma(TM)$ the set of smooth vector fields on M

Definition 1.1.1. Let M be a $n-$dimensional manifold and $\mathscr{D}$ a distribution locally spanned by the vector fields $X_1, \dots, X_r$. We define recursively the following family of subbundles of TM.

$$\mathscr{D}^1 := \mathscr{D}$$

$$\mathscr{D}^{i+1} := \mathscr{D}^i + [\mathscr{D}, \mathscr{D}^i],$$

where

$$[\mathscr{D}, \mathscr{D}^i] := \mathrm{span}\{[X, Y] |\ X \in \mathscr{D},\ y \in \mathscr{D}^i\}$$

and $[X, Y] = XY - YX$ is the commutator.
The Lie algebra generated by $X_1, \dots, X_r$ is defined as $\mathrm{Lie}(X_1, \dots, X_r) := \cup_{i \geq 1} \mathscr{D}^i$.

We observe that due to Jacobi identity $\mathrm{Lie}(X_1, \dots, X_r)$ is the smallest linear subspace of $\Gamma(TM)$ which contains $X_1, \dots, X_r$ and is invariant under the commutator.
Let $I = (i_1, \dots, i_k) \in \mathbb{N}^k$ be a multindex for some $k \geq 1$ we set $X_I := [X_{i_1}, X_{(i_2, \dots, i_k)}]$.
If $I \in \mathbb{N}^k$ we set $|I| := k$ Using Jacobi identity it can be proved that
$\mathscr{D}^i = \mathrm{span}\{X_I,\ |I| \leq i\}$. For $p \in M$ and $i \geq 1$ we define $\mathscr{D}^i(p) := \{X(p),\ X \in \mathscr{D}^i\}$ and $\mathrm{Lie}(X_1, \dots, X_r)(p) := \{X(p),\ X \in \mathrm{Lie}(X_1, \dots, X_r)\}$. $\mathscr{D}^i(p)$ is an increasing sequence of vector subspaces of T_pM whose union is $\mathrm{Lie}(X_1, \dots, X_r)(p)$.

Definition 1.1.2. We say that a distribution $\mathscr{D}$ is bracket generating (or satisfies the Hörmander condition) if for every $p \in M$ $\mathrm{Lie}(X_1, \dots, X_r)(p) = T_pM$.

Since at each p $\mathrm{Lie}(X_1, \dots, X_r)(p) = \cup_i \mathscr{D}^i(p)$, the bracket condition is equivalent to the existence of an index $i(p)$ such that $\mathscr{D}^{i(p)}(p) = T_pM$. A bracket generating distribution is not integrable (in the sense of Frobenius theorem), but this property tells us that we can fill all the tangent space at a point taking a finite number of iterated commutators (the number of commutators depends on the point). And this fact has remarkable consequences.

Definition 1.1.3. A subRiemannian manifold is a triple $(M, \mathscr{D}, \hat{g})$ where M is a manifold, $\mathscr{D}$ is a bracekt generating ditribution and $\hat{g}$ is a smooth metric on $\mathscr{D}$. For any $p \in M$ the dimension of $\mathscr{D}_p$ as a vector subspace of M is called rank of the distribution. The smallest index i such that $\mathscr{D}^i(p) = T_pM$ for every $p \in M$ is called step of the distribution.

Remark 1.1.4. Riemannian manifolds are special cases of subRiemannian manifolds in which the bracket generating distribution is the whole TM.

Definition 1.1.5. Let $\gamma : [0,T] \to M$ be a lipschitz curve, we say that γ is horizontal (or admissible) if $\dot{\gamma}(t) = \mathscr{D}_{\gamma(t)}$ for almost every $t \in [0,T]$

In other words, if $\mathscr{D} = \text{span}\{X_1, \dots, X_r\}$, horizontal curves satisfy almost everywhere

$$\dot{\gamma}(t) = \sum_{j=1}^{r} h_j(t) X_j(\gamma(t)),$$

where $h = (h_1, \dots, h_r) : [0,T] \to \mathbb{R}^r$ is a function in $L^\infty([0,T], R^r)$. $h_1, \dots, h_r$ are called controls of the curve γ. We can define a subRiemannian notion of length of a curve

Definition 1.1.6. Let γ be an horizontal curve, we define the length of γ by

$$L(\gamma) := \int_0^T \left(\hat{g}_{\gamma(t)}\left(\dot{\gamma}(t), \dot{\gamma}(t)\right)\right)^{\frac{1}{2}} dt$$

Horizontal curves are the "right" paths to follow in a subRiemannian manifold, in the sense that their velocities are constrained to live in the distribution. Due to the presence of this constraint it is not clear a priori if starting from any point on M we can reach any other point on the manifold through horizontal paths. As a consequence of the bracket generating assumption we have the following classical result

Theorem 1.1.7 (Chow-Rashevsky)**.** If $(M, \mathscr{D}, g)$ is a subRiemannian manifold and M is connected then every two points in M can be joined by an horizontal curve.

We introduce the Carnot-Caratheodory distance on our subRiemannian manifold.

Definition 1.1.8. Let $x, y \in M$, we set

$$d_{CC}(p,q) := \inf\{L(\gamma) \text{ s.t.} \gamma : [0,1] \to M, \text{ is horizonal and } \gamma(0) = p,\ \gamma(1) = q\},$$

and $d_{CC}(p,q) := +\infty$ if there is no horizontal path joining x and y.

Since the length of a curve is invariant under linear reparametrization, we will get an equivalent definition of d_{CC} if we take the interval $[0,T]$ instead of $[0,1]$.
As a consequence of the above theorem is that d_{CC} takes value on $[0,\infty)$. Moreover it can be proved (with some work) that d_{CC} is a distance. Another consequence of the bracket generating condition is that the the C-C distance induces the manifold topology on M. For a proof of all this facts see [Pig16].

Definition 1.1.9. A horizontal curve $\gamma : [0, T] \to M$ is called length minimizer if $L(\gamma) = d_{CC}(\gamma(0), \gamma(T))$

The following theorem tells us that length on subRiemannian manifolds length minimizers exist locally

Theorem 1.1.10. For any p in M there is an open neighborhood $U \subset M$ such that for any $q \in U$ there exists a length minimizer connecing p and q.

A proof of this theorem can be found in [Vit13]. In the paper it is assumed $M = \mathbb{R}^n$ but being the nature of the result local, this is not a restriction.

1.2 Carnot groups as subRiemannian manifolds

Definition 1.2.1. Let $\mathfrak{g}$ be a Lie algebra. We say that $\mathfrak{g}$ is stratified if there exist $V_1, \dots, V_l \subset \mathfrak{g}$, vector subspaces such that

$$g = V_1 \oplus \cdots \oplus V_l,$$

$$V_{i+1} = [V_1, V_i], \quad i = 1, \dots, l-1,$$

$$[V_1, V_r] = \{0\},$$

where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by elements of the form $[X, Y]$, for $X \in V_1$, $Y \in V_i$. l is called step of the stratification.

A Lie group $\mathbb{G}$ is called stratified if its Lie algebra is stratified. Similarly a Lie group is called nilpotent if its Lie algebra is nilpotent. It is easy to see that a stratified Lie algebra of step l is also a nilpotent Lie algebra of step l.

Definition 1.2.2. A Carnot group $\mathbb{G}$ is a connected, simply connected and stratified Lie group. We say that $\mathbb{G}$ is a Carnot group of step l if l is the step of the stratification of its Lie algebra.

Remark 1.2.3. A Carnot group can be naturally endowed with a subRiemannian structure as follows. Let $h \in \mathbb{G}$ and L_h denote the left translation by h, i.e. $L_h(g) = hg$, for $g \in \mathbb{G}$. We define the distribution $D_g := (dL_h)_e[V_1]$, where $(dL_h)_e V_1$ denotes $(dL)_e$ applied to the elements of V_1. Since $L_{hg} = L_h \circ L_g$, D_g is left invariant, indeed

$$(dL_h)_g[D_g] = (dL_h)_g(dL_g)_e[V_1] = d(L_{hg})_e[V_1] = D_{hg}.$$

This distribution is bracket generating, this follows by the naturality of the Lie bracket and the stratification.
We can define the metric on D taking a positive definite inner product $\hat{g}$ on V_1 and extending it on the whole distribution by the pullback of L, that is setting

$$\hat{g}_h(v, w) := ((L_{h^{-1}})^* \hat{g})(v, w) = \hat{g}((dL_{h^{-1}})_h[v], (dL_{h^{-1}})_h[w]),$$

for any $h \in \mathbb{G}$ and $v, w \in D_h$. We introduce the notation $|\cdot|_h := (\hat{g}_h(\cdot, \cdot))^{\frac{1}{2}}$ or simply $|\cdot|$ when there is no confusion on the point h.

Remark 1.2.4. The metric $\hat{g}$ is left invariant in the following sense if $a, h \in \mathbb{G}$, $v, w \in D_h$, then $\hat{g}_{ah}((dL_a)_h[v], (dL_a)_h[w]) = \hat{g}_h(v, w)$. The proof relies again on $L_a \circ L_h = L_{ah}$. We have

$$\begin{aligned}
\hat{g}_{ah}((dL_a)_h[v], (dL_a)_h[w]) &= \hat{g}\left((dL_{h^{-1}a^{-1}})_{ah}(dL_a)_h[v], (dL_{h^{-1}a^{-1}})_{ah}(dL_a)_h[w]\right) \\
&= \hat{g}\left((dL_{h^{-1}})_h(dL_{a^{-1}})_{ah}(dL_a)_h[v], (dL_{h^{-1}})_h(dL_{a^{-1}})_{ah}(dL_a)_h[w]\right) \\
&= \hat{g}(d(L_h \circ L_{a^{-1}} \circ L_a)_h[v], d(L_h \circ L_{a^{-1}} \circ L_a)_h[w]) \\
&= \hat{g}_h(v, w).
\end{aligned}$$

1.3 Dilations

The presence of a stratification allows us to define dilations on the Lie algebra of the group. In our setting these dilations can be transported to the group via the exponential map. We will show how dilations interact with the C-C distance. For the definition of the exponential map, its properties and the Backer-Hausdorff-Campbell formula (B-H-C from now) see [Kna02].

Definition 1.3.1. Let $\mathfrak{g}$ be a stratified Lie algebra and let $\lambda > 0$. We define the inhomogeneous dilation on $\mathfrak{g}$ of factor λ as the linear map $d_\lambda : \mathfrak{g} \to \mathfrak{g}$ such that

$$d_\lambda(v) = \lambda^j v \qquad \forall v \in V_j$$

We need the following fact about stratified Lie algebras.

Lemma 1.3.2. Let $\mathfrak{g} = V_1 \oplus \cdots \oplus V_l$ be a stratified Lie algebra of step l. Then

$$[V_i, V_j] \subset V_{i+j}$$

for all $i, j = 1 \dots, l$, where $V_k = 0$ fro $k > l$.

Proof. The proof is by induction on the index i. If $i = 1$, we know by definition of stratification $[V_1, V_j] \subset V_{j+1}$. We assume that $[V_i, V_j] \subset V_{i+j}$ for all j and i fixed. Being $\mathfrak{g}$ stratified, V_{i+1} is generated by elements of the form $[v_1, v_i]$ with $v_1 \in V_1$ and $v_i \in v_i$. If we take $v_j \in V_j$, by the Jacobi identity we get

$$[[v_1, v_i], v_j] = -[[v_i, v_j]v_1] - [[v_j, v_1], v_i].$$

By the inductive hypothesis $[v_i, v_j] \in V_{i+j}$, thus $-[[v_i, v_j]v_1] = [v_1, [v_i, v_j]] \in [V_1, V_{i+j}] = V_{i+j+1}$. Moreover $-[[v_j, v_1], v_i] = [v_i, [v_j, v_1]] \in [V_i, V_{j+1}] \subset V_{i+j+1}$, again by the inductive hypothesis. By linearity of the Lie brackets we have the claim. $\square$

Proposition 1.3.3. For any $\lambda > 0$, d_λ is an automorphism

Proof. d_λ is linear, we have only to prove that it behaves well with respect to the Lie bracket, i.e. for $X, Y \in \mathfrak{g}$

$$d_\lambda([X, Y]) = [d_\lambda(X), d_\lambda(Y)].$$

We can write the elements of $\mathfrak{g}$ as $X = X_1 + \cdots + X_l$, $Y = Y_1 + \cdots + Y_l$, with $X_i, Y_i \in V_i$. Since by lemma 1.3.2 $[X_i, Y_j] \in [V_i, V_j] \subset V_{i+j}$, we obtain

$$[d_\lambda(X), d_\lambda(Y)] = [\sum_{i=1}^{l} \lambda^i X_i, \sum_{j=1}^{l} \lambda^j Y_j] = \sum_{i,j=1}^{l} \lambda^{i+j}[X_i, Y_j] = \sum_{i,j=1}^{l} d_\lambda([X_i, Y_j]) = \lambda([X, Y]).$$

□

Since our Carnot group $\mathbb{G}$ is nilpotent and simply connected we can use the exponential map to define a notion of dilation on the group from the one on the lie algebra, by the formula

$$\delta_\lambda := \exp \circ d_\lambda \circ \exp^{-1}.$$

Remark 1.3.4. From the definition we obtain, for $\lambda, \mu > 0$

$$\begin{aligned}
\delta_{\lambda\mu}(h) &= \exp(d_{\lambda\mu}(\exp^{-1}(h))) = \exp(d_\lambda d_\mu(\exp^{-1}(h))) \\
&= \exp(d_\lambda \exp^{-1}((\exp d_\mu \exp^{-1})(h))) \\
&= (\delta_\lambda \delta_\mu)(h),
\end{aligned}$$

(where we suppressed the $\circ$ to improve readability). Using this formula together with the B-H-C we get that δ_λ is a group homomorphism

$$\begin{aligned}
\delta_\lambda(gh) &= \exp d_\lambda \exp^{-1}(gh) = \exp d_\lambda(P(\exp^{-1}(g), \exp^{-1}(h))) \\
&= \exp(P(d_\lambda(\exp^{-1}(g)), d_\lambda(\exp^{-1}(h)))) \\
&= (\exp(d_\lambda(\exp^{-1}(g)))(\exp(d_\lambda(\exp^{-1}(h))) = \delta_\lambda(g)\delta_\lambda(h),
\end{aligned}$$

where we used the fact that d_λ is a Lie algebra homomorphism to bring it inside the polynomial given by the B-H-C formula. We have also $\delta_1 = \mathrm{id}_{\mathbb{G}}$.

Remark 1.3.5. We show that $\mathscr{D}$ is invatiant under δ_λ, i.e. $d(\delta_\lambda)_h \mathscr{D} = \mathscr{D}_{\delta_\lambda(h)}$. We will use the simple identity

$$(\delta_\lambda L_a)(h) = \delta_\lambda(ah) = \delta_\lambda(a)\delta_\lambda(h) = (L_{\delta_\lambda(a)}\delta_\lambda)(h)$$

and the fact $d(\delta_\lambda)_e = d_\lambda$ (this is obtained from the properties of the exponential map, see [Kna02]. Let $v \in \mathscr{D}_h$, by definition of $\mathscr{D}_h$, v can be written as $d(L_h)_e[w]$, for some $w \in V_1$, thus

$$\begin{aligned}
d(\delta_\lambda)_h[v] &= d(\delta_\lambda)_h d(L_h)_e[w] = d(\delta_\lambda L_h)_e[w] = d(L_{\delta_\lambda(h)}\delta_\lambda)_e[w] \\
&= d(L_{\delta_\lambda(h)})_{\delta_\lambda(e)} d(\delta_\lambda)_e[w] = d(L_{\delta_\lambda(h)})_{\delta_\lambda(e)}[d_\lambda w] \in \mathscr{D}_{\delta_\lambda(h)}
\end{aligned}$$

On the other hand if we take an element $v \in \mathscr{D}_{\delta_\lambda(h)}$ we can write it as $d(L_{\delta_\lambda})_e[w]$, for some $w \in V_1$ and with a similar computation we show that v it is the image through $d(\delta_\lambda)_h$ of $d_\lambda^{-1}(w)$, which proves the other inclusion. Moreover, by the invariance of $\mathscr{D}$ under δ_λ and of the left-invariance of the metric we obtain (for $v \in \mathscr{D}_h,\ v = d(L_h)_e[w]$)

$$|d(\delta_\lambda)_h[v]| = |d(L_{\delta_\lambda(h)})_{\delta_\lambda(e)}[d_\lambda w]| = |d_\lambda(w)| = |\lambda w| = \lambda |d(L_h^{-1})_h[v]| = \lambda |v|$$

Proposition 1.3.6. Let $\gamma \in \mathrm{Lip}([0,T],\mathbb{G})$ be an horizontal curve, $h \in \mathbb{G}$, $\lambda > 0$. Then the curves $L_a \circ \gamma$, $\delta_\lambda \circ \gamma$ are horizontal and

$$L(L_a \circ \gamma) = L(\gamma), \quad L(\delta_\lambda \circ \gamma) = \lambda L(\gamma).$$

Moreover, for $g, h \in \mathbb{G}$

$$d_{CC}(ag, ah) = d_{CC}(g,h), \quad d_{CC}(\delta_\lambda(g), \delta_\lambda(h)) = \lambda d_{CC}(g,h)$$

Proof. γ horizontal means $\dot\gamma(t) \in \mathscr{D}_{\gamma(t)}$ for almost every t. Since $\frac{d}{dt}(L_a \circ \gamma)(t) = d(L_a)_{\gamma(t)}[\dot\gamma(t)]$ and $\mathscr{D}$ is left invariant we have $\frac{d}{dt}(L_a \circ \gamma)(t) \in \mathscr{D}_{a\gamma(t)}$ for almost every t. In the same fashion (using the invariance of $\mathscr{D}$ under δ_λ) we have $\frac{d}{dt}(\delta_\lambda \circ \gamma)(t) = d(\delta_\lambda)_{\gamma(t)}[\dot\gamma(t)] \in \mathscr{D}_{\delta_\lambda(\gamma(t))}$. By left-invariance of the metric

$$L(L \circ \gamma) = \int_0^T |\frac{d}{dt}(L_a \circ \gamma)(t)|dt = \int_0^T |\dot\gamma| dt = L(\gamma).$$

By the last remark

$$L(\delta_\lambda \circ \gamma) = \int_0^T |\frac{d}{dt}(\delta_\lambda \circ \gamma)(t)|dt = \int_0^T |d(\delta_\lambda)_{\gamma(t)}[\dot\gamma(t)]|dt = \lambda \int_0^T |\dot\gamma| dt = \lambda L(\gamma).$$

The first equality implies that $d_{CC}(g,h) \le d_{CC}(ag,ah)$ because to every curve in $\Omega_{g,h}$ we can associate a curve in $\Omega_{ag,ah}$ with the same length. For the opposite inequality it is enough to observe that to any curve in $\Omega_{ag,ah}$ we can associate a curve in $\Omega_{g,h}$ with the same length, simply taking $L_{a^{-1}} \circ \gamma$. The last claim is proved by the same argument. $\square$

1.4 Exponential coordinates

In this section we want to define a special system of coordinates on $\mathbb{G}$, called exponential coordinates of the first kind.

In our assumptions ($\mathfrak{g}$ nilpotent and simply connected) the the exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ is a diffeomorphism.
Thus if we identify $\mathfrak{g}$ and $\mathbb{R}^n$ throught a previously fixed basis of $\mathbb{R}^n$, say $\{e_1, \dots, e_n\}$, we obtain a global chart on G,

$$F \colon \mathbb{R}^n \to G$$
$$(x_1, \dots, x_n) \longmapsto \exp(x_1 X_1 + \cdots + x_n X_n).$$

The coordinates given by this chart are called exponential coordinates of the first kind, or simply exponential coordinates. Using these coordinates we can think at our Carnot group as being just $\mathbb{R}^n$ with a noncommutative group law defined by

$$x \cdot y := z \Longleftrightarrow \sum_{j=1} z_j X_j = P\left(\sum_{i=1}^n x_i X_i, \sum_{i=1}^n y_i X_i\right).$$

This group law makes $\mathbb{R}^n$ a Lie group and F an isomorphism of Lie groups. P is given by the B-H-C formula.

We introduce a basis on $\mathfrak{g}$, which keeps track of the stratification.

Definition 1.4.1. We call a basis $X_1, \dots, X_n$ of $\mathfrak{g}$ adapted to the stratification if $X_1, \dots, X_r$ is a basis of V_1, $X_{r+1}, \dots, X_{r+dim_{V_2}}$ is a basis of V_2 and so on.

For the rest of the chapter we will assume to be working with an adapted basis. We will also take the basis such that $X_1, \dots, X_r$ are orthonormal with respect to the metric.

Definition 1.4.2. For $i = 1, \dots, n$ there is an unique $1 \le j \le l$ such that $X_i \in V_j$. We define $d(i) := j$ and we call $d(i)$ the degree of i. Moreover given exponential coordinates $(x_1, \dots, x_n)$ we say x_i is a coordinate of the $j-$th layer if $d(i) = j$.

Definition 1.4.3. Given a monomial $x_1^{\alpha_1}, \dots, x_n^{\alpha_n}$ we define its weighted degree as $\sum_{i=1}^n d(i)\alpha_i$. A polynomial in the variables $x_1, \dots, x_n$ is called homogeneous if the monomials which compose it have the same weighted degree. If the monomial has two different variables, like, $x_1^{\alpha_1}, \dots, x_n^{\alpha_n} y_1^{\beta_1}, \dots, y_n^{\beta_n}$, its weighted degree is defined by $\sum_{i=1}^n d(i)(\alpha_i + \beta_i)$. Similarly for polynomials in more varaibles.

We observe that in the definition of weighted degree the terms $d(i)$ appear. This is due to the fact that coordinates belonging to different layers scale differently when dilated. Indeed, the dilation δ_λ in exponential coordinates is

$$\begin{aligned}
(F^{-1}\delta_\lambda F)(x_1, \dots, x_n) &= F^{-1}\delta_\lambda \exp(x_1 X_1 + \dots + x_n X_n) \\
&= F^{-1} \exp(\sum_{i=1}^n x_i \lambda^{d(i)} X_i) \\
&= (\lambda x_1, \dots, \lambda x_r, \dots, \lambda^l x_{n-dimV_l} + \dots + \lambda^l x_n)
\end{aligned}$$

Thus, in this coordinates, the right notion of degree to have homogeneity is the one of weighted degree.
Now we want to rewrite P given by the B-H-C formula in the form $P(Y, Z) = \sum_{i=1}^n P_i(y, z) X_i$, for $Y = \sum_{i=1}^n y_i X_i$, $Z = \sum_{i=1}^n z_i X_i \in \mathfrak{g}$, where $y = (y_1, \dots, y_n)$, $z = (z_1, \dots, z_n)$ and $P_i(y, z)$ a suitable polynomial with properties to be discussed. We recall

$$P(Y, Z) = Y + Z + \sum_{p=1}^{l-1} \frac{(-1)^p}{p+1} \sum_{\substack{0 \le i_1, \dots, i_p < s \\ 0 \le j_1, \dots, j_p < s \\ i_k + j_k \ge 1}} \frac{(\mathrm{ad} Y)^{i_1} (\mathrm{ad} Z)^{k_1} \cdots (\mathrm{ad} Y)^{i_p} (\mathrm{ad} Z)^{j_p}}{(i_1 + \dots + i_p + 1) i_1! \dots i_p! j_1! \dots j_p!} Y.$$

We observe that for $i = 1, \dots, l$, the polynomial $P_i(y, z)$ which gives the cofficient of X_i is of the form $y_i + z_i$ because all the commutators coming from the second half of the formula lie in $V_2 \oplus \dots \oplus V_l$. We observe also that the weighted degree of these polynomials is 1. Let's see what is the idea to get some information on the weighted degree of P_i, for i of degree 2. Clearly this polynomial will contain a term of the form $y_i + z_i$, which is a polynomial of weighted degree 2. The only bracket which has a component in V is $[Z, Y]$ times some constant. By linearity of the bracket, the components in V_2 are of the form $z_j y_k [X_j, X_k]$ for $d(i) + d(k) = 2$. This means that each monomial multiplying $[X_j, X_k]$

is of weighted degree 2. We conclude that for i of degree 2, $P_i(y,z)$ has weighted degree 2. In general $P(Y,Z)$ is a linear combination of terms of the form

$$[y_{i_{1,1}}, X_{i_{1,1}}, [y_{i_{1,2}} X_{i_{1,2}}, \dots, [z_{j_{1,1}} X_{j_{1,1}}, [z_{j_{1,2}} X_{j_{1,2}}, \dots, [\dots, y_i X_i] \dots],$$

where the first block contains k_1 commutators the second block contains l_1 commutators and so on. Each term equals a constant vector in some V_m multiplied by some monomial in $y_1, \dots, y_n, z_1 \dots, z_n$ with weighted degree m (which comes from $[V_a, V_b] \subset V_{a+b}$ recalling that $X_i \in V_{d(i)}$). Thus only terms containing a monomial of weighted degree $d(k)$ contribute to $P_k(Y,Z)$. Thus we have that for each $i = 1, \dots, n$ $P_i(y,z)$ is a polynomial of weighted degree $d(i)$.

Remark 1.4.4. Let's see how left translation behaves in exponential coordinates. If $(x_1, \dots, x_n)$, $(y_1, \dots, y_n) \in \mathbb{R}^n$, we have

$$F(x_1, \dots, x_n) F(y_1, \dots, y_n) = \exp\left(P\left(\sum_{i=1}^n x_i X_i, \sum_{j=1}^n y_i X_i\right)\right),$$

where P is a polynomial given by the Cambpell-Hausdorff-Backer formula. This implies

$$L_x y = x \cdot y = F^{-1}\left(P\left(\sum_{i=1}^n x_i X_i, \sum_{j=1}^n y_i X_i\right)\right) =$$
$$F^{-1}\left(\sum_{k=1}^n P_k(x,y) X_k\right) = \sum_{k=1}^n P_k(x,y) e_k,$$

where $P_k(x,y)$ is an homogeneous polynomial of weighted degree $d(k)$, in the variables x, y.

We denote with X_i^L the left invariant vector field on $\mathbb{G}$ associated to X_i. We want to write $X_1^L, \dots, X_n^L$ in a suitable way in exponential coordinates.

Definition 1.4.5. We set $\widetilde{X}_j := (F^{-1})_* X_j^L$, for $j = 1, \dots, n$.

Proposition 1.4.6. For $x \in \mathbb{R}^n$ we have

$$\widetilde{X}_j(x) = \frac{\partial}{\partial x_j}|_x + \sum_{i | d(i) > d(j)} c_{ji}(x) \frac{\partial}{\partial x_i}|_x,$$

where $c_{ji}(x)$ are polynomials of weighted degree $d(i) - d(j)$.

Proof.

$$\widetilde{X}_j(x) = d(F^{-1})_{F(x)} X_j^L(F(x)) = d(F^{-1})_{F(x)} \left(dL_{F(x)}\right)_e \left(\frac{d}{dt}\left(\exp(tX_j)\right)|_{t=0}\right) =$$
$$d(F^{-1})_{F(x)}\left(\frac{d}{dt}\left(F(x)\exp(tX_j)\right)|_{t=0}\right) = d(F^{-1})_{F(x)}\left(\frac{d}{dt}\left(F(x)\,F(te_j)\right)|_{t=0}\right) =$$

$$d(F^{-1})_{F(x)}\left(\frac{d}{dt}\left(F\left(x\cdot te_j\right)\right)|_{t=0}\right)=\frac{d}{dt}\left(x\cdot te_j\right)|_{t=0}.$$

On the other hand we can write $\widetilde{X}_j(x)=\sum_{i=1}^{n}c_{ji}(x)\frac{\partial}{\partial x_i}|_x$, where $c_{ji}\in C^\infty(\mathbb{R}^n)$, for $i=1,\dots,n$. By the above remark we know that

$$x\cdot te_j=\sum_{k=1}^{n}P_i(x,y)e_i.$$

Putting these two facts together (identifying $\mathbb{R}^n$ and its tangent space at x), we obtain

$$c_{ji}(x)=\frac{d}{dt}P_i(x,te_j)|_{t=0}.$$

By the Cambpell-Hausdorff-Backer formula, when $d(j)\geqslant d(i)$, $P_i(x,te_j)=x_i+\delta_{ij}t$ (because if $d(j)\geq d(i)$ the commutators of the form $[X,X_j]$ live in $V_{d(j)+1}$), thus $c_{ji}(x)=\delta_{ij}$. If $d(j)\leq d(i)$, c_{ji} is a polynomial of weighted degree $d(i)-d(j)$, indeed t is a variable of degree $d(j)$, $P_i(x,t)$ is homogeneous of weighted degree $d(i)$ and we are differentiating and evaluating it at $t=0$. Thus

$$\widetilde{X}_j(x)=\frac{\partial}{\partial x_j}|_x+\sum_{i|d(i)>d(j)}c_{ji}(x)\frac{\partial}{\partial x_i}|_x.$$

□

Remark 1.4.7. We highlight a direct consequence of the above result: since $c_{ji}(x)$ is a polynomial of weighted degree $d(i)-d(j)$, the only components of $x=(x_1,\dots,x_n)$ appearing in $c_{ji}(x)$ are those with degree smaller than $d(i)$.

1.5 Ball-Box estimate

In this section we prove the so called Ball-Box theorem in the context of Carnot groups. The theorem will give us an estimate on $d(e,g)$, for $g\in\mathbb{G}$, which will be used in the next chapter. The theorem holds in more general subRiemannian manifolds, for a proof and a precise statement see [Jea14], [NSW85] . The proof presented here is simpler than the one for a general subRiemannian manifold and it is taken from [Pig16].

Proposition 1.5.1. There exists a constant $C>1$ depending only on $\mathbb{G}$ such that for any $g\in\mathbb{G}$, $g=\exp(x_1X_1+\dots x_nX_n)$ we have

$$C^{-1}\max_{i=1,\dots,n}|x_i|^{\frac{1}{d(i)}}\leq d_{CC}(g,e)\leq C\max_{i=1,\dots,n}|x_i|^{\frac{1}{d(i)}}\tag{1}$$

Proof. We have already seen how dilation acts in exponential coordinates, that is for each $i=1,\dots,n$ the i-th component of $\widetilde{\delta}_\lambda(g)$ is $\lambda^{d(i)}x_i$ (where we set $\widetilde{\delta}_\lambda:=F^{-1}\delta_\lambda F$). We define

$$f(g):=\max_{i=1,\dots,n}|x_i|^{\frac{1}{d(i)}}.$$

We observe that this function is well defined because F is injective and surjective. We have $f\delta_\lambda(g) = \lambda f(g)$, indeed the action of f on a point g is defined in terms of the exponential coordinates of g and we have just recalled the action of δ_λ in exponential coordinates. We have seen in proposition 1.3.6 that the distance from e satisifes the same property, that is $d_{CC}(\delta_\lambda(g), e) = d_{CC}(\delta_\lambda(g), \delta_\lambda(e)) = \lambda d_{CC}(g, e)$. Thus, since the thesis is trivial when $g = e$ we are left to show that the thesis holds for some C when $f(g) = 1$. Indeed the general case is obtained just by a dilation. Let $K := f^{-1}(1)$. This set, viewed ix exponential coordinates is compact with respect to the standard topology of $\mathbb{R}^n$. Since the standard topology of $\mathbb{R}^n$ coincides with the topology induced by d_{CC}, K is compact with respect to the topology induced by d_{CC}. This implies that $d(\cdot, e)$ has a maximum and a minimum on K, which is the thesis. $\square$

Remark 1.5.2. The above estimate is called Ball-Box esitmate because from (1) it follows that

$$\mathrm{Box}_{\frac{\lambda}{C}} \subset F^{-1}(B_{CC}(0, \lambda)) \subset \mathrm{Box}_{C\lambda},$$

where

$$\mathrm{Box}_\mu := \left\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \max_{i=1,\dots,n} |x_i|^{\frac{1}{d(i)}} < \mu\right\}$$

and $B_{CC}(0, \lambda)$ is the Carnot-Carathéodory ball centered at the origin.

1.6 The Endpoint map: normal and abnormal extremals

In this section we introduce the Endpoint map (and its extended and modified versions) and show how first order necessary conditions are deduced for length minimizers. The property of not being locally open will be crucial to obtain these necessary conditions. We will also show how this property leaves the door open to the existence of the so called abnormal length minimizers. Even if we won't discuss them, we want to observe that also the Goh conditions (which are second order conditions for strictly abnormal geodesics) are related to this property. Some well known facts about the Endpoint map will be presented without proof, which can be found in the literature, see for instance [ABB15].

Let $g_0 \in \mathbb{G}$ be fixed. We call $\mathscr{U}_{g_0}$ the set of controls h such that the Cauchy problem

$$\begin{cases} \dot\gamma(t) = \sum_{i=1}^r h_i(t) X_i(\gamma(t)) \\ \gamma(0) = g_0. \end{cases} \tag{2}$$

has a unique solution in $\mathrm{Lip}([0,1], \mathbb{G})$ solving the equation a.e.. The choice of $[0,1]$ simplifies the notation but we could have carried out the same discussion in for $[0, T]$ (in that case $\mathscr{U}_{g_0}$ depends also on T, and the same is true for the Endpoint map).

Definition 1.6.1. Let $g_0 \in \mathbb{G}$ be fixed. The Endpoint map

$$\mathrm{End}_1 : \mathscr{U}_{g_0} \subset L^\infty([0,1], \mathbb{R}^r) \to \mathbb{G}$$

is defined by $\mathrm{End}_1(h) = \gamma(1)$ where γ is the solution of the above Cauchy problem with control h. We will use the notation End to denote End_1. We define also the extended Endpoint map $\mathrm{extEnd}(h) := (E(h), \mathrm{End}(h))$, where $E(h) = \frac{1}{2}||h||_2^2$

At this point we can make the key observation. Let γ be a constant speed length minimizing curve, joining g_0 and $\gamma(1)$ with associated control h. Then extEnd cannot be locally open at h. Indeed, if we assume extEnd to be locally open at h then we can find a control v (as near as we want to h, by local openness), such that $\mathrm{extEnd}(v) = \mathrm{extEnd}(h)$ and $E(v) < E(h)$, but this contradicts the minimality of γ.
For the rest of the paragraph we fix a length minimizer $\bar{\gamma}$ with associated control $\bar{h} \in \mathscr{U}_{g_0}$.

Definition 1.6.2. We define the flow associate to $\bar{h}$ at time $0 \leq t \leq 1$ as the map $\Phi_t : \mathbb{G} \to \mathbb{G}$ defined by the following cauchy problem

$$\begin{cases} \dot{\Phi}_t(g) = \sum_{i=1}^r \bar{h}_i(t) X_i(\Phi_t(g)) \\ \Phi_0(g) = g \end{cases} \tag{3}$$

We observe that $\Phi(g_0) = \bar{\gamma}(t)$, since in this case the two Cauchy problems (2) and (3) coincide. We define the modified Endpoint map as

$$\widehat{\mathrm{End}}(h) := \Phi_1^{-1} \circ \mathrm{End}(h).$$

This map takes a control h and gives a point on $\mathbb{G}$. This point is obtained in this way: first we follow the curve γ associated to the control h and with initial point g_0 until we reach $\gamma(1)$; then we follow the curve associated with control $\bar{h}$ and starting point $\gamma(1)$ until we reach its final point, which is the output of $\widehat{\mathrm{End}}(h)$. We observe also that $\bar{h}$ is implicit in the definition of $\widehat{\mathrm{End}}$ and that $\widehat{\mathrm{End}}(\bar{h}) = g_0$. Moreover if we define the modified extended Endpoint map $\widehat{\mathrm{extEnd}} := \left(\widehat{\mathrm{End}}, E\right)$ it is clear that also this map it is not locally open at length minimizers. This fact together with a consequence of the open map theorem implies that the differential of $\widehat{\mathrm{extEnd}}$ cannot be surjective computed on controls associated to length minimizers, in particular when computed in $\bar{h}$.

Lemma 1.6.3. The differential of $\widehat{\mathrm{End}}$ at $\bar{h}$ is given by

$$d(\widehat{\mathrm{End}})_{\bar{h}}(v) = \int_0^1 (d(\Phi_t)_{g_0})^{-1} \left[\sum_{i=1}^r v_i(t) X_i(\bar{\gamma}(t)) \right] dt.$$

Using this lemma and the fact that

$$dE_{\bar{h}}[v] = \int_0^1 \sum_{i=1}^r \bar{h}_i(t) v_i(t) dt$$

we get an expression for the differential of $\widehat{\mathrm{extEnd}}$ at $\bar{h}$.

Theorem 1.6.4. Let $\bar{\gamma}$ and $\bar{h}$ be as above. There is a nonzero covector $(\bar{\xi}, \bar{\nu}) \in T_{g_0}\mathbb{G} \times \{0,1\}$ such that

$$\langle \bar{\xi}, ((d\Phi_t)_{g_0})^{-1}\left[X_i^L(\bar{\gamma}(t))\right]\rangle + \bar{\nu}\bar{h}_i(t) = 0,$$

a.e. and for every $i = 1, \dots, r$.

Proof. Since $\widehat{\text{extEnd}}_{\bar{h}}$ non surjective we can find a nonzero covector $(\bar{\xi}, \bar{\nu}) \in T^*_{g_0}G \times \mathbb{R}$ which is zero on the image of $\widehat{\text{extEnd}}_{\bar{h}}$. Explicitly, for every $v \in L^\infty([0,1])$ (controllare),

$$0 = \langle (\bar{\xi}, \nu), d\,\widehat{\text{extEnd}}_{\bar{h}}[v]\rangle = \int_0^1 \langle \bar{\xi}, ((d\Phi_t)_{g_0})^{-1}\left[\sum_{i=1}^r v_i(t) X_i^L(\bar{\gamma}(t))\right]\rangle dt + \bar{\nu}\int_0^1 \sum_{i=1}^r \bar{h}_i(t) v_i(t) dt.$$

Since v is arbitrary we conclude by the fundamental lemma of calculus of variations that

$$\langle \bar{\xi}, ((d\Phi_t)_{g_0})^{-1}\left[X_i^L(\bar{\gamma}(t))\right]\rangle + \bar{\nu}\bar{h}_i(t) = 0,$$

a.e. and for every $i = 1, \dots, r$. Up to rescaling $(\bar{\xi}, \bar{\nu})$ we can assume $\bar{\nu}$ to be 0 or 1. $\square$

We can rewrite the first order conditions in another form, by rewriting the term $\langle \bar{\xi}, ((d\Phi_t)_{g_0})^{-1}\left[X_i^L(\bar{\gamma}(t))\right]\rangle$. We define a curve associated to the covector $\bar{\xi}$ pulling it back with Φ_t^{-1}, that is

$$\xi(t) := \left(\Phi_t^{-1}\right)^* \bar{\xi} \in T^*_{\bar{\gamma}(t)}\mathbb{G}.$$

This curve is called dual curve associated to $\bar{\xi}$. Now we observe that

$$\langle \bar{\xi}(t), X_i(\bar{\gamma}(t))\rangle = \langle \bar{\xi}, ((d\Phi_t)_{g_0})^{-1}\left[X_i^L(\bar{\gamma}(t))\right]\rangle,$$

by definition of pullback of a covector (indeed $((d\Phi_t)_{g_0})^{-1}$ is the pushforward through Φ_t^{-1}). Thus we can write the necessary conditions as

$$\langle \bar{\xi}(t), X_i^L(\bar{\gamma}(t))\rangle + \bar{\nu}\bar{h}_i(t) = 0.$$

Now we will declare when an admissible curve (not necessarly a length minimizer) with control $h \in \mathscr{U}_{g_0}$ is a normal or abnormal extremal.

Definition 1.6.5. Let γ be an admissible curve with control h. We say that γ is an extremal, if γ is a critical point for $\widehat{\text{extEnd}}$. We say that γ is a normal extremal if there exists a covector $\bar{\xi} \in T^*_{g_0}\mathbb{G}$ such that $(\bar{\xi}, 1)$ vanishes on $\text{im}(d\,\widehat{\text{extEnd}}_h)$. We say that it is an abnormal extremal if there exists a nonzero covector $\bar{\xi} \in T^*_{g_0}\mathbb{G}$, such that $(\bar{\xi}, 0)$ vanishes on $\text{im}(d\,\widehat{\text{extEnd}}_h)$.

From the definition it clear that a curve is an extremal if and only if it is a normal or abnormal extremal. Anyway the the two cases are not mutually exclusive, indeed there are (nontrivial) examples of curves being both normal and abnormal extremals with respect to different covectors $\bar{\xi}$. This motivates the following

Definition 1.6.6. An extremal γ is said to be strictly abnormal if it is not normal.

Remark 1.6.7. In the Riemannian case abnormal extremals do not exist. Indeed if we assume $\bar{\nu} = 0$, the necessary condition takes the form $\langle \xi(t), X_i^L(\gamma(t)) \rangle = 0$, for every i and also for every t (being everything at least continuous). Since in this case the distribution is the whole tangent bundle this implies $\xi(t) = 0$ for every t, and in particular $\bar{\xi} = 0$, which is a contradiction.

It is known that in the Riemannian case geodesics are smooth. It is natural at this point to ask what regularity do extremals in this framework have. Just to illustrate the idea used to prove that normal extremals are $\mathscr{C}^\infty$ regular we assume that our manifold is $\mathbb{R}^n$. The proof is based on the fact that if γ is a normal extremal with associated dual curve ξ, it is possible to prove that the pair (γ, ξ) solves the hamiltonian system

$$\begin{cases} \dot{\gamma} = H_\xi(\gamma, \xi) \\ \dot{\xi} = -H_\gamma(\gamma, \xi), \end{cases}$$

where $H(y, \xi) = \frac{1}{2} \sum_{i=1}^r \langle X_i(x), \xi \rangle^2$ for $y, \xi \in \mathbb{R}^n$. Using this reformulation we obtain regularity by a bootstrap procedure. Since both γ and ξ are continuous, $\dot{\gamma}$, $\dot{\xi}$ are sums of products of continuous function, thus they are continuous, this means that γ and ξ are $\mathscr{C}^1$. This means that $\dot{\gamma}$, $\dot{\xi}$ are sums of products of $\mathscr{C}^1$ functions, thus they are $\mathscr{C}^1$, and so on. This approach does not work for strictly abnormal extremals since the necessary condition does not allow us to use a bootstrap procedure. Actually, the problem of regularity for strictly abnormal exremals is still open, and in general we know only that strictly abnormal extremals are Lipschitz regular (but since we started with Lipschitz regular curves this is not a great achievement). Only partial results are know in the framework of Carnot groups.

1.7 Horizontal curves

In this section we want to show that there is a correspondence between horizontal curves in $\mathbb{G}$ starting at e and curves in V_1 starting at 0.

Definition 1.7.1. We call $\bar{\pi}_j : \mathfrak{g} \to V_j$ the canonical projection, $j = 1, \ldots, l$. We will write $\bar{\pi}$ instead of $\bar{\pi}_1$. We set also $W_j := V_j \oplus \cdots \oplus V_l$

Lemma 1.7.2. $\pi : \mathbb{G} \to (V_1, +)$ given by $\pi := \bar{\pi} \circ \exp^{-1}$ is a group homomorphism.

Proof. π is smooth and by definition does not depend on the choice of the basis on $\mathfrak{g}$ but only on the stratification. To show that it is a group homomorphism we take $g = \exp(x_1 X_1 + \cdots + x_n X_n)$ and $h = \exp(y_1 X_1 + \cdots + y_n X_n)$ and compute $\pi(gh)$. By the B-H-C formula we have

$$\begin{aligned} gh &= \exp(x_1 X_1 + \cdots + x_n X_n) \exp(y_1 X_1 + \cdots + y_n X_n) = \exp\left(\sum_{i=1}^n P_i(x, y) X_i \right) \\ &= \exp\left((x_1 + y_1) X_1 + \cdots + (x_r + y_r) X_r + \sum_{i=r+i}^n P_i(x, y) X_i \right). \end{aligned}$$

Thus $\pi(gh) = (x_1 + y_1)X_1 + \cdots + (x_r + y_r)X_r = \pi(g) + \pi(h)$ because the other terms live in W_2. $\square$

If $g = \exp(x_1X_1 + \cdots + x_nX_n)$ then $\pi(g) = x_1X_1 + \cdots + x_rX_r$. To prove a similar statement for the other projections, we introduce $\mathbb{G}_i := \exp(W_i)$.

Lemma 1.7.3. For $1 \leq i \leq l$, $\mathbb{G}_i$ is a closed subgroup of G and $\pi_i : \mathbb{G}_i \to (V_i, +)$ defined by $\bar{\pi} \circ \exp^{-1}$ is a group homorphism

Proof. To show that $\mathbb{G}_i$ is a subgroup we take $g = \exp(x_{n_i}X_{n_i} + \cdots + x_nX_n)$,
$h = \exp(y_{n_1}X_{n_i} + \cdots + y_nX_n) \in \mathbb{G}_i$, where $n_i := \dim(V_1) + \cdots + \dim(V_i)$. By the B-H-C formula we have

$$gh = \exp\left(P\left(\sum_{j=n_i+1}^{n} x_jX_j, \sum_{j=n_i+1}^{n} y_jX_j\right)\right),$$

and since P involves only $\sum_{j=n_i+1}^{n} x_jX_j$, $\sum_{j=n_i+1}^{n} y_jX_j$ and their commutators it lives in W_i, thus $gh \in \mathbb{G}_i$. $\mathbb{G}_i$ is closed since it is the image under a diffeomorphism of a closed subspace. π_i does not depend on the choice of the basis and it can be shown that it is an homeomorphism as in the previous proof. $\square$

Lemma 1.7.4. For any $g \in \mathbb{G}$, $X \in V_1$ we have $d\pi_g(X^L(g)) = X$, where we have identified V_1 and $T_{\pi(g)}V_1$.

Proof. By definition $\exp(tX)$ is the integral curve of X^L through e, thus for each g, $L_g \exp(tX)$ is the integral curve of X^L through g. This means

$$X^L(g) = \frac{d}{dt}\left(L_g \exp(tX)\right)|_{t=0}.$$

Since π is a homorphism and $X \in V_1$

$$\begin{aligned} d\pi_g(X^L(g)) &= d\pi_g\left[\frac{d}{dt}\left(L_g \exp(tX)\right)|_{t=0}\right] = \frac{d}{dt}\left(\pi(L_g \exp(tX))\right)|_{t=0} \\ &= \frac{d}{dt}\left(\pi(g) + \pi(\exp(tX))\right)|_{t=0} = \frac{d}{dt}(tX)|_{t=0} \\ &= X \end{aligned}$$

$\square$

To prove our correspondece we need the following result, which is true on Carnot groups

Proposition 1.7.5. Every control in $L^\infty([0,T], \mathbb{G})$ is admissible.

Theorem 1.7.6. Let $\gamma \in \mathrm{Lip}([0,T], \mathbb{G})$ horizontal such that a.e.

$$\dot{\gamma}(t) = \sum_{i=1}^{r} h_i(t)X_i^L(\gamma(t)),$$

then it holds a.e.

$$\frac{d}{dt}(\pi \circ \gamma)(t) = \sum_{i=1}^{r} h_i(t) X_i.$$

Moreover given $\hat{\gamma} : [0, T] \to V_1$ lipschitz, with $\hat{\gamma}(0) = 0$, there is a unique horizontal curve $\gamma \in \mathrm{Lip}([0, T], \mathbb{G})$ such that $\hat{\gamma} = \pi \circ \gamma$ and $\gamma(0) = e$.

Proof. Using the lemma just proved we can compute

$$\frac{d}{dt}(\pi \circ \gamma)(t) = d\pi_{\gamma(t)}[\dot{\gamma}(t)] = \sum_{i=1}^{r} h_i(t) d\pi_{\gamma(t)}[X_i^L(\gamma(t))] = \sum_{i=1}^{r} h_i(t) X_i.$$

For the second part of the statement we write $\hat{\gamma}(t) = \sum_{i=1}^{r} \hat{h}(t) X_i$ for some control $\hat{h} \in L^\infty([0, T], \mathbb{R}^r)$, which exists by the assumption on $\hat{\gamma}$. By the above proprositon the control $\hat{h}$ is admissible, thus there is an horizontal curve γ such that a.e.

$$\dot{\gamma}(t) = \sum_{i=1} \hat{h}(t) X_i^L(\gamma(t)).$$

By the first part of the statement $\frac{d}{dt}(\pi \circ \gamma)(t) = \hat{\gamma}(t)$ a.e.. Since $\hat{\gamma}(0) = 0 = \pi(e) = \pi \circ \gamma(0)$ we have $\hat{\gamma} = \pi \circ \gamma$. For uniqueness, it's enough to observe that if γ_1 is such that $\gamma_1(0) = e$ and has the same control of γ then it satisfies the same Cauchy problem as γ. $\square$

Remark 1.7.7. By left translation the same statement can be proved for any initial point $g \in \mathbb{G}$. (In this case $\hat{\gamma}(0)$ will be $\pi(g)$).

1.8 Cut and Cor: how to modify curves in a Carnot group

Now we describe an inductive technique that will be used in the following chapter to prove the non minimality of the double logarithm spiral. This technique has been introduced in [LM08] to prove that for special classes of subRiemannian manifolds minimal curves don't have corner-like singularities. The idea in that paper was to show that if a horizontal curve has a corner like singularity, then we can find a shorter (admissible) competitor modifying the original curve in an appropriate way. We will follow the description of [MPV18]. We introduce some notation.
For any $Y \in \mathfrak{g}$ we take an unit speed geodesic $\delta_Y : [0, \ell_Y] \to \mathbb{G}$, joining e and $\exp(Y)$. Being the curve unit speed it holds $\ell_Y = d(e, \exp Y)$. For our purposes any geodesic joining the two points will work, since we are interested only in its length.

Definition 1.8.1. Let $\gamma \in \mathrm{Lip}([a, a + a'], \mathbb{G})$ and $\nu \in \mathrm{Lip}([b, b + b'], \mathbb{G})$ be two curves we define their join by $\gamma * \nu : [a, a + (a' + b')] \to \mathbb{G}$

$$\gamma * \nu := (t) \begin{cases} \gamma(t) & t \in [a, a + a'] \\ \gamma(a + a')\beta(b)^{-1}\beta(t + b - (a + a')) & t \in [a + a', a + (a' + b')] \end{cases}$$

The meaning of the join is clear, we follow first γ and once arrived at the endpoint of γ we follow the curve ν properly traslated. We observe that the join is horizontal if and only if the two curves are horizontal.

Definition 1.8.2. Let $\gamma : [a,b] \to \mathbb{G}$ be an horizontal curve. Let $[s,s'] \subset [a,b]$ be a subinterval. If $\underline{\gamma}(s) \neq \underline{\gamma}(s')$ let $w := \frac{\gamma(s')-\gamma(s)}{|\underline{\gamma}(s')-\underline{\gamma}(s)|}$, we define the cutted curve $\mathrm{Cut}(\gamma,[s,s']) : [a, b-(s-s')+|\underline{\gamma}(s')-\underline{\gamma}(s)|] \to \mathbb{G}$

$$\mathrm{Cut}(\gamma,[s,s'])(t) := \gamma|_{[a,s]} * \exp(\cdot w)|_{[0,|\underline{\gamma}(s')-\underline{\gamma}(s)|]} * \gamma_{[s',b]}.$$

If $\underline{\gamma}(s') = \underline{\gamma}(s)$ the cutted curve is defined as γ.

We observe that the cutted curve is horizontal, being defined as join of horizontals. Moreover the length of the cutted curve is smaller than the initial curve. To see the geometric interpretation of the Cut we have to focus on the first layer. We consider $\pi(\mathrm{Cut}(\gamma,[s,s'])) = \underline{\gamma}|_{[a,s]} * (\cdot t)|_{[0,|\underline{\gamma}(s')-\underline{\gamma}(s)|]} * \underline{\gamma}|_{[s',b]}$, which is the curve in V_1 obtained following $\underline{\gamma}$ from $\underline{\gamma}(a)$ to $\underline{\gamma}(s)$, then a line segment from $\underline{\gamma}(s)$ to $\underline{\gamma}(s')$ and then $\underline{\gamma}$ from $\underline{\gamma}(s')$ to $\underline{\gamma}(b)$.

Remark 1.8.3. $\pi(\mathrm{Cut}(\gamma,[s,s']))$ has the same final point of $\underline{\gamma}$. Indeed

$$\mathrm{Cut}(\gamma,[s,s'])\left(b-(s-s')+|\underline{\gamma}(s')-\underline{\gamma}(s)|\right) = \gamma(s)\exp(|\underline{\gamma}(s')-\underline{\gamma}(s)|w)\gamma(s')^{-1}\gamma(b).$$

Thus, using the fact that π is a homomorphism,

$$\begin{aligned}\pi(\mathrm{Cut}(\gamma,[s,s'])\left(b-(s-s')+|\underline{\gamma}(s')-\underline{\gamma}(s)|\right)) &= \underline{\gamma}(s)+\underline{\gamma}(s')-\underline{\gamma}(s)+\underline{\gamma}(s')+\underline{\gamma}(b)\\ &= \underline{\gamma}(b).\end{aligned}$$

Nevertheless, in general, the final point of $\mathrm{Cut}(\gamma,[s,s'])$ in $\mathbb{G}$ is different from $\gamma(b)$. Thus if we want to construct a shorter competitor joining $\gamma(a)$ and $\gamma(b)$ we can start cutting the curve but we need a way to fix the final point.

Definition 1.8.4. Let $\gamma : [a,b] \to \mathbb{G}$ be a horizontal curve. For any subinterval $[s,s'] \subset [a,b]$ and $Y \in \mathfrak{g}$, we define the corrected curve $\mathrm{Cor}(\gamma,[s,s'],Y) : [a, b+2\ell_Y]$ by

$$\mathrm{Cor}(\gamma,[s,s'],Y)(t) := \gamma|_{[a,s]} * \delta_Y * \gamma|_{[s,s']} * \delta_Y(\ell_Y - \cdot) * \gamma|_{[s',b]}.$$

The process of transforming γ into $\mathrm{Cor}(\gamma,[s,s'],Y)$ is called application of the correction device associated with $[s,s']$ and Y.

The process of correcting the curve makes it longer, indeed $L(\mathrm{Cor}(\gamma,[s,s'],Y)) = L(\gamma)+2\ell_Y$. It is convenient to define the displacement of the final point

$$\mathrm{Dis}(\gamma,[s,s'],Y) := \gamma^{-1}(b)\mathrm{Cor}(\gamma,[s,s'],Y)(b+2\ell_y)$$

to keep track of the difference between the final point of γ and that of the corrected curve. The displacement is the error (at the level of the Lie group) that we make when we correct a curve. We will show how this error behaves under the projections π_j. We need sime preliminary result, in which the notation and assumptions on γ, $[s, s']$ and Y will be the same as above. We denote by $C_g(h) = ghg^{-1}$ the conjugation and by $[g, h] = ghg^{-1}h^{-1}$ the commutator in $\mathbb{G}$. It will be clear from the context if $[\cdot, \cdot]$ is used to denote the Lie bracket or the commutator. Let $Ad(g) = d(C_g)_e$. It is known that Ad is an automorphism of $\mathfrak{g}$ and that the formulas $Ad(\exp(X)) = e^{ad(X)}$, $C_g(\exp(Y)) = \exp(Ad(g)Y)$ hold, for $g \in \mathbb{G}$, $X,\ Y \in \mathfrak{g}$. We introduce also the notation $\gamma|_a^b := \gamma(a)^{-1}\gamma(b)$ and call $b_1 = b + 2\ell_Y$

Lemma 1.8.5. We have the following formula

$$\mathrm{Dis}(\gamma, [s, s'], Y) = C_{\gamma|_b^s}\left(\left[\exp(Y), \gamma|_s^{s'}\right]\right),$$

Proof. Writing explicitly $\mathrm{Cor}(\gamma, [s, s'], Y)$ we have

$$\begin{aligned}\gamma(b)^{-1}\mathrm{Cor}(\gamma, [s, s'], Y)(b_1) &= \gamma(b)^{-1}\gamma(s)\exp(Y)\gamma(s)^{-1}\gamma(s')\exp(-Y)\gamma(s')^{-1}\gamma(b)\\ &= \gamma|_b^s \exp(Y)\gamma|_s^{s'}\exp(-Y)\gamma|_{s'}^b\end{aligned}$$

The right hand side gives

$$\begin{aligned}C_{\gamma|_b^s}\left(\left[\exp(Y), \gamma|_s^{s'}\right]\right) &= \gamma|_b^s\left[\exp(Y), \gamma|_s^{s'}\right]\gamma|_s^b = \gamma|_b^s\exp(Y)\gamma|_s^{s'}\exp(-Y)\gamma|_{s'}^s\gamma|_s^b\\ &= \gamma|_b^s\exp(Y)\gamma|_s^{s'}\exp(-Y)\gamma|_{s'}^b.\end{aligned}$$

□

Lemma 1.8.6. For any $g \in \mathbb{G}$, $h \in \mathbb{G}_j$ $C_g(h) \in \mathbb{G}_j$, i.e. G_j is a normal subgroup, and $\pi_j(C_g(h)) = \pi_j(h)$

Proof. We write $g = \exp(X)$, $h = \exp(Y)$. Using the properties of the exponential map

$$\exp^{-1}(C_g(h)) = \exp^{-1}\exp(\mathrm{Ad}_g(h)) = e^{adX}Y = \sum_{k=0}^{\infty}\frac{(\mathrm{ad}X)^k}{k!}Y = Y + \sum_{k=1}^{\infty}\frac{(\mathrm{ad}X)^k}{k!}Y,$$

which is an element of W_j since $Y \in W_j$ and $\sum_{k=1}^{\infty}\frac{(\mathrm{ad}X)^k}{k!}Y \in W_{j+1}$. Thus $C_g(h) \in \mathbb{G}_j$ and

$$\pi_j(C_g(h)) = \bar{\pi}_j\exp^{-1}(C_g(h)) = \bar{\pi}_j\left(Y + \sum_{k=1}^{\infty}\frac{(\mathrm{ad}X)^k}{k!}Y\right) = \bar{\pi}_j(Y) = \pi_j(h).$$

□

Lemma 1.8.7. For any $g \in \mathbb{G}$, $h \in \mathbb{G}_j$ with $1 \le j < l$ we have

$$[g, h] \in \mathbb{G}_{j+1} \quad \text{and} \quad \pi_{j+1}([g, h]) = [\pi(g), \pi_j(h)].$$

Similarly, if $g \in \mathbb{G}_j$ and $h \in \mathbb{G}$, then

$$[g, h] \in \mathbb{G}_{j+1} \quad \text{and} \quad \pi_{j+1}([g, h]) = [\pi_j(g), \pi(h)].$$

Proof. We prove the statement in the case $g \in \mathbb{G}$, $h \in \mathbb{G}_j$. $[g,h] = ghg^{-1}h^{-1} = C_g(h)h^{-1} \in \mathbb{G}_j$ since $C_g(h) \in \mathbb{G}_j$ by the previous lemma and by lemma 1.7.3 $\mathbb{G}$ is a subgroup. Moreover, by the previous lemma and the fact that $\pi_j : \mathbb{G}_j \to W_j$ is a homomorphism

$$\pi_j([g,h]) = \pi_j(C_g(h)) + \pi_j(h^{-1}) = \pi_j(h) - \pi_j(h) = 0,$$

which means that $\exp^{-1}([g,h])$ has no component in V_j, thus it lies in W_{j+1}. We write $g = \exp(X)$, $h = \exp(Y)$. We want $\exp^{-1}(C_g(h)h^{-1})$. As in the proof of the previous lemma we have

$$\exp^{-1}(C_g(h)) = \sum_{k=0}^{\infty} \frac{(\mathrm{ad}X)^k}{k!} Y = Y + [X,Y] + R'$$

where $R' = \sum_{k=2}^{\infty} \frac{(\mathrm{ad}X)^k}{k!} Y \in W_{j+2}$. Being $h^{-1} = \exp(-Y)$ the B-H-C formula gives

$$\exp^{-1}(C_g(h)h^{-1}) = P(Y+[X,Y]+R', -Y) = Y+[X,Y]+R'-Y+R'' = [X,Y]+R'+R''$$

where R'' is the double sum given by the formula. We show that $R'' \in W_{j+2}$. Expanding each factor of the double sum we obtain that R'' is a linear combination of elements of the form

$$(\mathrm{ad}Z_1)\cdots(\mathrm{ad}Z_k)\, Z_{k+1},$$

where $k \geq 1$ and $Z_i \in \{Y, [X,Y], R'\}$. The elements containing only Y vanish, while the other terms of the sum belong to W_{j+2} because k is at least 1 and $[X,Y]$, $R' \in W_{j+1}$. This means that $R'' \in W_{j+2}$. Thus we have

$$\pi_{j+1}([g,h]) = \bar{\pi}_{j+1} \exp^{-1}([g,h]) = \bar{\pi}_{j+1}([X,Y] + R' + R'') = \bar{\pi}_{j+1}([X,Y]).$$

To compute the $j+1-$component of $[X,Y]$ we write $X = \pi(X)+R_X$ and $Y = \bar{\pi}_j(Y)+R_Y$ where $R_X \in W_2$ and $R_Y \in W_{j+1}$. By linearity of the bracket

$$[X,Y] = [\pi(X), \bar{\pi}_j(Y)] + [\pi(X), R_Y] + [R_X, \bar{\pi}_j] + [R_X, R_Y],$$

and being the last three terms of the equality in W_{j+2} we have $\bar{\pi}_{j+1}([X,Y]) = [\pi(X), \bar{\pi}_j(Y)] = [\pi(g), \pi_j(h)]$. $\square$

Finally, we prove

Lemma 1.8.8. If $Y \in V_j$, then $\mathrm{Dis}(\gamma, [s,s'], Y) \in \mathbb{G}_{j+1}$ and

$$\pi_{j+1}(\mathrm{Dis}(\gamma, [s,s'], Y) = [Y, \underline{\gamma}(s') - \underline{\gamma}(s)].$$

Proof. From lemma 1.8.5 and lemma 1.8.7 we have

$$\mathrm{Dis}(\gamma, [s,s'], Y) = C_{\gamma|_b^s}\left(\left[\exp(Y), \gamma|_s^{s'}\right]\right) \in \mathbb{G}_{j+1}.$$

Moreover, again by lemma 1.8.7

$$\begin{aligned}\pi_{j+1}\left(\mathrm{Dis}(\gamma,[s,s'],Y)\right)\right) &= \pi_{j+1}\left(C_{\gamma|_b^s}\left(\left[\exp(Y),\gamma|_s^{s'}\right]\right)\right)\\ &= \left[\pi_j\left(\exp(Y)\right),\pi\left(\gamma|_s^{s'}\right)\right]\end{aligned}$$

Since $Y \in V_j$, $\pi_j(Y) = Y$. Being π and homomorphism we have

$$\pi(\gamma|_s^{s'}) = \pi(\gamma(s)^{-1}\pi(\gamma(s')) = \underline{\gamma}(s') - \underline{\gamma}(s),$$

which concludes the proof. □

We introduce a compact notation to deal with iterated applications of the correction devices (for our purposes two consecutive iterations will be enough since in the following chapter we will work in a Carnot group of rank 2. The definition can be adapted to more consecutive iterations).

Definition 1.8.9. Given $\gamma : [a,b] \to \mathbb{G}$ a horizontal curve, two subintervals $[s,s']$, $[t,t']$ of $[a,b]$ with $s' \le t$ and Y, $Y' \in \mathfrak{g}$ we set

$$\mathrm{Cor}(\gamma,[s,s'],Y,[t,t'],Y') := \mathrm{Cor}(\mathrm{Cor}(\gamma,[s,s'],Y),[t+2\ell_Y,t'+2\ell_Y],Y').$$

2 Non-minimality of the double logarithm spiral

In this chapter we want to study the non minimality of a curve in a Carnot group G of rank 2 and step 3. We consider the horizontal curve $\gamma : [0,1) \longrightarrow G$, s.t.

$$\underline{\gamma}(t) := (\pi \circ \gamma)(t) = t\cos(\phi(t))X_1 + t\sin(\phi(t))X_2,$$

where $\underline{\gamma}(0) := 0$ and $\phi : (0,1) \to (0,\infty)$

$$\phi(t) := \log(-\log(t)).$$

We consider a basis adapted to the stratification $\{X_1, \dots, X_n\}$. As metric on $\mathfrak{g}$ we will take g that makes $X_1, \dots, X_n$ orthonormal.
Using the technique introduced in the previous chapter, the idea is to look for $a, b \in (0,1)$ such that we can cut the curve in the interval $[a,b]$ in a way that will allow us to restore the final point with a gain of length. We will show that for $a, b < T < 1$ the curve γ restricted to $[0,T]$ is not a length minimizer. The correction of the final point $\gamma(T)$ will be carried on manually in two steps. In the first we will fix the component of the error on the second layer. This will be done applying a correction device to the cutted curve which will create a new curve, longer than the cutted one. In the second step we will fix the component of the error on the third layer applying a correction device to the curve obtained in the second step. This operation will produce a new curve with the same endpoint of γ. In both steps we have to check that for b small the amount of length added (aaplying Cor) is smaller than the length obtained cutting the curve.
We set $\gamma^{(1)}(t) :=$Cut$(\gamma, [a,b])(t)$. $\gamma^{(1)}$ is horizontal and defined on $[0, T_1]$, where $T_1 = T - (b-a) + |\underline{\gamma}(b) - \underline{\gamma}(a)|$. The aim of the chapter is to prove the following

Theorem 2.0.1. Let G be a Carnot group of step 3 and rank 2 and $\gamma : [0,1[\to \mathbb{G}$ a horizontal curve with $\gamma(0) = e$,

$$\pi(\gamma(t)) = t\cos(\phi(t))X_1 + t\sin(\phi(t))X_2,$$

where $\phi : (0,1) \to (0,\infty)$

$$\phi(t) := \log(-\log(t)).$$

Then, for $T < 1$, $\gamma|_{[0,T]}$ is not a length minimizer between $\gamma(0)$ and $\gamma(T)$.

The proof will be completed by remark 2.4.5 but the constructions and estimates of the last two paragraphs are part of the proof as well. For this reason we don't state explicitly where the proof begins This chapter will be divided in paragraphs to make exposition clearer.

2.1 Gain of length obtained by the cut

In this section we want to estimate from below the difference

$$\Delta L(a,b) = L\left(\underline{\gamma}|_{[a,b]}\right) - |\underline{\gamma}(a) - \underline{\gamma}(b)|$$

that is the gain of length produced by the cut. The estimates will be made for $0 < a < b < T$, where $a = b - b^{1+\tau}$, for a suitable $0 < \tau < 1$ which will be fixed at the end of the argument. We will make two kinds of approximation to get the estimates, the first will be for some a fixed and b near a, the second will be for a near 0. The validity of the approximations is justified by $\frac{b-a}{a} = \frac{b^{1+\tau}}{b(1-b^{1+\tau})} \to 0$ for b going to zero. This means that the distance between a and b is smaller than the distance between a and 0 when b is small enough. Thus we will take b sufficiently small such that all the approximations work. Moreover we will use both a and b in our estimates.

Lemma 2.1.1. We have

$$L\left(\underline{\gamma}|_{[a,b]}\right) \geqslant (b-a)\left[1 + \frac{1}{2}\left(\frac{a(\phi(b)-\phi(a))}{b-a} + \phi(b) - ⨍_a^b \phi(t)dt\right)^2 + ⨍_a^b o\left(t^2\dot{\phi}(t)^2\right)dt\right].$$

Proof.

$$\begin{aligned} L\left(\underline{\gamma}|_{[a,b]}\right) &= \int_a^b \sqrt{1 + t^2\dot{\phi}(t)^2}dt = \int_a^b \left(1 + \frac{1}{2}t^2\dot{\phi}(t)^2 + o\left(t^2\dot{\phi}(t)^2\right)\right)dt \\ &= b - a + \frac{1}{2}\int_a^b t^2\dot{\phi}(t)^2 dt + \int_a^b o\left(t^2\dot{\phi}(t)^2\right)dt. \end{aligned}$$

We use Hölder inequality to estimate $⨍_a^b t^2\dot{\phi}(t)^2 dt$

$$\begin{aligned} ⨍_a^b t^2\dot{\phi}(t)^2 dt &\geqslant \left(⨍_a^b t\dot{\phi}(t)dt\right)^2 = \left(⨍_a^b \left[\frac{d}{dt}(t\phi(t)) - \phi(t)\right]dt\right)^2 \\ &\geqslant \frac{1}{(b-a)^2}\left(b\phi(b) - a\phi(a) - \int_a^b \phi(t)dt\right)^2 \\ &= \frac{1}{(b-a)^2}\left((b-a)\phi(b) - a(\phi(b)-\phi(a)) - \int_a^b \phi(t)dt\right)^2 \\ &= \left(\frac{a(\phi(b)-\phi(a))}{b-a} + \phi(b) - ⨍_a^b \phi(t)dt\right)^2. \end{aligned}$$

Thus

$$L\left(\underline{\gamma}|_{[a,b]}\right) \geqslant (b-a)\left[1 + \frac{1}{2}\left(\frac{a(\phi(b)-\phi(a))}{b-a} + \phi(b) - ⨍_a^b \phi(t)dt\right)^2 + ⨍_a^b o\left(t^2\dot{\phi}(t)^2\right)dt\right].$$

□

Lemma 2.1.2. We have

$$|\underline{\gamma}(a)-\underline{\gamma}(b)| = |a-b|\left[1+\frac{1}{2}\left(\frac{ab}{(a-b)^2}(\phi(a)-\phi(b))^2+R_3(a,b)\right)\right],$$

where $R_3(a,b) = o\left(\frac{a^2}{(a-b)^2}(\phi(a)-\phi(b))^2\right)$, $R_4(a,b)$ is negligible with respect to $\left(\frac{ab}{(a-b)^2}(\phi(a)-\phi(b))^2+R_3(a,b)\right)$ for a small enough

Proof.

$$\begin{aligned} |\underline{\gamma}(a)-\underline{\gamma}(b)| &= \Big|\,[a\cos(\phi(a))-b\cos(\phi(b))]\,X_1+[a\sin(\phi(a))-b\sin(\phi(b))]\,X_2\Big| \\ &= \Big|\,[a\cos(\phi(a)-\phi(b))-b]\,X_1+a\sin(\phi(a)-\phi(b))X_2\Big| \end{aligned}$$

where in the last equality we applied a rotation of angle $\phi(b)$. We recall

$$\cos(\phi(a)-\phi(b)) = 1-\frac{(\phi(a)-\phi(b))^2}{2}+R_1(a,b),$$

$$\sin(\phi(a)-\phi(b)) = \phi(a)-\phi(b)+R_2(a,b),$$

where $R_1(a,b)=o((\phi(a)-\phi(b))^3)$ and $R_2(a,b)=o((\phi(a)-\phi(b))^2)$ for $b\to a$. Substituting in the above expression and remembering that $X_1,\ X_2$ are orthogonal with respect to the metric

$$\begin{aligned} &\left|\left(a+a\left(-\frac{(\phi(a)-\phi(b))^2}{2}+R_1(a,b)\right)-b\right)X_1+a\left(\phi(a)-\phi(b)+R_2(a,b)\right)X_2\right| \\ &= |a-b|\left|\left(1+\frac{a}{a-b}\left(-\frac{(\phi(a)-\phi(b))^2}{2}+R_1(a,b)\right)\right)X_1+\frac{a}{a-b}\left(\phi(a)-\phi(b)+R_2(a,b)\right)X_2\right| \\ &= |a-b|\sqrt{\left(1+\frac{a}{a-b}\left(-\frac{(\phi(a)-\phi(b))^2}{2}+R_1(a,b)\right)\right)^2+\left(\frac{a}{a-b}\left(\phi(a)-\phi(b)+R_2(a,b)\right)\right)^2} \\ &= |a-b|\sqrt{1-\frac{a}{(a-b)}(\phi(a)-\phi(b))^2+\frac{a^2}{(a-b)^2}(\phi(a)-\phi(b))^2+R_3(a,b)} \\ &= |a-b|\sqrt{1+\frac{ab}{(a-b)^2}(\phi(a)-\phi(b))^2+R_3(a,b)}. \end{aligned}$$

Where

$$\begin{aligned} R_3(a,b) := \frac{a^2\left(\phi(a)-\phi(b)\right)^4}{4(b-a)^2}+a^2\frac{R_1(a,b)^2}{(b-a)^2}+2\frac{a}{b-a}R_1(a,b) \\ -\frac{a^2\left(\phi(a)-\phi(b)\right)^2R_1(a,b)}{(b-a)^2}+2\frac{a^2R_2(a,b)^2}{(b-a)^2} \end{aligned}$$

and clearly $R_3(a,b) = o\left((\phi(a)-\phi(b))^2\right)$. We use the first order expansion

$$\sqrt{1+\frac{ab}{(a-b)^2}(\phi(a)-\phi(b))^2+R_3(a,b)} = 1+\frac{1}{2}\left(\frac{ab}{(a-b)^2}(\phi(a)-\phi(b))^2+R_3(a,b)\right)+R_4(a,b),$$

where $R_4(a,b)$ is negligible with respect to $\frac{1}{2}\left(\frac{ab}{(a-b)^2}(\phi(a)-\phi(b))^2+R_3(a,b)\right)$ if a is small enough. $\square$

Putting together these two facts we get

Proposition 2.1.3.
$$\Delta L(a,b) \geqslant \frac{(b-a)}{2}\left[\frac{a}{b-a}(\phi(b)-\phi(a))(\ddot{\phi}(a)(b-a)^2)\right) + R_5(a,b)\right],$$

where $R_5(a,b) \in o\left(\frac{a}{b-a}(\phi(b)-\phi(a))(\ddot{\phi}(a)(b-a)^2)\right)$ for $b \to a$.

Proof.
$$\begin{aligned}
\Delta L(a,b) \geqslant \frac{(b-a)}{2}\Bigg[&\left(\frac{a(\phi(b)-\phi(a))}{b-a}+\phi(b)-\fint_a^b \phi(t)dt\right)^2 \\
&+\fint_a^b o\left(t^2\dot{\phi}(t)^2\right)dt - \frac{ab}{(b-a)^2}(\phi(a)-\phi(b))^2 - R_3(a,b)\Bigg] \\
= \frac{(b-a)}{2}\Bigg[&\frac{a}{(a-b)}(\phi(b)-\phi(a))^2+\left(\phi(b)-\fint_a^b \phi(t)dt\right)^2 \\
&+2\frac{a}{b-a}(\phi(b)-\phi(a))\left(\phi(t)-\fint_a^b \phi(t)dt\right)+\fint_a^b o\left(t^2\dot{\phi}(t)^2\right)dt - R_3(a,b)\Bigg] \\
= \frac{(b-a)}{2}\Bigg[&\frac{a}{b-a}(\phi(b)-\phi(a))\left(\phi(a)-\phi(b)+2\phi(b)-2\fint_a^b \phi(t)dt\right) \\
&+\left(\phi(b)-\fint_a^b \phi(t)dt\right)^2+\fint_a^b o\left(t^2\dot{\phi}(t)^2\right)dt - R_3(a,b)\Bigg]
\end{aligned}$$

We substitute the expansion

$$\int_a^b \phi(t)dt = \phi(a)(b-a)+\frac{1}{2}\dot{\phi}(a)(b-a)^2+\frac{1}{6}\ddot{\phi}(a)(b-a)^3+o((b-a)^3)$$

inside

$$\left(\phi(a)-\phi(b)+2\phi(b)-2\fint_a^b \phi(t)dt\right),$$

which yields

$$\phi(b)-\phi(a)-\dot{\phi}(a)(b-a)-\frac{1}{3}\ddot{\phi}(a)(b-a)^2-o((b-a)^2).$$

We expand also

$$\phi(b)-\phi(a)=\dot\phi(a)(b-a)+\frac{1}{2}\ddot\phi(a)(b-a)^2+o((b-a)^2).$$

Thus

$$\left(\phi(a)-\phi(b)+2\phi(b)-2\fint_a^b\phi(t)dt\right)=\frac{1}{6}\ddot\phi(a)(b-a)^2+o((b-a)^2).$$

To conclude, we prove that $R_3(a,b)$ and $\fint_a^b o(t^2\phi(t)^2)dt$ are $o\left(a\frac{\phi(a)-\phi(b)}{b-a}\ddot\phi(a)(b-a)^2\right)$, for $b\to 0$, while $\left(\phi(b)-\fint_a^b\phi(t)dt\right)^2$ goes to 0 like $a\frac{\phi(a)-\phi(b)}{b-a}\ddot\phi(a)(b-a)^2$ (times a positive constant) and for our purposes it can be ignored, being positive. For $R_3(a,b)$ we have

$$\frac{(\phi(b)-\phi(a))^2}{a\frac{\phi(a)-\phi(b)}{b-a}\ddot\phi(a)(b-a)^2}=\frac{\phi(b)-\phi(a)}{a(b-a)\ddot\phi(a)}=\frac{\dot\phi(a)+\frac{o(b-a)}{b-a}}{a\ddot\phi(a)}=\frac{a^2\log(a)^2}{a^2\log(a)(-1-\log(a))},$$

which is bounded for a small.

$$\fint_a^b o(t^2\phi(t)^2)dt=a^2\phi(a)^2+\frac{o(b-a)}{b-a},$$

and

$$\frac{a^2\phi(a)^2}{a\frac{\phi(a)-\phi(b)}{b-a}\ddot\phi(a)(b-a)^2}\to 0$$

for $b\to 0$. We write

$$\begin{aligned}\left(\phi(b)-\fint_a^b\phi(t)dt\right)^2&=\left(\phi(b)-\phi(b)-\frac{1}{2}\dot\phi(b)(b-a)+\frac{o(b-a)}{b-a}\right)^2\\&=\frac{1}{4}\dot\phi(b)^2(b-a)^2+\frac{o(b-a)}{b-a}.\end{aligned}$$

Thus

$$\frac{\dot\phi(b)^2(b-a)^2}{a\frac{\phi(a)-\phi(b)}{b-a}\ddot\phi(a)(b-a)^2}=\frac{-b^3\log(b)^3}{b^3\log(b)^2(-1-\log(b))}\xrightarrow{b\to 0}1,$$

where we used $\frac{\phi(a)-\phi(b)}{b-a}=-\dot\phi(b)+\frac{o(b-a)}{b-a)}$ and the fact that a behaves like b in the limit. $\square$

2.2 Estimate of the error produced by the cut

The aim of this second section is to study the error $\gamma(T)^{-1}\gamma^{(1)}(T_1)$. We observe that by the definition of Cut it follows $\pi\left(\gamma^{-1}(T)\gamma^{(1)}(T_1)\right)=0$, thus the projection of the error on the first layer is zero. For the second layer we need to rewrite it in a more convenient form.

Lemma 2.2.1. It holds
$$\gamma(T)^{-1}\gamma^{(1)}(T_1) = C_{\gamma|_b^T}(g_1),$$
where $g_1 = \gamma(b)^{-1}\gamma^{(1)}(a + |\underline{\gamma}(b) - \underline{\gamma}(a)|)$.

Proof. We use the fact that γ and $\gamma^{(1)}$ have the same final projection on V_1, formally
$$\underline{\gamma}^{(1)}|_{[a+|\underline{\gamma}(b)-\underline{\gamma}(a)|,T_1]}(t) = \underline{\gamma}|_{[a+|\underline{\gamma}(b)-\underline{\gamma}(a)|,T_1]}\left(t + (b-a) - |\underline{\gamma}(b) - \underline{\gamma}(a)|\right)$$
and the uniqueness of the horizontal lift to get
$$\gamma^{(1)}|_{a+|\underline{\gamma}(b)-\underline{\gamma}(a)|}^{T_1} = \gamma|_b^T.$$
From which
$$\begin{aligned}\gamma^{(1)}(T_1) &= \gamma^{(1)}(a + |\underline{\gamma}(b) - \underline{\gamma}(a)|)\gamma|_b^T \\ &= \gamma(b)g_1\gamma|_b^T \\ &= \gamma(b)\gamma|_b^T C_{\gamma|_b^T}(g_1) \\ &= \gamma(T)C_{\gamma|_b^T}(g_1).\end{aligned}$$
□

Recalling how the projection π_2 interacts with the conjugation we get
$$\pi_2(\gamma(T)^{-1}\gamma^{(1)}(T_1)) = \pi_2(g_1).$$

Moreover, we define the curve
$$\alpha(t) := \begin{cases}\gamma(b)^{-1}\gamma(b-t) & t \in [0, b-a] \\ \gamma(b)^{-1}\gamma(a)\exp(t-(b-a))w) & t \in [b-a, b-a+|\underline{\gamma}(b) - \underline{\gamma}(a)|].\end{cases}$$
Where w is as in definition 1.8.2

Remark 2.2.2. This curve joins e to g_1. The idea behind it is to first follow the curve γ from $\gamma(b)$ to $\gamma(a)$ and then the curve $\gamma^{(1)}$ from $\gamma^{(1)}(b-a) = \gamma(a)$ to $\gamma^{(1)}(a+|\underline{\gamma}(b)-\underline{\gamma}(a)|)$. The translation $\gamma(b)^{-1}$ is there just to make the curve start from e.
In addition, $\pi(\alpha(t))$ is a closed curve on V_1. We call A the region inside this curve

With these facts in mind we are ready to estimate the error.

Proposition 2.2.3. It holds $\pi_2(\gamma(T)^{-1}\gamma^{(1)}(T_1)) = c_1\mathscr{L}^2(A)X_3$ for some constant c_1 which does not depend on the parameters of the Cut.

Proof. As observed above it is enough to prove $\pi_2(g_1) = c_1\mathscr{L}^2(A)X_3$.
Since α is horizontal (because it is defined joining horizontal curves) we have

$$\dot{\alpha}(t) = h_1(t)X_1^L(\alpha(t)) + h_2(t)X_2^L(\alpha(t)).$$

We want to find an expression for $\dot{\alpha}(t)$ in exponential coordinates. In coordinates we have $\widetilde{\alpha}(t) := \left(F^{-1}\circ\alpha\right)(t) = (\widetilde{\alpha}_1(t),\ldots,\widetilde{\alpha}_n(t))$ as a curve on $\mathbb{R}^n$. We denote $\dot{\widetilde{\alpha}}(t) = \left(\dot{\widetilde{\alpha}}_1(t),\ldots,\dot{\widetilde{\alpha}}_n(t)\right) = \frac{d}{dt}(F^{-1}\circ\alpha)(t)$.
We have

$$\begin{aligned}\dot{\widetilde{\alpha}}(t) = d(F^{-1})_{\alpha(t)}\dot{\alpha}(t) = d(F^{-1})_{\alpha(t)}\left(h_1(t)X_1^L(\alpha(t)) + h_2(t)X_2^L(\alpha(t))\right) =\\ h_1(t)\widetilde{X}_1(F(\alpha(t))) + h_2(t)\widetilde{X}_2(F(\alpha(t))).\end{aligned}$$

Plugging in the expression in exponential coordinates for $\widetilde{X}_1,\ \widetilde{X}_2$ found in the previous chapter we obtain

$$\begin{aligned}\dot{\widetilde{\alpha}}_1(t) &= h_1(t),\\ \dot{\widetilde{\alpha}}_2(t) &= h_2(t),\\ \dot{\widetilde{\alpha}}_k(t) &= h_1(t)c_{1k}\left(\widetilde{\alpha}(t)\right) + h_2(t)c_{2k}\left(\widetilde{\alpha}(t)\right), \qquad k = 3,\ldots,n,\end{aligned} \tag{4}$$

where the components of $\widetilde{\alpha}(t)$ appearing in $c_{ik}\left(\widetilde{\alpha}(t)\right)$, $i = 1,2$, are those with degree strictly smaller than $d(k)$. In particular if we integrate $\widetilde{\alpha}_3$

$$\begin{aligned}&\int_0^{b-a+|\underline{\gamma}(b)-\underline{\gamma}(a)|}\dot{\widetilde{\alpha}}_3(t)dt =\\ &\int_0^{b-a+|\underline{\gamma}(b)-\underline{\gamma}(a)|} h_1(t)c_{13}\left(\widetilde{\alpha}_1(t),\widetilde{\alpha}_2(t)\right) + h_2(t)c_{23}\left(\widetilde{\alpha}_1(t),\widetilde{\alpha}_2(t)\right)dt =\\ &\int_0^{b-a+|\underline{\gamma}(b)-\underline{\gamma}(a)|} \dot{\widetilde{\alpha}}_1(t)c_{13}\left(\widetilde{\alpha}_1(t),\widetilde{\alpha}_2(t)\right) + \dot{\widetilde{\alpha}}_2(t)c_{23}\left(\widetilde{\alpha}_1(t),\widetilde{\alpha}_2(t)\right)dt =\\ &\int_\Gamma c_{13}(x_1,x_2)dx_1 + \int_\Gamma c_{23}(x_1,x_2)dx_2,\end{aligned}$$

where $\Gamma := \left\{\left(\widetilde{\alpha}_1(t),\widetilde{\alpha}_2(t)\right),\ \ t\in\left[0, b-a+|\underline{\gamma}(b)-\underline{\gamma}(a)|\right]\right\}$.
The first integral gives

$$\begin{aligned}&\widetilde{\alpha}_3\left(b-a+|\underline{\gamma}(b)-\underline{\gamma}(a)\right) - \widetilde{\alpha}_3\left(0\right) =\\ &\left[F^{-1}\left(\gamma(b)^{-1}\gamma^{(1)}(b-a+|\underline{\gamma}(b)-\underline{\gamma}(a)|)\right) - F^{-1}\left(\gamma(b)^{-1}\gamma(b)\right)\right]_3 =\\ &\left[F^{-1}\left(g_1\right)\right]_3,\end{aligned}$$

where $\left[F^{-1}\left(g_1\right)\right]_3$ is the third component of g_1 in coordinates. This component lives in the second layer V_2 and it is equal to $\pi_2(g_1)$. ($[\ ]_3$ has been used to denote that we are

considering the third component of the vector).
By Stokes, the last integral gives

$$\int_\Gamma c_{13}(x_1,x_2)dx_1 + \int_\Gamma c_{23}(x_1,x_2)dx_2$$
$$\int_A \left(\frac{\partial}{\partial x_1} c_{23}(x_1,x_2) - \frac{\partial}{\partial x_2} c_{13}(x_1,x_2)\right) dx_1 dx_2 =$$
$$c\int_A dx_1 dx_2 = c^1 \mathscr{L}^2(A).$$

The second equality is true because $c_{i3}(x_1,x_2)$ is a polynomial of weighted degree 1 in $x_1,\ x_2$. □

Now we approximate $\mathscr{L}^2(A)$. Identifying V_1 and $\mathbb{R}^2$ observe that A can be written as Sec − Tri, where Sec is

$$\mathrm{Sec} := \left\{(r,\theta) \in (0,\infty)\times(0,2\pi), \text{s.t. } \ r < |\underline{\gamma}(t)|,\ t\in[a,b], \ \ \phi(a) < \theta < \phi(b)\right\},$$

and Tri is the triangle with vertices 0, $\underline{\gamma}(a)$ and $\underline{\gamma}(b)$.

Proposition 2.2.4. We have

$$\mathscr{L}^2(A) = -\frac{1}{2}(b-a)^2\left[b\frac{\phi(b)-\phi(a)}{b-a} - \bar{\sigma}\dot{\phi}(\bar{\sigma}) + \phi(a) - \phi(\bar{\sigma}) + \frac{abR_6(b)}{(b-a)^2}\right],$$

for some $\bar{\sigma}\in[a,b]$, where $R_6(b) \in o(\phi(b)-\phi(a)^2)$ for $b\to a$.

Proof. We compute $\mathscr{L}^2(\mathrm{Tri})$ and $\mathscr{L}^2(\mathrm{Sec})$. The area of the triangle is simply

$$\mathscr{L}^2(\mathrm{Tri}) = \frac{1}{2}ab\sin(\phi(a)-\phi(b)).$$

Using the first order expansion of sin, we have

$$\mathscr{L}^2(\mathrm{Tri}) = \frac{1}{2}ab\left[\phi(a)-\phi(b)+R_6(a,b)\right],$$

where $\frac{R_6(a,b)}{(\phi(b)-\phi(a))^2} \to 0$ for $b\to a$.

For Sec we have

$$\begin{aligned}
\mathscr{L}^2(\mathrm{Sec}) &= \int_{\phi(a)}^{\phi(b)}\int_0^{\phi^{-1}(\theta)} r\,dr\,d\theta = \frac{1}{2}\int_a^b t^2|\dot{\phi}(t)|dt = -\frac{1}{2}\int_a^b t^2\dot{\phi}(t)dt \\
&= -\frac{1}{2}\left[b^2\phi(b) - a^2\phi(a) - 2\int_a^b t\phi(t)dt\right] \\
&= -\frac{1}{2}(b-a)\left[\frac{b^2\phi(b)-b^2\phi(a)}{b-a} + \frac{b^2\phi(a)-a^2\phi(a)}{b-a} - 2\frac{\int_a^b t\phi(t)dt}{b-a}\right]
\end{aligned}$$

$$= -\frac{1}{2}(b-a)\left[b^2\frac{\phi(b)-\phi(a)}{b-a} + (b+a)\phi(a) - 2\frac{\int_a^b t\phi(t)dt}{b-a}\right]$$

$$\stackrel{(*)}{=} -\frac{1}{2}(b-a)\left[b^2\frac{\phi(b)-\phi(a)}{b-a} + (b-a)\phi(a) - \frac{(b-a)^2(\phi(\bar\sigma)+\bar\sigma\dot\phi(\bar\sigma))}{b-a}\right]$$

$$= -\frac{1}{2}(b-a)\left[b^2\frac{\phi(b)-\phi(a)}{b-a} - (b-a)\bar\sigma\dot\phi(\bar\sigma) + (b-a)\phi(a) - (b-a)\phi(\bar\sigma)\right].$$

In $(*)$ we used

$$\int_a^b t\phi(t)dt = a\phi(a)(b-a) + \frac{\phi(\bar\sigma)+\bar\sigma\dot\phi(\bar\sigma)}{2}(b-a)^2,$$

where $\bar\sigma \in (a,b)$ is given by the Lagrange formula for the remainder.
Thus

$$\mathscr{L}^2(A) = -\frac{1}{2}(b-a)^2\left[b\frac{\phi(b)-\phi(a)}{b-a} - \bar\sigma\dot\phi(\bar\sigma) + \phi(a) - \phi(\bar\sigma) + ab\frac{R_6(b)}{(b-a)^2}\right].$$

□

Remark 2.2.5. We observe that in the expression for $\mathscr{L}^2(A)$ given by the above proposition the term $-\frac{1}{2}(b-a)^2 b\frac{\phi(b)-\phi(a)}{b-a}$ is positive, while the term $-\frac{1}{2}(b-a)^2(-\bar\sigma\dot\phi(\bar\sigma) + \phi(a) - \phi(\bar\sigma))$ is negative, for b small.

2.3 Correction of the error in the second layer

In this section we will show in detail how to correct the error produced by the cut in the second layer. To this aim we want to use Cor to define a curve $\gamma^{(2)}$ such that $\pi(\gamma^{-1}(T)\gamma^{(2)}(T_2)) = \bar\pi_2(E_2)$, where $E_2 \in W_2$ satisfies $\gamma(T)^{-1}\gamma^{(1)}(T_1) = \exp(E_2)$ (such an element exists since $\gamma(T)^{-1}\gamma^{(1)}(T_1) \in \mathbb{G}_2$). We observe that since $\dim(V_2) = 1$ to correct the error in the second layer it is enough to consider only one $Y \in V_1$. More precisely, we set

$$\gamma^{(2)} := \mathrm{Cor}\left(\gamma^{(1)}, [b^\beta, 2b^\beta], -Y\right)$$

where $0 < \beta < 1$ and Y have to be fixed in order to get

$$\pi_2\left(\mathrm{Dis}\left(\gamma^{(1)}, [b^\beta, 2b^\beta], -Y\right)\right) = \bar\pi_2(E_2) \tag{5}$$

(we will omit the dependence of $\gamma^{(2)}$ on β and Y). Actually, Y will depend on two parameters ε, β. Indeed we will look for Y orthogonal to $|\underline{\gamma}(2b^\beta) - \underline{\gamma}(b^\beta)|$ and with length ε. We will highlight this dependece writing $Y_\varepsilon(\beta)$ (or simply Y_ε to avoid heavy notation). Moreover we use the estimates of the first section for $a = b - b^{1+\tau}$, for $\tau > 0$. τ will be fixed at the very end (actually any sufficiently small choice of τ will work), and in this section we will show that for some τ we can find ε, β that allows us to correct

the error on the second layer. We will find an explicit formula for ε in terms of β and an interval of existence for β.
We start making some observation

Remark 2.3.1. We recall that $\gamma^{(2)}$ is defined as a join of curves from which it follows

$$\pi\left(\gamma^{(2)}(t)|_{[2b^\beta+2\ell_{Y_\varepsilon},2\ell_{Y_\varepsilon}+T_1]}\right)=\pi\left(\gamma^{(1)}(t)|_{[2b^\beta,T_1]}\right)=\pi\left(\gamma(t+b-a-|\underline{\gamma}(b)-\underline{\gamma}(a)|)\right),$$

which means that $\gamma^{(2)}$ has the same final projection of γ on V_1.
Moreover $T_2:=2\ell_{Y_\varepsilon}+T_1>T_1$.

Remark 2.3.2. Let us focus a little bit more on the nature of equation (5). As we have already seen, the error created cutting the curve γ lies in $\mathbb{G}_2$. The idea in this first step is to fix the error in the second layer producing another error which lies is $\mathbb{G}_3$. Formally this means

$$\gamma(T)^{-1}\gamma^{(2)}(T_2)\in\mathbb{G}_3.$$

By lemma 1.8.8 we already know that

$$\gamma(T)^{-1}\gamma^{(2)}(T_2)=\left(\gamma(T)^{-1}\gamma^{(1)}(T_1)\right)\left(\gamma^{(1)}(T_1)^{-1}\gamma^{(2)}(T_2)\right)\in\mathbb{G}_2,$$

thus it is enough to show

$$\begin{aligned}0&=\pi_2\left(\gamma(T)^{-1}\gamma^{(2)}(T_2)\right)=\pi_2(\exp(E_2)+\pi_2\left(\gamma^{(1)}(T_1)^{-1}\gamma^{(2)}(T_2)\right)\\&=\bar{\pi}_2(E_2)+\pi_2\left(\mathrm{Dis}\left(\gamma^{(1)},[b^\beta,2b^\beta],-Y_\varepsilon\right)\right),\end{aligned}$$

which is equation (5).

Using the same idea used to prove Proposition 2.2.3, we can estimate $\pi_2\left(\mathrm{Dis}\left(\gamma^{(1)},[b^\beta,2b^\beta],-Y_\varepsilon\right)\right)$. By remark 2.3.1 we have that also $\gamma^{(1)}$ and $\gamma^{(2)}$ have the same final projection on V_1, thus by lemma 2.2.1 we get

$$\gamma^{(1)}(T_1)^{-1}\gamma^{(2)}(T_2)=C_{\gamma^{(1)}|_{2b^\beta}^{T_1}}(g_2),$$

where $g_2=\gamma^{(1)}(2b^\beta)^{-1}\gamma^{(2)}(2b^\beta+2\ell_{Y_\varepsilon})$. As in the previous section, we consider the horizontal curve joining e and g_2, given by

$$\alpha^1(t):=\begin{cases}\gamma^{(1)}(b^\beta)^{-1}\gamma^{(1)}(2b^\beta-t) & t\in[0,b^\beta]\\ \gamma^{(1)}(b^\beta)^{-1}\gamma^{(2)}(t) & t\in[b^\beta,2b^\beta+2\ell_{Y_\varepsilon}].\end{cases}$$

Remark 2.3.3. We observe that when projected on V_1 this is a closed curve. Let B be the region inside the curve (in V_1). Moreover, by construction the two line segments

$$(\pi\circ\alpha^1|_{[b^\beta,b^\beta+\ell_{Y_\varepsilon}]})(t)$$

and

$$(\pi \circ \alpha^1|_{[2b^\beta+\ell_{Y_\varepsilon}, 2b^\beta+2\ell_{Y_\varepsilon}]})(t)$$

are parallel and the support of

$$(\pi \circ \alpha^1|_{[b^\beta+\ell_{Y_\varepsilon}, 2b^\beta+\ell_{Y_\varepsilon}]})(t)$$

is just the support of

$$(\pi \circ \alpha^1|_{[0,b^\beta]})(t)$$

translated of ε along Y_1.

We now adapt the proof of proposition 2.2.3 to this case and refer to that proof for the missing details.

Proposition 2.3.4. It holds $\pi_2\left(\mathrm{Dis}\left(\gamma^{(1)}, [b^\beta, 2b^\beta], Y_\varepsilon\right)\right) = c_1 \mathscr{L}^2(B) X_3$, where c_1 is independent of the parameters of the cut and the correction device (and it is the same constant of the previous paragraph).

Proof. We just need to prove $\pi_2(g_2) = c_1 \mathscr{L}^2(B) X_3$. We can write

$$\dot{\alpha}^1(t) = h_1^1(t) X_1^L(\alpha^1(t)) + h_2^1(t) X_2^L(\alpha^1(t)).$$

Reasoning as in 2.2.3, in exponential coordinates

$$\dot{\widetilde{\alpha}}_1^1(t) = h_1^1(t),$$

$$\dot{\widetilde{\alpha}}_2^1(t) = h_2^1(t),$$

$$\dot{\widetilde{\alpha}}_k^1(t) = h_1^1(t) c_{1k}(\widetilde{\alpha}^1(t)) + h_2^1(t) c_{2k}(\widetilde{\alpha}^1(t)), \quad \text{for} \quad k = 3, \dots, n,$$

where the components of $\widetilde{\alpha}^1$ appearing in $c_{ik}(\widetilde{\alpha}^1(t))$, $i = 1, 2$ are those with weighted degree strictly smaller than $d(k)$. Thus if we set

$$\Gamma := \{(\widetilde{\alpha}_1^1(t), \widetilde{\alpha}_2^1(t)), \quad t \in [0, 2b^\beta + 2\ell_{Y_\varepsilon}]\},$$

we can write

$$\int_0^{2b^\beta+2\ell_{Y_\varepsilon}} \dot{\widetilde{\alpha}}_3^1(t) dt = \int_\Gamma c_{13}(x_1, x_2) dx_1 + \int_\Gamma c_{23}(x_1, x_2) dx_2.$$

The right hand side gives

$$[F^{-1}(g_2)]_3,$$

where $[F^{-1}(g_2)]_3$ lives on V_2 and is equal to $\pi_2(g_2)$. While the left hand side gives

$$\int_\Gamma c_{13}(x_1, x_2) dx_1 + \int_\Gamma c_{23}(x_1, x_2) dx_2 = c_1 \int_B dx_1 dx_2 = c_1 \mathscr{L}^2(B).$$

$\square$

The next step is to compute $\mathscr{L}^2(B)$. By remark 2.3.3 we can compute it putting an orthonormal frame centered in $\underline{\gamma}(2b^\beta)$ with the $x-$axes pointing in direction of $\underline{\gamma}(b^\beta) - \underline{\gamma}(2b^\beta)$ and the $y-$axes pointing in the direction of Y. In this frame $[b^\beta, 2b^\beta] \ni t \to \underline{\gamma}^{(1)}(t)$ is the graph of a function, say f, and B is the region

$$\{(x,y), \quad \text{s.t. } \ x \in [0, |\underline{\gamma}^{(1)}(2b^\beta) - \underline{\gamma}^{(1)}(b^\beta)|], \quad f(x) \leqslant y \leqslant f(x) + \varepsilon\}$$

Thus we can compute $\mathscr{L}^2(B)$ using Fubini

$$\begin{aligned}\int_B d\mathscr{L}^2 &= \int_0^{|\underline{\gamma}^{(1)}(2b^\beta)-\underline{\gamma}^{(1)}(b^\beta)|} \int_0^{f(x)+\varepsilon} dxdy - \int_0^{|\underline{\gamma}^{(1)}(2b^\beta)-\underline{\gamma}^{(1)}(b^\beta)|} \int_0^{f(x)} dxdy \\ &= \int_0^{|\underline{\gamma}^{(1)}(2b^\beta)-\underline{\gamma}^{(1)}(b^\beta)|} \varepsilon dx = \varepsilon |\underline{\gamma}^{(1)}(2b^\beta) - \underline{\gamma}^{(1)}(b^\beta)|.\end{aligned}$$

Remark 2.3.5. Requiring equation (5) to hold we get

$$\varepsilon = \frac{c_1}{c_1} \frac{\mathscr{L}^2(A)}{|\underline{\gamma}^{(1)}(2b^\beta) - \underline{\gamma}^{(1)}(b^\beta)|},$$

which is one of the parameters we wanted. Even if it was clear from the beginning, we can now see explicitly that ε depends on β.

To conclude our argument we want to find a small β such that $L\left(\underline{\gamma}|_{[0,T]}\right) > L\left(\underline{\gamma}^{(2)}|_{[0,T_2]}\right)$. We rewrite the second term as

$$\begin{aligned}L\left(\underline{\gamma}^{(2)}|_{[0,T_2]}\right) &= L\left(\underline{\gamma}^{(1)}|_{[0,b^\beta]}\right) + \varepsilon + L\left(\underline{\gamma}^{(1)}|_{[b^\beta,2b^\beta]}\right) + \varepsilon + L\left(\underline{\gamma}^{(1)}|_{[2b^\beta,T_1]}\right) \\ &= L\left(\underline{\gamma}^{(1)}|_{[0,a]}\right) + |\underline{\gamma}(b) - \underline{\gamma}(a)| + L\left(\underline{\gamma}^{(1)}|_{[a,b^\beta]}\right) \\ &\qquad +2\varepsilon + L\left(\underline{\gamma}^{(1)}|_{[b^\beta,2b^\beta]}\right) + L\left(\underline{\gamma}^{(1)}|_{[2b^\beta,T_1]}\right).\end{aligned}$$

Thus the inequality reduces to

$$\Delta L(a,b) > 2\varepsilon = 2\frac{\mathscr{L}^2(A)}{|\underline{\gamma}^{(1)}(2b^\beta) - \underline{\gamma}^{(1)}(b^\beta)|}.$$

We introduced the notation $\Delta\phi = \frac{\phi(a)-\phi(b)}{b-a} > 0$, and $\eta = b - a$. We know that

$$|\underline{\gamma}^{(1)}(2b^\beta) - \underline{\gamma}^{(1)}(b^\beta)| = |\underline{\gamma}(2b^\beta + b - a - |\underline{\gamma}(b) - \underline{\gamma}(a)|) - \underline{\gamma}(b^\beta + b - a - |\underline{\gamma}(b) - \underline{\gamma}(a)|)|$$

from the definition of Cut. We will actually prove an inequality that implies the wanted one, namely

$$\frac{\eta}{2}\left[\frac{a}{\eta}(\phi(b) - \phi(a))\ddot{\phi}(a)\eta^2 + R_5(a,b)\right] > \frac{\frac{1}{2}\eta^2 b\Delta\phi - abR_6(b)}{b^\beta}.$$

This inequality is stronger than the one we need to prove that we can correct the error with a gain of length. Indeed $\frac{1}{2}\eta^2 b\Delta\phi - abR_6(b) > \mathscr{L}^2(A)$ for b small due to the remark 2.2.5. $\Delta L(a,b) \geq \frac{\eta}{2}\left[\frac{a}{\eta}(\phi(b)-\phi(a))\ddot{\phi}(a)\eta^2 + R_5(a,b)\right]$ by proposition 2.1.3. Moreover $b^\beta < |\underline{\gamma}(2b^\beta + b - a - |\underline{\gamma}(b) - \underline{\gamma}(a)|) - \underline{\gamma}(b^\beta + b - a - |\underline{\gamma}(b) - \underline{\gamma}(a)|)|$ for b small, by 2.1.2. Dividing by $\Delta\phi$ and η and multiplying by $b^{1-\beta}$ we can rewrite

$$b^{\beta-1}b(1-b^\tau)\ddot{\phi}(a)\eta^2 > \eta - \frac{b(1-b^\tau)R_6(a,b)}{\eta\Delta\phi} - \frac{R_5(a,b)}{\Delta\phi}b^{\beta-1}.$$

To conclude we look for which τ and β the term of the right goes to zero faster than the term of the left. To avoid heavy notation we will also use the fact that a behaves like b for b small, indeed for our purposes we can forget about $1-\beta^{1+\tau}$ in the expression of a, being this term approximately 1 for b small. For the term containing $R_6(a,b)$

$$\frac{\frac{b(\phi(b)-\phi(a))^2}{\eta\Delta\phi}}{b^{\beta-1}b\eta^2\ddot{\phi}(a)} = \frac{b(\phi(a)-\phi(b))}{b^\beta b^{2\tau}\frac{-1-\log(b)}{\log(b)^2}} = \frac{b^2\Delta\phi}{b^\beta b^\tau \frac{-1-\log(b)}{\log(b)^2}}.$$

In the last equality we multiplied and divided by b and wrote $\frac{\phi(a)-\phi(b)}{b^{1+\tau}} = \Delta\phi$. Now using $\Delta\phi = \dot{\phi}(b) + o(1)$ for $b \to 0$, the dominant term in the last expression is given by

$$\frac{b\log(b)}{b^\beta b^\tau \frac{-1-\log(b)}{\log(b)^2}},$$

which goes to 0 for $b \to 0$ if $\beta + \tau < 1$. For the term containing $R_5(a,b)$

$$\frac{R_5(a,b)b^{\beta-1}}{b\Delta\phi\ddot{\phi}\eta^2 b^{\beta-1}} \xrightarrow{b\to 0} 0.$$

For the term containing η

$$\frac{\eta}{b\ddot{\phi}\eta^2 b^{\beta-1}} = \frac{b^{1+\tau}}{b\frac{-1-\log(b)}{b^2\log(b)^2}b^{2+2\tau}b^{\beta-1}}$$

This goes to 0 if the exponent of b (at the numerator) is strictly larger that 0, that is

$$1+\tau-1-2\tau-\beta+1 > 0.$$

Which means $\beta+\tau < 1$. This implies that it is possible to correct the error on the first layer.

Remark 2.3.6. We have checked explicitly that the terms containing the remainders were small o of the term on the left hand side, but we could have forgotten them. This is clear for $R_5(a,b)$, being a small o of the other term inside the bracket independently of β. It is true also for $R_6(a,b)$ because $b^2R_6(a,b) \in o(\eta^2 b\Delta\phi)$ and, for $\beta+\tau<1$, $\eta^2 b\Delta\phi$

goes to zero faster than $b^{\beta-1}b(\phi(b)-\phi(a))\ddot{\phi}(a)\eta^2$. Thanks to this remark we will forget about the terms containing the remainders in the following, because we will always have a situation of this type: a term which is negligible independently of the parameters and a term which will be negligible as long as the parameters satisfy the right relations. Moreover, we can also forget about the costants in the final inequality, since we will always end up proving that there exists a choice for the parameters that makes the on the left side of $>$ unbounded.

2.4 Correction of the error in the third layer

The application of the correction device on $\gamma^{(1)}$ fixes the error on the second layer but it produces an error on the third layer. We will apply two correction devices on $\gamma^{(2)}$ to fix this error. Formally, $\gamma(T)^{-1}\gamma^{(2)}(T_2) \in \mathbb{G}_3$, thus it exists $E_3 \in V_3$ such that $\exp(E_3) = \gamma(T)^{-1}\gamma^{(2)}(T_2)$. We want to estimate the size of the error. We prove that it holds $\bar{\pi}_3(E_3) = \sum_{j|d(j)=3}(y_j - x_j)X_j$, where

$$\gamma(T) = \exp(x_1X_1 + \cdots + x_nX_n),$$

$$\gamma^{(2)}(T_2) = \exp(y_1X_1 + \cdots + y_nX_n).$$

We compute

$$\begin{aligned}
\pi_3\left(\gamma(T)^{-1}\gamma^{(2)}(T_2)\right) &= \bar{\pi}_3 \exp^{-1}(\exp(-x_1X_1 - \cdots - x_nX_n)\exp(y_1X_1 + \cdots + y_nX_n)) \\
&= \bar{\pi}_3(P(-x_1X_1 - \cdots - x_nX_n, y_1X_1 + \cdots + y_nX_n))
\end{aligned}$$

The B-H-C formula in our framework reduces to

$$P(X,Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] - [Y,[X,Y]])$$

for generic $X,\ Y \in \mathfrak{g}$. Before computing the commutators we observe that since the projection on the first layer of the final piece of γ coincides with the projection on the first layer of the final piece of $\gamma^{(2)}$, this implies $x_1 = y_1$, $x_2 = y_2$. Moreover, since we chose Y to fix the component on the third layer of E_2 we have $x_3 = y_3$. For the first commutator

$$[-x_1X_1 - \cdots - x_nX_n, y_1X_1 + \cdots + y_nX_n]$$

we observe that the only contribution to the third layer comes from

$$\begin{aligned}
[-x_1X_1 - x_2X_2, y_3X_3] - [y_1X_1 + y_2X_2, -x_3X_3] &= (y_1x_3 - x_1y_3)[X_1,X_3] + (y_2x_3 - x_2y_3)[X_2,X_3] \\
&= x_1(x_3 - y_3)[X_1,X_3] + x_2(x_3 - y_3)[X_2,X_3],
\end{aligned}$$

and this is zero because $x_3 = y_3$. We turn to the first iterated commutator. In this case the only contribution to the third layer is given by

$$[-x_1X_1 - x_2X_2, [-x_1X_1 - x_2X_2, y_1X_1 + y_2X_2]] = 0,$$

since $[-x_1X_1 - x_2X_2, y_1X_1 + y_2X_2] = -x_1y_2[X_1, X_2] + y_1x_2[X_1, X_2] = 0$. Similarly the second iterated commutator gives zero. Thus

$$\pi_3(\gamma(T)^{-1}\gamma^{(2)}(T_2)) = \sum_{j|d(j)=3} (y_j - x_j)X_j.$$

which is what we wanted. We estimate $|\pi_3(\gamma(T)^{-1}\gamma^{(2)}(T_2))|$. To do this we estimate the components in exponential coordinates of the third layer. Let $\widetilde{\gamma} = (\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n)$ and $\widetilde{\gamma}^{(2)} = (\widetilde{\gamma}_1^{(2)}, \dots, \widetilde{\gamma}_n^{(2)})$ be the curves in exponential coordinates. We want to know how the quantities $|\widetilde{\gamma}_j^{(2)}(T_2) - \widetilde{\gamma}_j(T)|$ go to zero for $b \to 0$, for j such that $d(j) = 3$.

Proposition 2.4.1. For $b \to 0$ we have

$$|\widetilde{\gamma}_j^{(2)}(T_2) - \widetilde{\gamma}_j(T)| = O(b^\beta \mathscr{L}^2(A)),$$

for any j such that $d(j) = 3$.

Proof. We use the equations (4) to reconstruct the error for the coordianates of the third layer. We have

$$\widetilde{\gamma}_j(T) - \widetilde{\gamma}_j(a) = \int_a^T \dot{\widetilde{\gamma}}_1 c_{1j}(\widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_3) + \dot{\widetilde{\gamma}}_2 c_{2j}(\widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_3)dt,$$

$$\widetilde{\gamma}_j^{(2)}(T_2) - \widetilde{\gamma}_j^{(2)}(a) = \int_a^{T_2} \dot{\widetilde{\gamma}}_1^{(2)} c_{1j}(\widetilde{\gamma}_1^{(2)}, \widetilde{\gamma}_2^{(2)}, \widetilde{\gamma}_3^{(2)}) + \dot{\widetilde{\gamma}}_2^{(2)} c_{2j}(\widetilde{\gamma}_1^{(2)}, \widetilde{\gamma}_2^{(2)}, \widetilde{\gamma}_3^{(2)})dt.$$

We have $\gamma(a) = \gamma^{(1)}(a) = \gamma^{(2)}(a)$ because the cutted curve coincides with γ until a and the corrected curve coincides with the cutted one until b^β. Thus

$$\widetilde{\gamma}_j^{(2)}(T_2) - \widetilde{\gamma}_j(T) =$$
$$\int_a^{T_2} \dot{\widetilde{\gamma}}_1^{(2)} c_{1j}(\widetilde{\gamma}_1^{(2)}, \widetilde{\gamma}_2^{(2)}, \widetilde{\gamma}_3^{(2)}) + \dot{\widetilde{\gamma}}_2^{(2)} c_{2j}(\widetilde{\gamma}_1^{(2)}, \widetilde{\gamma}_2^{(2)}, \widetilde{\gamma}_3^{(2)})dt - \int_a^T \dot{\widetilde{\gamma}}_1 c_{1j}(\widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_3) + \dot{\widetilde{\gamma}}_2 c_{2j}(\widetilde{\gamma}_1, \widetilde{\gamma}_2, \widetilde{\gamma}_3)dt.$$

In the above difference some terms cancels out, while some terms reduce to integration along closed curves. To see this we need to split the polynomials c_{1j}, c_{2j}. Due to the structure of these polynomials we can write

$$c_{1j}(z_1, z_2, z_3) = p_{1j}(z_1, z_2) + \rho_{1j} z_3,$$

$$c_{2j}(z_1, z_2, z_3) = p_{2j}(z_1, z_2) + \rho_{2j} z_3$$

where p_{1j}, p_{2j} are polynomials of homogeneous degree 2 in the variables of the first layer and ρ_{1j}, ρ_{2j} are constants. Using this splitting we can rearrange the difference between the two intgrals as

$$\int_a^{T_2} \dot{\widetilde{\gamma}}_1^{(2)} p_{1j}(\widetilde{\gamma}_1^{(2)}, \widetilde{\gamma}_2^{(2)})dt + \int_a^{T_2} \dot{\widetilde{\gamma}}_2^{(2)} p_{2j}(\widetilde{\gamma}_1^{(2)}, \widetilde{\gamma}_2^{(2)})dt - \int_a^T \dot{\widetilde{\gamma}}_1 p_{1j}(\widetilde{\gamma}_1, \widetilde{\gamma}_2)dt - \int_a^T \dot{\widetilde{\gamma}}_2 p_{2j}(\widetilde{\gamma}_1, \widetilde{\gamma}_2)dt$$

$$+\int_a^{T_2}\dot{\widetilde{\gamma}}_1^{(2)}\rho_{1j}\widetilde{\gamma}_3^{(2)}dt+\int_a^{T_2}\dot{\widetilde{\gamma}}_2^{(2)}\rho_{2j}\widetilde{\gamma}_3^{(2)}dt-\int_a^{T}\dot{\widetilde{\gamma}}_1\rho_{1j}\widetilde{\gamma}_3 dt-\int_a^{T}\dot{\widetilde{\gamma}}_2\rho_{2j}\widetilde{\gamma}_3 dt. \qquad (6)$$

Now in the first row we have only integrals involving only coordinates of the first layer We show that this row gives a sum of integrals integral of 1−forms along a closed curves (and we will use Stokes). By definition of cut and cor we deduce that

$$\begin{aligned}\underline{\gamma}^{(2)}|_{[a+|\underline{\gamma}(b)-\underline{\gamma}(a)|,b^\beta]}(t) &= \underline{\gamma}^{(1)}|_{[a+|\underline{\gamma}(b)-\underline{\gamma}(a)|,b^\beta]}(t)\\ &= \underline{\gamma}|_{[a+|\underline{\gamma}(b)-\underline{\gamma}(a)|,b^\beta]}(t+b-a-|\underline{\gamma}(b)-\underline{\gamma}(a)|), \end{aligned} \qquad (7)$$

$$\begin{aligned}\underline{\gamma}^{(2)}|_{[2b^\beta+2\ell_Y,T_2]}(t) &= \underline{\gamma}^{(1)}|_{[2b^\beta+2\ell_Y,T_2]}(t-2\ell_Y)\\ &= \underline{\gamma}|_{[2b^\beta+2\ell_Y,T_2]}(t-2\ell_Y+b-a-|\underline{\gamma}(b)-\underline{\gamma}(a)|). \end{aligned} \qquad (8)$$

We recall that $T_2=T_1+2\ell_Y=T-b+a+|\underline{\gamma}(b)-\underline{\gamma}(a)|+2\ell_Y$. We can write the first integral in (6) as

$$\begin{aligned}\int_a^{T_2}\dot{\widetilde{\gamma}}_1^{(2)}p_{1j}(\widetilde{\gamma}_1^{(2)},\widetilde{\gamma}_2^{(2)})dt &= \int_a^{a+|\underline{\gamma}(b)-\underline{\gamma}(a)|}\dot{\widetilde{\gamma}}_1^{(2)}p_{1j}(\widetilde{\gamma}_1^{(2)},\widetilde{\gamma}_2^{(2)})dt+\int_{a+|\underline{\gamma}(b)-\underline{\gamma}(a)|}^{b^\beta}\dot{\widetilde{\gamma}}_1^{(2)}p_{1j}(\widetilde{\gamma}_1^{(2)},\widetilde{\gamma}_2^{(2)})dt\\ &+\int_{b^\beta}^{2b^\beta+2\ell_Y}\dot{\widetilde{\gamma}}_1^{(2)}p_{1j}(\widetilde{\gamma}_1^{(2)},\widetilde{\gamma}_2^{(2)})dt+\int_{2b^\beta+2\ell_Y}^{T_2}\dot{\widetilde{\gamma}}_1^{(2)}p_{1j}(\widetilde{\gamma}_1^{(2)},\widetilde{\gamma}_2^{(2)})dt.\end{aligned}$$

In the integral

$$\int_{a+|\underline{\gamma}(b)-\underline{\gamma}(a)|}^{b^\beta}\dot{\widetilde{\gamma}}_1^{(2)}p_{1j}(\widetilde{\gamma}_1^{(2)},\widetilde{\gamma}_2^{(2)})dt$$

we substitute (7) and use the change of variable $s=t+b-a-|\underline{\gamma}(b)-\underline{\gamma}(a)|$ to obtain

$$\int_b^{b^\beta+b-a-|\underline{\gamma}(b)-\underline{\gamma}(a)|}\dot{\widetilde{\gamma}}_1(s)p_{1j}(\widetilde{\gamma}_1(s),\widetilde{\gamma}_2(s))ds. \qquad (9)$$

In the integral

$$\int_{2b^\beta+2\ell_Y}^{T_2}\dot{\widetilde{\gamma}}_1^{(2)}p_{1j}(\widetilde{\gamma}_1^{(2)},\widetilde{\gamma}_2^{(2)})dt$$

we substitute (8) and use the change of variable $s=t-2\ell_Y+b-a-|\underline{\gamma}(b)-\underline{\gamma}(a)$ to obtain

$$\int_{2b^\beta+b-a-|\underline{\gamma}(b)-\underline{\gamma}(a)|}^{T_2}\dot{\widetilde{\gamma}}_1(s)p_{1j}(\widetilde{\gamma}_1(s),\widetilde{\gamma}_2(s))ds \qquad (10)$$

(9) and (10) cancels out with the corresponding terms in the third integral in (6). We can split in the same way the second integral in (6) and perform the same change of variables that led to (9) and (10). In this case the two terms that we obtain cancels out

with the corresponding terms in the fourth integral in (6). After these cancelations the first row in (6) becomes

$$\begin{aligned}
&\int_a^{a+|\underline{\gamma}(b)-\underline{\gamma}(a)|} \dot{\widetilde{\gamma}}_1^{(2)} p_{1j}(\widetilde{\gamma}_1^{(2)}, \widetilde{\gamma}_2^{(2)}) + \dot{\widetilde{\gamma}}_2^{(2)} p_{2j}(\widetilde{\gamma}_1^{(2)}, \widetilde{\gamma}_2^{(2)}) dt \\
&- \int_a^b \dot{\widetilde{\gamma}}_1 p_{1j}(\widetilde{\gamma}_1, \widetilde{\gamma}_2) + \dot{\widetilde{\gamma}}_2 p_{2j}(\widetilde{\gamma}_1, \widetilde{\gamma}_2) dt \\
&+ \int_{b^\beta}^{2b^\beta+2\ell_Y} \dot{\widetilde{\gamma}}_1^{(2)} p_{1j}(\widetilde{\gamma}_1^{(2)}, \widetilde{\gamma}_2^{(2)}) + \dot{\widetilde{\gamma}}_2^{(2)} p_{2j}(\widetilde{\gamma}_1^{(2)}, \widetilde{\gamma}_2^{(2)}) dt \\
&- \int_{b^\beta+b-a-|\underline{\gamma}(b)-\underline{\gamma}(a)|}^{2b^\beta+b-a-|\underline{\gamma}(b)-\underline{\gamma}(a)|} \dot{\widetilde{\gamma}}_1 p_{1j}(\widetilde{\gamma}_1, \widetilde{\gamma}_2) + \dot{\widetilde{\gamma}}_2 p_{2j}(\widetilde{\gamma}_1, \widetilde{\gamma}_2) dt.
\end{aligned} \tag{11}$$

Now each of the rows in the above expression can be seen as a curvilinear integal along a closed curve (in the plane). We focus on the first two rows. We make the change of variable $s = t + b - a$ in

$$\begin{aligned}
&\int_a^{a+|\underline{\gamma}(b)-\underline{\gamma}(a)|} \dot{\widetilde{\gamma}}_1^{(2)}(t) p_{1j}(\widetilde{\gamma}_1^{(2)}(t), \widetilde{\gamma}_2^{(2)}(t)) + \dot{\widetilde{\gamma}}_2^{(2)}(t) p_{2j}(\widetilde{\gamma}_1^{(2)}(t), \widetilde{\gamma}_2^{(2)}(t)) dt \\
&= \int_{b-a}^{b-a+|\underline{\gamma}(b)-\underline{\gamma}(a)|} \dot{\widetilde{\gamma}}_1^{(2)} p_{1j}(\widetilde{\gamma}_1^{(2)}, \widetilde{\gamma}_2^{(2)}) + \dot{\widetilde{\gamma}}_2^{(2)} p_{2j}(\widetilde{\gamma}_1^{(2)}, \widetilde{\gamma}_2^{(2)}) ds
\end{aligned}$$

(we did not display the argument $(s + 2a - b)$ of the functions due to lack of space). If we write explicitly for $s \in [b - a, b - a + |\underline{\gamma}(b) - \underline{\gamma}(a)|]$

$$\gamma^{(2)}((s + 2a - b) = \gamma^{(1)}(s + 2a - b) = \gamma(a) \exp((s + a - b)w)$$

which look suspiciously similar to the final piece of the curve α defined in section 2.2 to estimate the error produced by the cut ($\widetilde{\gamma}_1^{(2)}(s+2a-b)$, $\widetilde{\gamma}_2^{(2)}(s+2a-b)$ are the first two components of that curve in exponential coordinates, for $s \in [b-a, b-a+|\underline{\gamma}(b)-\underline{\gamma}(a)|]$. We make the change of variable $s = b - t$ in

$$\begin{aligned}
-\int_a^b \dot{\widetilde{\gamma}}_1 p_{1j}(\widetilde{\gamma}_1, \widetilde{\gamma}_2) + \dot{\widetilde{\gamma}}_2 p_{2j}(\widetilde{\gamma}_1, \widetilde{\gamma}_2) dt &= -\int_{b-a}^0 \dot{\widetilde{\gamma}}_1 p_{1j}(\widetilde{\gamma}_1, \widetilde{\gamma}_2) + \dot{\widetilde{\gamma}}_2 p_{2j}(\widetilde{\gamma}_1, \widetilde{\gamma}_2) ds \\
&= \int_0^{b-a} (-\dot{\widetilde{\gamma}}_1) p_{1j}(\widetilde{\gamma}_1, \widetilde{\gamma}_2) + (-\dot{\widetilde{\gamma}}_2) p_{2j}(\widetilde{\gamma}_1, \widetilde{\gamma}_2) ds
\end{aligned}$$

(again we did not display the argument $(b - s)$ due to lack of space). Now if we define the planar curve $\widetilde{\kappa}(s) = (\widetilde{\kappa}_1(s), \widetilde{\kappa}_2(s))$

$$\widetilde{\kappa}(s) := \begin{cases} (\widetilde{\gamma}_1(b - s), \widetilde{\gamma}_2(b - s)) & s \in [0, b - a] \\ (\widetilde{\gamma}_1^{(2)}(s + 2a - b), \widetilde{\gamma}_2^{(2)}(s + 2a - b)) & s \in [b - a, b - a + |\underline{\gamma}(b) - \underline{\gamma}(a)|] \end{cases}$$

we have

$$\int_0^{b-a+|\underline{\gamma}(b)-\underline{\gamma}(a)|} \dot{\widetilde{\kappa}}_1(s) p_{1j}(\widetilde{\kappa}_1(s), \widetilde{\kappa}_1(s)) + \dot{\widetilde{\kappa}}_2(s) p_{2j}(\widetilde{\kappa}_1(s), \widetilde{\kappa}_1(s)) ds =$$

$$\int_0^{b-a}(-\dot{\widetilde{\gamma}}_1)p_{1j}(\widetilde{\gamma}_1,\widetilde{\gamma}_2)+(-\dot{\widetilde{\gamma}}_2)p_{2j}(\widetilde{\gamma}_1,\widetilde{\gamma}_2)ds+\int_{b-a}^{b-a+|\underline{\gamma}(b)-\underline{\gamma}(a)|}\dot{\widetilde{\gamma}}_1^{(2)}p_{1j}(\widetilde{\gamma}_1^{(2)},\widetilde{\gamma}_2^{(2)})+\dot{\widetilde{\gamma}}_2^{(2)}p_{2j}(\widetilde{\gamma}_1^{(2)},\widetilde{\gamma}_2^{(2)})ds.$$

Thus

$$\int_a^{a+|\underline{\gamma}(b)-\underline{\gamma}(a)|}\dot{\widetilde{\gamma}}_1^{(2)}p_{1j}(\widetilde{\gamma}_1^{(2)},\widetilde{\gamma}_2^{(2)})+\dot{\widetilde{\gamma}}_2^{(2)}p_{2j}(\widetilde{\gamma}_1^{(2)},\widetilde{\gamma}_2^{(2)})dt$$

is the curvilinear integral of the one form $p_{1j}(x_1,x_2)dx_1+p_{2j}(x_1,x_2)dx_2$ along the closed curve κ. The region inside the curve κ is exactly $\widetilde{A}:=A+\underline{\gamma}(b)$, where A is defined in remark 2.2.2. Using Stokes we can write

$$\begin{aligned}\int_0^{b-a+|\underline{\gamma}(b)-\underline{\gamma}(a)|}&\dot{\widetilde{\kappa}}_1(s)p_{1j}(\widetilde{\kappa}_1(s),\widetilde{\kappa}_1(s))+\dot{\widetilde{\kappa}}_2(s)p_{2j}(\widetilde{\kappa}_1(s),\widetilde{\kappa}_1(s))ds\\ &=\int_{\partial\widetilde{A}}p_{1j}(x_1,x_2)dx_1+\int_{\partial\widetilde{A}}p_{2j}(x_1,x_2)dx_2\\ &=\int_{\widetilde{A}}\left(\frac{\partial}{\partial x_1}p_{2j}(x_1,x_2)-\frac{\partial}{\partial x_2}p_{1j}(x_1,x_2)\right)dx_1dx_2.\end{aligned}$$

For the last two rows in (11) we can do a similar thing (we will find a curviliear integral along the boundary of $\widetilde{B}:=B+\underline{\gamma}^{(1)}(b^\beta)$ defined in remark 2.3.3). We define the curve $\widetilde{\zeta}(t)=(\widetilde{\zeta}_1(t),\widetilde{\zeta}_2(t))$ by

$$\widetilde{\zeta}(t)=\begin{cases}(\widetilde{\gamma}_1,\widetilde{\gamma}_2)(2b^\beta-t+b-a-|\underline{\gamma}(b)-\underline{\gamma}(a)|) & t\in[0,b^\beta]\\ (\widetilde{\gamma}_1^{(2)},\widetilde{\gamma}_2^{(2)})(t) & t\in[b^\beta,2b^\beta+2\ell_Y]\end{cases}$$

and we observe that up to a traslation these are the first two coordinates of the curve α^1 defined in paragraph 2.3. The second row in (11) is equal to

$$\int_0^{2b^\beta+2\ell_Y}\dot{\widetilde{\zeta}}(t)_1p_{1j}(\widetilde{\zeta}(t)_1,\widetilde{\zeta}(t)_2)+\dot{\widetilde{\zeta}}(t)_2p_{2j}(\widetilde{\zeta}(t)_1,\widetilde{\zeta}(t)_2)dt.$$

To see this it is enough to make the change of variable $s=2b^\beta+b-a-|\underline{\gamma}(b)-\underline{\gamma}(a)|-t$ in

$$\int_{b^\beta+b-a-|\underline{\gamma}(b)-\underline{\gamma}(a)|}^{2b^\beta+b-a-|\underline{\gamma}(b)-\underline{\gamma}(a)|}\dot{\widetilde{\gamma}}_1p_{1j}(\widetilde{\gamma}_1,\widetilde{\gamma}_2)+\dot{\widetilde{\gamma}}_2p_{2j}(\widetilde{\gamma}_1,\widetilde{\gamma}_2)dt.$$

Again by Stokes

$$\begin{aligned}\int_0^{2b^\beta+2\ell_Y}&\dot{\widetilde{\zeta}}(t)_1p_{1j}(\widetilde{\zeta}(t)_1,\widetilde{\zeta}(t)_2)+\dot{\widetilde{\zeta}}(t)_2p_{2j}(\widetilde{\zeta}(t)_1,\widetilde{\zeta}(t)_2)dt\\ &=\int_{\partial\widetilde{B}}p_{1j}(x_1,x_2)dx_1+\int_{\partial\widetilde{B}}p_{2j}(x_1,x_2)dx_2\\ &=\int_{\widetilde{B}}\left(\frac{\partial}{\partial x_1}p_{2j}(x_1,x_2)-\frac{\partial}{\partial x_2}p_{1j}(x_1,x_2)\right)dx_1dx_2.\end{aligned}$$

Thus we have written the first row of (6) as sum of integrals of $2-$forms on a region of the plane. We focus on the second row on (6). We integrate by parts once and use $\widetilde{\gamma}_i(T) = \widetilde{\gamma}_i^{(2)}(T_2)$, $i = 1, 2, 3$, $\gamma(a) = \gamma^{(2)}(a)$, to obtain

$$\int_a^{T_2} \dot{\widetilde{\gamma}}_1^{(2)} \rho_{1j} \widetilde{\gamma}_3^{(2)} dt + \int_a^{T_2} \dot{\widetilde{\gamma}}_2^{(2)} \rho_{2j} \widetilde{\gamma}_3^{(2)} dt - \int_a^{T} \dot{\widetilde{\gamma}}_1 \rho_{1j} \widetilde{\gamma}_3 dt - \int_a^{T} \dot{\widetilde{\gamma}}_2 \rho_{2j} \widetilde{\gamma}_3 dt =$$
$$\int_a^{T_2} \widetilde{\gamma}_1^{(2)} \rho_{1j} \dot{\widetilde{\gamma}}_3^{(2)} dt + \int_a^{T_2} \widetilde{\gamma}_2^{(2)} \rho_{2j} \dot{\widetilde{\gamma}}_3^{(2)} dt - \int_a^{T} \widetilde{\gamma}_1 \rho_{1j} \dot{\widetilde{\gamma}}_3 dt - \int_a^{T} \widetilde{\gamma}_2 \rho_{2j} \dot{\widetilde{\gamma}}_3 dt.$$

We can use (4) to write $\dot{\widetilde{\gamma}}_3$ and $\dot{\widetilde{\gamma}}_3^{(2)}$ in terms of the coordinates of the first layer.

Thus we reduced to the case in which we have only integrals involving the coordinates of the first layer. Reasoning as we did to evaluate the first row of (6) we find

$$\int_a^{T_2} \widetilde{\gamma}_1^{(2)} \rho_{1j} \dot{\widetilde{\gamma}}_3^{(2)} dt + \int_a^{T_2} \widetilde{\gamma}_2^{(2)} \rho_{2j} \dot{\widetilde{\gamma}}_3^{(2)} dt - \int_a^{T} \widetilde{\gamma}_1 \rho_{1j} \dot{\widetilde{\gamma}}_3 dt - \int_a^{T} \widetilde{\gamma}_2 \rho_{2j} \dot{\widetilde{\gamma}}_3 dt$$
$$= \int_{\partial \widetilde{a}} \dot{\widetilde{\kappa}}_1 \omega_1(\widetilde{\kappa}_1, \widetilde{\kappa}_2) dt + \int_{\partial \widetilde{A}} \dot{\widetilde{\kappa}}_2 \omega_2(\widetilde{\kappa}_1, \widetilde{\kappa}_2) dt + \int_{\partial \widetilde{B}} \dot{\widetilde{\zeta}}_1 \omega_1(\widetilde{\zeta}_1, \widetilde{\zeta}_2) dt + \int_{\partial \widetilde{B}} \dot{\widetilde{\zeta}}_2 \omega_2(\widetilde{\zeta}_1, \widetilde{\zeta}_2) dt,$$

where

$$\omega_1(y_1, y_2) := \rho_{1j} y_1 c_{13}(y_1, y_2) + \rho_{2j} y_2 c_{13}(y_1, y_2),$$
$$\omega_2(y_1, y_2) := \rho_{1j} y_1 c_{23}(y_1, y_2) + \rho_{2j} y_2 c_{23}(y_1, y_2).$$

Using Stokes in the last equality we obtain

$$\int_a^{T_2} \widetilde{\gamma}_1^{(2)} \rho_{1j} \dot{\widetilde{\gamma}}_3^{(2)} dt + \int_a^{T_2} \widetilde{\gamma}_2^{(2)} \rho_{2j} \dot{\widetilde{\gamma}}_3^{(2)} dt - \int_a^{T} \widetilde{\gamma}_1 \rho_{1j} \dot{\widetilde{\gamma}}_3 dt - \int_a^{T} \widetilde{\gamma}_2 \rho_{2j} \dot{\widetilde{\gamma}}_3 dt$$
$$= \int_{\widetilde{A}} \left(\frac{\partial}{\partial x_1} \omega_2(x_1, x_2) - \frac{\partial}{\partial x_2} \omega_1(x_1, x_2) \right) dx_1 dx_2 + \int_{\widetilde{B}} \left(\frac{\partial}{\partial x_1} \omega_2(x_1, x_2) - \frac{\partial}{\partial x_2} \omega_1(x_1, x_2) \right) dx_1 dx_2.$$

After all these computations we can write (6) as

$$\int_{\widetilde{A}} \left(\frac{\partial}{\partial x_1} p_{2j}(x_1, x_2) - \frac{\partial}{\partial x_2} p_{1j}(x_1, x_2) \right) dx_1 dx_2 + \int_{\widetilde{B}} \left(\frac{\partial}{\partial x_1} p_{2j}(x_1, x_2) - \frac{\partial}{\partial x_2} p_{1j}(x_1, x_2) \right) dx_1 dx_2$$
$$\int_{\widetilde{A}} \left(\frac{\partial}{\partial x_1} \omega_2(x_1, x_2) - \frac{\partial}{\partial x_2} \omega_1(x_1, x_2) \right) dx_1 dx_2 + \int_{\widetilde{B}} \left(\frac{\partial}{\partial x_1} \omega_2(x_1, x_2) - \frac{\partial}{\partial x_2} \omega_1(x_1, x_2) \right) dx_1 dx_2.$$

Now we can make the estimates. To do this we will think to V_1 as $\mathbb{R}^2$. Indeed we have identified $\mathfrak{g}$ and $\mathbb{R}^n$ when we introduced the exponential coordinates, we just restrict this identification to V_1 and $\mathbb{R}^2$. Moreover we take on $\mathbb{R}^n$ the inner product that makes the images of $X_1, \dots, X_n$, under this identification, orthonormal. Since p_{1j}, p_{2j} are polynomials in x_1, x_2 of weighted degree 2, $\frac{\partial}{\partial x_2} p_{1j}$ and $\frac{\partial}{\partial x_1} p_{2j}$ are polynomials in x_1, x_2 of weighted degree 1. Moreover ω_1 and ω_2 are polynomials of degree 2 in x_1, x_2, thus $\frac{\partial}{\partial x_2} \omega_1$ and $\frac{\partial}{\partial x_1} \omega_2$ are polynomials of degree 1 in x_1, x_2 (in this case, since we have only variables if the first layer the notions of degree and weighted degree conincide, being the weights 1). We observe that the region $\widetilde{A}$ is contained in a ball in the plane of radius b centered at the origin. This comes from the fact that $|\underline{\gamma}(t)| = t$ for $t \in [0, T]$ and the fact that $\underline{\gamma}(b))$ is the most distant point of $\widetilde{A}$ from the origin. Among the points of $\widetilde{B}$ the most distant from the origin is $\underline{\gamma}^{(2)}(2b^\beta + \ell_Y)$, which is at distance at most ℓ_Y from $\underline{\gamma}^{(2)}(2b^\beta)$. Thus $\widetilde{B}$ is contained in the two dimensional ball centered at the origin and of radius $2b^\beta + \ell_Y$. From these geometrical facts we have

$$\begin{aligned}\left|\int_{\widetilde{A}} \left(\frac{\partial}{\partial x_1} p_{2j}(x_1, x_2) - \frac{\partial}{\partial x_2} p_{1j}(x_1, x_2)\right) dx_1 dx_2\right| &\leq \int_{\widetilde{A}} \left|\frac{\partial}{\partial x_1} p_{2j}(x_1, x_2) - \frac{\partial}{\partial x_2} p_{1j}(x_1, x_2)\right| dx_1 dx_2 \\ &= \int_{\widetilde{A}} |c_2 x_1 + c_3 x_2| dx_1 dx_2 \leq c_4 b \int_{\widetilde{A}} dx_1 dx_2 \\ &= c_4 b \mathscr{L}^2(\widetilde{A}) = c_4 b \mathscr{L}^2(A),\end{aligned}$$

where c_2, c_3 and c_4 are structural constants that depend only on the Lie group G. Similarly

$$\begin{aligned}\left|\int_{\widetilde{B}} \left(\frac{\partial}{\partial x_1} p_{2j}(x_1, x_2) - \frac{\partial}{\partial x_2} p_{1j}(x_1, x_2)\right) dx_1 dx_2\right| &\leq c_4(2b^\beta + \ell_Y)\mathscr{L}^2(B), \\ \left|\int_{\widetilde{A}} \left(\frac{\partial}{\partial x_1} \omega_2(x_1, x_2) - \frac{\partial}{\partial x_2} \omega_1(x_1, x_2)\right) dx_1 dx_2\right| &\leq c_5 b \mathscr{L}^2(A), \\ \left|\int_{\widetilde{B}} \left(\frac{\partial}{\partial x_1} \omega_2(x_1, x_2) - \frac{\partial}{\partial x_2} \omega_1(x_1, x_2)\right) dx_1 dx_2\right| &\leq c_5(2b^\beta + \ell_Y)\mathscr{L}^2(B),\end{aligned}$$

where c_5 is a constant that depends only on the Lie group G. We show that ℓ_Y goes to zero faster than b^β for $b \to 0$. From the Ball-Box estimate

$$\ell_Y \leq |Y| = \varepsilon = \frac{\mathscr{L}^2(A)}{|\underline{\gamma}^{(1)}(2b^\beta) - \underline{\gamma}^{(1)}(b^\beta)|} \leq \frac{\mathscr{L}^2(A)}{b^\beta} \leq \frac{\eta^2 b \left(\Delta\phi + \frac{aR_6(a,b)}{(b-a)^2}\right)}{b^\beta}.$$

The quantity

$$\frac{\ell_Y}{b^\beta} \leq \frac{\eta^2 b \left(\Delta\phi + \frac{aR_6(a,b)}{(b-a)^2}\right)}{b^{2\beta}}$$

goes to zero as soon as $2 + 2\tau - 2\beta > 0$, which is always satisfied since $\beta < 1$ and $\tau > 0$. Being interested in the leading order term we will neglect ℓ_Y. We show that $b\mathscr{L}^2(A)$ goes to zero faster than $b^\beta \mathscr{L}^2(B)$ for $b \to 0$.

$$\frac{b\mathscr{L}^2(A)}{b^\beta \mathscr{L}^2(B)} = \frac{b\mathscr{L}^2(A)}{b^\beta \mathscr{L}^2(A)} = b^{1-\beta}$$

which goes to 0 since $\beta < 1$.
In conclusion we have seen that for each j such that $d(j) = 3$ and $b \to 0$, we have

$$|\widetilde{\gamma}_j^{(2)}(T_2) - \widetilde{\gamma}_j(T)| = O(b^\beta \mathscr{L}^2(B)).$$

$\square$

Remark 2.4.2. Using the trivial inequality

$$|\pi_3(\gamma(T)^{-1}\gamma^{(2)}(T_2))| \leq \sum_{j|d(j)=3} |\widetilde{\gamma}_j^{(2)}(T_2) - \widetilde{\gamma}(T)|$$

and the last proposition we obtain

$$|\bar{\pi}_3(E_3)| = |\pi_3(\gamma(T)^{-1}\gamma^{(2)}(T_2))| = O(b^{\beta}\mathscr{L}^2(B)). \tag{12}$$

We fix a parameter $0 < \beta_{21} < \beta$. β_{21} will be choosen at the end of the argument in order to fix the error on the third layer. For now, the only condition imposed on β_{21} implies $b^{\beta} < b^{\beta_{21}} =: b_{21}$. We fix also a second parameter β_{22}, which is the solution of the equation

$$\phi(b_{21}) - \phi(b^{\beta_{22}}) = \frac{\pi}{2}.$$

For b small for instance $b < \frac{1}{e}$ this equation has a unique solution since ϕ is invertible (actually b will be taken small enough to have $b_{22} > 2b_{21}$).
We call $b_{22} := b^{\beta_{22}}$. We choose

$$\widetilde{V}_1 = \frac{\underline{\gamma}(2b_{21}) - \underline{\gamma}(b_{21})}{|\underline{\gamma}(2b_{21}) - \underline{\gamma}(b_{21})|}, \qquad \widetilde{V}_2 = \frac{\underline{\gamma}(2b_{22}) - \underline{\gamma}(b_{22})}{|\underline{\gamma}(2b_{22}) - \underline{\gamma}(b_{22})|}.$$

Remark 2.4.3. We explain briefly the geometric idea behind the chioce of b_{22}, $\widetilde{V}_1$ and $\widetilde{V}_2$. We will implicitly assume b really small. In V_1, the angle between $\gamma(2t)$ and $\gamma(t)$ is $\phi(t) - \phi(2t)$, which is going to zero for $t \to 0$ since $t\dot{\phi}(t) \to 0$ for t going to 0. $\widetilde{V}_1$ is a unit vector in the direction $\underline{\gamma}(2b_{21}) - \underline{\gamma}(b_{21})$. As we have observed the angle between these two vector is going to zero, which means that for b small they are pointing almost in the same direction. On the other hand $|\underline{\gamma}(2b_{21})| = 2b_{21}$ and $|\underline{\gamma}(b_{21})| = b_{21}$ and this means that $\underline{\gamma}(2b_{21})$ is twice as distant from the origin as $\underline{\gamma}(2b_{22})$. These two facts combined say that the paralellogram spanned by $\underline{\gamma}(2b_{21})$ and $\underline{\gamma}(2b_{22})$ is almost flat with one side twice the other, thus the shorter diagonal (which is $\widetilde{V}_1$) points almost in the same direction as the two sides. Thus we can think that the angle of $\widetilde{V}_1$ with X_1 is roughly $\phi(b_{21})$ (which is actually the angle between X_1 and $\underline{\gamma}(b_{21})$). Similarly the angle between $\widetilde{V}_2$ and X_1 is approximately $\phi(b_{22})$. Now we recall that b_{22} has been choosen such that the difference between these two angles is $\frac{\pi}{2}$, thus we can think $\widetilde{V}_1$, $\widetilde{V}_2$ as being almost orthogonal.

We take $Z_{21},\ Z_{22} \in V_2$ (to be found) and define

$$\gamma^{(3)}(t) := \mathrm{Cor}(\gamma^{(2)}, [b^{\beta_{21}}, 2b^{\beta_{21}}], -Z_{21}; [b^{\beta_{22}}, 2b^{\beta_{22}}], -Z_{22})(t).$$

Actually, as for Y_{ε}, we won't find Z_{21} and Z_{22} but only a condition on their norms (which of course will be in terms of β_{21}, and β_{22}) to have a gain of length.
We know from lemma 1.8.8 that $\exp^{-1}\left(\gamma^{(2)}(T_2)^{-1}\gamma^{(3)}(T_3)\right) \in V_3 \oplus \cdots \oplus V_s$ and

$$\pi_3\left(\gamma^{(2)}(T_2)^{-1}\gamma^{(3)}(T_3)\right) = -[Z_{21}, \underline{\gamma}(2b_{21}) - \underline{\gamma}(b_{21})] - [Z_{22}, \underline{\gamma}(2b_{22}) - \underline{\gamma}(b_{22})].$$

We want the new corrected curve to fix the error in the third layer, thus we ask

$$\begin{aligned}0 &= \pi_3(\gamma(T)^{-1}\gamma^{(3)}(T_3))) = \pi_3(\gamma(T)^{-1}\gamma^{(2)}(T_2))) + \pi_3(\gamma^{(2)}(T_2)^{-1}\gamma^{(3)}(T_3))) \\ &= \bar{\pi}_3(E_3) - [Z_{21}, \underline{\gamma}(2b_{21}) - \underline{\gamma}(b_{21})] - [Z_{22}, \underline{\gamma}(2b_{22}) - \underline{\gamma}(b_{22})] \end{aligned} \tag{13}$$

(which is a condition on β_{21} and β_{22}).
Since $V_3 = [V_1, V_2]$ there are $Y_{21},\ Y_{22} \in V_2$ such that

$$\bar{\pi}_3(E_3) = [X_1, Y_{21}] + [X_2, Y_{22}] \tag{14}$$

with $|Y_{21}|,\ |Y_{22}| = O(b^\beta \mathscr{L}^2(A))$. The estimates on the order of $|Y_{21}|$ and $|Y_{22}|$ come from equation (12).
To put together equation (13) and equation (14) we write

$$X_i = w_{i1}\widetilde{V}_1 + w_{i2}\widetilde{V}_2$$

for constants $w_{i1},\ w_{i2}$ and $i = 1,\ 2$. Substituting these expressions in (14) and using (13) we find the equations

$$Z_{21}|\underline{\gamma}(2b_{21}) - \underline{\gamma}(b_{21})| = w_{11}Y_{21} + w_{21}Y_{22},$$

$$Z_{22}|\underline{\gamma}(2b_{22}) - \underline{\gamma}(b_{22})| = w_{12}Y_{21} + w_{22}Y_{22}.$$

We prove the following simple lemma that will be used to obtain estimates for the constants w_{ij}.

Lemma 2.4.4. Let $u_1,\ u_2 \in \mathbb{R}^2$ be linearly independent vectors. Let

$$e_1 = k_{11}u_1 + k_{12}u_2 \qquad e_2 = k_{21}u_1 + k_{22}u_2$$

for some constants $k_{ij},\ i,j = 1,2$. Then, for $i,j = 1,2$ we have

$$|k_{ij}| \le \frac{\max\{|u_1|,\ |u_2|\}}{|\det(u_1,u_2)|}.$$

Proof. Denoting by u_{ij} the j-th component of u_i it is immediate to check that

$$e_1 = \frac{u_{22}u_1 - u_{12}u_2}{\det(w_1,w_2)} \qquad e_2 = \frac{-u_{21}u_1 + u_{11}u_2}{\det(w_1,w_2)}.$$

$\square$

With this lemma we can estimate the $|w_{i1}|,\ |w_{i2}|$. Indeed for $i = 1,2$, we can apply the lemma to $X_i = w_{i1}\widetilde{V}_1 + w_{i2}\widetilde{V}_2$, obtaining

$$|w_{ij}| \le \frac{\max\{|\widetilde{V}_1|,\ |\widetilde{V}_2|\}}{|\det(\widetilde{V}_1,\widetilde{V}_2)|} = \frac{1}{\sin(\frac{\pi}{2})}$$

(where the determinant is computed idenfying $\mathfrak{g}$ and $\mathbb{R}^2$ as we have done above).

Using this estimate and the estimates on the order of $|Y_{21}|$ and $|Y_{22}|$ we obtain

$$|Z_{21}| = O\left(\frac{b^{\beta}\mathscr{L}^2(A)}{b^{\beta_{21}}}\right),$$

$$|Z_{22}| = O\left(\frac{b^{\beta}\mathscr{L}^2(A)}{b^{\beta_{22}}}\right).$$

Now we require that the length of the corrected curve is smaller than the length of γ. We can write

$$L(\gamma^{(3)}) = L(\gamma^{(2)}) + 2d(e, \exp(Z_{21})) + 2d(e, \exp(Z_{22})),$$

from which the correction is fine if for b small

$$\Delta L(a,b) > 2\varepsilon + 2d(e, \exp(Z_{21})) + 2d(e, \exp(Z_{22})).$$

We recall

$$\Delta L(a,b) \geqslant \frac{\eta}{2}\Big[a(\Delta\phi)\ddot{\phi}(a)\eta^2 + R_5(a,b)\Big].$$

By remark 2.3.6 we will forget about $R_5(a,b)$ being negligible with respect to the other term inside the bracket.
Using the Ball-Box estimate we can estimate from above $d(e, \exp(Z_{21}))$ and $d(e, \exp(Z_{22}))$.

$$d(e, \exp(Z_{21})) \leqslant C|Z_{21}|^{\frac{1}{2}} = O\left(\frac{b^{\frac{\beta}{2}}\mathscr{L}^2(A)^{\frac{1}{2}}}{b^{\frac{\beta_{21}}{2}}}\right),$$

$$d(e, \exp(Z_{22})) \leqslant C|Z_{22}|^{\frac{1}{2}} = O\left(\frac{b^{\frac{\beta}{2}}\mathscr{L}^2(A)^{\frac{1}{2}}}{b^{\frac{\beta_{22}}{2}}}\right).$$

Since $\beta_{21} > \beta_{22}$ we actually have $d(e, \exp(Z_{22})) = O\left(\frac{b^{\frac{\beta}{2}}\mathscr{L}^2(A)^{\frac{1}{2}}}{b^{\frac{\beta_{21}}{2}}}\right)$. Thus the length needed to use the correction devices is $O\left(\frac{b^{\frac{\beta}{2}}\mathscr{L}^2(A)^{\frac{1}{2}}}{b^{\frac{\beta_{21}}{2}}}\right)$. Moreover using the inequality given by proposition 2.2.4 and the remark 2.2.5, we obtain that the elongation produced by the correction is

$$O\left(\frac{b^{\frac{\beta}{2}}\eta b^{\frac{1}{2}}\left(\Delta\phi + \frac{aR_6(a,b)}{\eta^2}\right)^{\frac{1}{2}}}{b^{\frac{\beta_{21}}{2}}}\right).$$

By remark 2.3.6 we will forget about $\frac{aR_6(a,b)}{\eta^2}$ being this term negligible as long as the parameters satisfy the right conditions (see below). To fix the error with a gain of length we want to find β_{21} such that for b going to zero

$$\frac{b^{\frac{\beta}{2}}\eta(b\Delta\phi)^{\frac{1}{2}}}{b^{\frac{\beta_{21}}{2}}} = o\left(\frac{\eta}{2}a(\Delta\phi)\ddot{\phi}(a)\eta^2\right)$$

(we already know from the previous paragraph that ε goes to zero faster than the right hand side for our choice of β. Thus it is enough to control this term). As we did for the correction of the second layer we will use the fact that a behaves like b in the limit. We will also use the fact that also $\Delta\phi$ behaves like $\dot{\phi}(b)$ in the limit (formally $\Delta\phi = \dot{\phi}(b) + o(1)$ for $b \to 0$). Using these approximations and writing η, $\dot{\phi}(b)$ and $\ddot{\phi}(b)$ explicitly, we are interested in the limiting behaviour of the following expression

$$\frac{\frac{b^{\frac{\beta}{2}} b^{1+\tau}}{b^{\frac{\beta_{21}}{2}}} \frac{1}{\log(b)^{\frac{1}{2}}}}{b^{1+\tau} \frac{1}{\log(b)} \frac{-1-\log(b)}{b^2 \log(b)^2} b^{2+2\tau}} = \frac{b^{\frac{\beta}{2} - 2\tau - \frac{\beta_{21}}{2}} \log(b)^{\frac{5}{2}}}{-1 - \log(b)}.$$

This expression goes to 0 as long as $\frac{\beta}{2} - 2\tau - \frac{\beta_{21}}{2} > 0$, that is $4\tau < \beta - \beta_{21}$. This inequality is compatible with $\beta + \tau < 1$ found in the previous paragraph.

Remark 2.4.5. This concludes the proof of theorem . Indeed, if we take b small and τ, β, β_{21} satisfying the above inequality, then $\gamma^{(3)}$ defined above has smaller length than γ, the same initial point and the same final point. This last fact is true because $\gamma(T)^{-1}\gamma^{(3)}(T)$ is $\exp(V_4) = \exp(0) = e$.

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