Università degli Studi di Padova

Dipartimento di Matematica

Corso di Laurea Magistrale in Matematica

Intrinsic Lipschitz approximation of H-perimeter minimizing boundaries

Tesi di Laurea Magistrale

Candidato:

Giorgio Stefani

Relatore:

Prof. Roberto Monti

Controrelatore:

Prof. Luigi Ambrosio

Sessione di Laurea del 22 luglio 2016

Anno Accademico 2015 – 2016

Università degli Studi di Padova

Dipartimento di Matematica

Corso di Laurea Magistrale in Matematica

Intrinsic Lipschitz approximation of H-perimeter minimizing boundaries

Tesi di Laurea Magistrale

Candidato:
Giorgio Stefani
matricola n. 1089974

Relatore:
Prof. Roberto Monti

. . . .

. . . .

Controrelatore:

Prof. Luigi Ambrosio

Sessione di Laurea del 22 luglio 2016
Anno Accademico 2015 – 2016

Abstract. In this thesis, we prove two new approximation results of H-perimeter minimizing boundaries by means of intrinsic Lipschitz functions in the setting of the Heisenberg group $\mathbb{H}^n$ with $n \geq 2$. The first one is an improvement of a recent result of Monti [**50**] and is the natural reformulation in $\mathbb{H}^n$ of the classical Lipschitz approximation in $\mathbb{R}^n$. The second one is an adaptation of the approximation via maximal function developed by De Lellis and Spadaro [**24**, **25**].

MINIO: How do you select a problem to study?

ATIYAH: I think that presupposes an answer. I don't think that's the way I work at all. Some people may sit back and say, "I want to solve this problem" and they sit down and say, "How do I solve this problem." I don't. I just move around in the mathematical waters, thinking about things, being curious, interested, talking to people, stirring up ideas; things emerge and I follow them up. Or I see something which connects up with something else I know about, and I try to put them together and things develop. I have practically never started off with any idea of what I'm going to be doing or where it's going to go. I'm interested in mathematics; I talk, I learn, I discuss and then interesting questions simply emerge. I have never started off with a particular goal, except the goal of understanding mathematics.

(*An interview with Michael Atiyah*, [**48**])

Contents

Introduction

The general framework. The study of Geometric Measure Theory in the Heisenberg group $\mathbb{H}^n$ started from the pioneering work [**37**] and today the literature in this area has become rather wide. Among the open problems in this field, the regularity of sets that are minimizers for the horizontal perimeter has gained bigger and bigger attention, since its solution would play a key role in the development of this research area, especially in the resolution of the Heisenberg isoperimetric problem.

The n-dimensional Heisenberg group $(\mathbb{H}^n, *)$, $n \in \mathbb{N}$, is the manifold $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ endowed with the group law $(z,t)*(w,s) = (z+w, t+s+P(z,w))$ for $(z,t),(w,s) \in \mathbb{H}^n$, where $z, w \in \mathbb{C}^n$, $t, s \in \mathbb{R}$ and $P\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$ is the (antisymmetric) bilinear form

$$P(z,w) = 2\,\mathrm{Im}\left(\sum_{j=1}^{n} z_j \bar{w}_j\right), \quad z, w \in \mathbb{C}^n.$$

The Lie algebra of left invariant vector fields in $\mathbb{H}^n$ is spanned by the vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}, \quad j = 1, \dots, n,$$

and for any $p = (z,t) \in \mathbb{H}^n$ the *horizontal sub-bundle* H of $T\mathbb{H}^n$ is given by

$$H_p = \mathrm{span}\{X_1(p), \dots, X_n(p), Y_1(p), \dots, Y_n(p)\} \equiv \mathbb{R}^{2n}. \tag{0.1}$$

The H-perimeter of a Lebesgue measurable set $E \subset \mathbb{H}^n$ is the total variation of its characteristic function χ_E in the horizontal directions (0.1).

At the present stage of the theory, most of the known regularity results assume some strong *a priori* regularity and/or some restrictive geometric structure of the minimizer, see [**11**, **12**, **14**, **58**]. On the other hand, examples of minimal surfaces in the first Heisenberg group $\mathbb{H}^1$ that are only Lipschitz continuous in the Euclidean sense have been constructed, see, e.g., [**55**, **56**], but no similar examples of non-smooth minimizers are known in $\mathbb{H}^n$ with $n \geq 2$.

The most natural approach to develop a general regularity theory for H-perimeter minimizing sets in the Heisenberg group $\mathbb{H}^n$ is to reformulate the classical De Giorgi's regularity theory for perimeter minimizers in $\mathbb{R}^n$ in this context.

De Giorgi's regularity theory for perimeter minimizers in $\mathbb{R}^n$ was developed in the revolutionary series of papers [**18**–**20**] and was later codified in [**21**, **42**]. During the last fifty years, De Giorgi's ideas have been improved and generalized by several authors, see, e.g., [**1**, **2**, **9**, **32**–**35**, **57**] and the recent monograph [**46**]. In particular, one of the most important achievements of this field is the powerful Almgren's regularity theory of area minimizing integral currents in $\mathbb{R}^n$ of general codimension, [**2**]. We also refer to the long term program undertaken by De Lellis and Spadaro to make Almgren's work more readable and exploitable for a larger community of specialists, [**23**–**30**, **59**], and to the recent extension of the theory to infinite dimensional spaces, [**4**, **5**].

De Giorgi's scheme. Nowadays De Giorgi's regularity theory has a well-defined underlining scheme which is divided in four main steps. Below we outline this scheme in the context of the Heisenberg group $\mathbb{H}^n$, summarizing the state of the art on the regularity of H-perimeter minimizing boundaries.

Step 1: Lipschitz approximation. The first step in the regularity theory of perimeter minimizing sets in $\mathbb{R}^n$ is a good approximation of minimizers.

In De Giorgi's original approach, the approximation is made by convolution and the estimates are based on a monotonicity formula, see [**42**]. In the Heisenberg group, the validity of a monotonicity formula is not completely clear, see [**17**].

A more flexible approach is the approximation of minimizing boundaries by means of Lipschitz graphs, see [**57**]. This scheme works also in the Heisenberg group. The boundary of sets with finite H-perimeter is not rectifiable in the standard sense and, in fact, may have fractional Hausdorff dimension, [**44**]. Nevertheless, the notion of intrinsic graph in the sense of [**38**] turns out to be effective in the approximation and leads to the following result, see [**50**, Theorem 5.1]. Here $\mathbf{e}(E, B_r, \nu)$ is De Giorgi's excess in the fixed direction $\nu = -X_1$, that is, the L^2-averaged oscillation from the direction ν in the ball B_r of ν_E, the inner horizontal unit normal to E; the set $\mathbb{W} = \mathbb{R} \times \mathbb{H}^{n-1}$ is the hyperplane passing through the origin orthogonal to the direction ν; the ball B_r and the s-dimensional spherical Hausdorff measure $\mathcal{S}^s$ are both induced by the box norm in $\mathbb{H}^n$ (see Chapter 1 for precise definitions).

THEOREM 0.1. *Let $n \geq 2$. For any $L > 0$, there are constants $k = k(n) > 1$ and $c = c(n, L) > 0$ with the following property. For any set $E \subset \mathbb{H}^n$ that is H-perimeter minimizing in the ball B_{kr} with $0 \in \partial E$, $r > 0$, $\nu_E(0) = \nu$, there exists an L-intrinsic Lipschitz function $\varphi \colon \mathbb{W} \to \mathbb{R}$ such that*

$$\mathcal{S}^{2n+1}\Big((\partial E \,\triangle\, \mathrm{gr}(\varphi)) \cap B_r\Big) \leq c r^{2n+1} \mathbf{e}(E, B_{kr}, \nu).$$

Theorem 0.1 holds also for $n = 1$ but, in this case, the Lipschitz constant L has to be suitably large.

Step 2 : Harmonic approximation. The second step in the regularity theory in $\mathbb{R}^n$ is the existence of a harmonic function: the minimal set can be blown-up at a point of its (reduced) boundary by a quantity depending on the excess and the corresponding approximating functions weakly converge in $W^{1,2}$ to a harmonic function.

In the Heisenberg group $\mathbb{H}^n$, the intrinsic Lipschitz functions $\{\varphi_l\}_l$ approximating the corresponding rescaled sets $\{E_l\}_l$ weakly converge in a suitable intrinsic Sobolev class $W^{1,2}_H$ to a limit function ψ, see [**51**, Theorem 2.5]. This holds when $n \geq 2$ thanks to the Poincaré inequality valid on the vertical hyperplane $\mathbb{W}$ proved in [**16**]. Moreover, the limit function ψ is independent of the variable y_1 of the factor $\mathbb{R}$ in the hyperplane $\mathbb{W} = \mathbb{R} \times \mathbb{H}^{n-1}$, see the first claim of [**51**, Theorem 3.2]. This fact seems to have no counterpart in the classical theory and is a consequence of the first order Taylor expansion of H-perimeter proved in [**36**]. However, it is a completely open problem to prove that this limit function ψ is harmonic for the natural *linear* sub-Laplacian of the hyperplane $\mathbb{W}$, because it is not clear how to control the linearization of the *non-linear* intrinsic gradients $\nabla^{\varphi_l}\varphi_l$ during the limit procedure under the sole H-perimeter minimizing property.

Step 3: Decay estimate for excess and Hölder regularity. The third step in the regularity theory in $\mathbb{R}^n$ is the decay estimate for the spherical excess,

$$\operatorname{Exc}(E, B_r(x)) = \min_{|v|=1} \mathbf{e}(E, B_r(x), \nu).$$

Indeed, the crucial result of De Giorgi's regularity theory for perimeter minimizers in $\mathbb{R}^n$ is the following *excess decay lemma*: there exists a critical threshold $\varepsilon_0 > 0$ such that, if E is a perimeter minimizer in an open set $\Omega \subset \mathbb{R}^n$ and $x \in \partial E$, then

$$\operatorname{Exc}(E, B_r(x)) < \varepsilon_0 \implies \operatorname{Exc}(E, B_{\alpha r}(x)) \leq \frac{1}{2} \operatorname{Exc}(E, B_r(x)), \tag{0.2}$$

for some $\alpha \in (0, 1)$ sufficiently small. In fact, by *Step 2*, the renormalized Lipschitz approximations tend to a harmonic function, and the well-known decay property of harmonic functions leads to (0.2).

By a standard iteration scheme, the excess decay (0.2) shows that the unit normal ν_E is Hölder continuous, which in turn implies that the boundary ∂E is locally the graph of a $C^{1,\gamma}$ function for some $\gamma \in (0, 1)$.

At the present stage of the theory, the decay estimate (0.2) is not available for H-perimeter minimizers in the Heisenberg group, since the harmonic nature of the limit function ψ in *Step 2* has not been established yet. However, it is known that the continuity of the normal ν_E implies that the boundary of the H-perimeter minimizer is a C^1_H-regular surface in the sense of [**37**], see [**53**, Theorem 1.2].

Step 4: Schauder-type regularity. The fourth and last step in the regularity theory for perimeter minimizers in $\mathbb{R}^n$ is the smoothness of the minimal boundary. Indeed, by *Step 3*, the boundary of a perimeter minimizer in $\mathbb{R}^n$ is locally the graph of a $C^{1,\gamma}$ function g. Since, by the minimality of E, g solves the minimal surface equation in the weak sense, one eventually gets the smoothness of g by the regularity theory for quasilinear elliptic equations (Schauder's estimates).

In the context of the Heisenberg group, it is an open problem to deduce further regularity properties for an intrinsic Lipschitz function $\varphi\colon D \to \mathbb{R}$ on an open set $D \subset \mathbb{W}$ under the sole hypothesis that φ minimizes the intrinsic area functional,

$$A(\varphi) = \int_D \sqrt{1 + |\nabla^{\varphi}\varphi|^2}\, d\mathcal{L}^{2n}.$$

In fact, it is not even clear how to prove that φ solves the intrinsic minimal surface equation

$$\nabla^{\varphi} \cdot \left(\frac{\nabla^{\varphi}\varphi}{\sqrt{1 + |\nabla^{\varphi}\varphi|^2}} \right) = 0 \quad \text{in } D \subset \mathbb{W}. \tag{0.3}$$

Indeed, the first variation of the area functional can be performed only if φ is sufficiently regular, see [**52**]. In addition, formulas for the first and second variation of the H-perimeter have been recently established, see [**36**], but they can be computed only along special *contact flows*, which cannot be used to variate the area functional in the usual way. On the other hand, Euclidean Lipschitz continuous vanishing viscosity solutions of the minimal surface equation (0.3) are known to be Hölder continuous in $\mathbb{H}^1$ and smooth in $\mathbb{H}^n$ for all $n \geq 2$, see [**11**, **12**].

Content of the thesis. In this thesis, we prove two new intrinsic Lipschitz approximation theorems for H-perimeter minimizers in the setting of the Heisenberg group $\mathbb{H}^n$ with $n \geq 2$.

Improved Lipschitz approximation. The first result is an improvement of Theorem 0.1 and is the natural reformulation in $\mathbb{H}^n$ of the classical Lipschitz approximation in $\mathbb{R}^n$, see [**46**, Theorem 23.7]. Here the disk $D_r \subset \mathbb{W}$ is induced by the restriction of the box norm of $\mathbb{H}^n$ to $\mathbb{W}$ and the cylinder $C_r(p)$, $p \in \mathbb{H}^n$, is defined as $C_r(p) = p * C_r$, where $C_r = D_r * (-r, r)$ (see Chapter 1 for precise definitions).

THEOREM 0.2. *Let $n \geq 2$. There exist positive dimensional constants $C_1(n)$, $\varepsilon_1(n)$ and $\delta_1(n)$ with the following property. If $E \subset \mathbb{H}^n$ is an H-perimeter minimizer in the cylinder C_{642} with $0 \in \partial E$ and $\mathbf{e}(E, C_{642}, \nu) \leq \varepsilon_1(n)$ then, setting for brevity*

$$M = C_1 \cap \partial E, \qquad M_0 = \left\{ q \in M : \sup_{0<s<64} \mathbf{e}(E, C_s(q), \nu) \leq \delta_1(n) \right\},$$

there exists an intrinsic Lipschitz function $\varphi\colon \mathbb{W} \to \mathbb{R}$ such that

$$\sup_{\mathbb{W}} |\varphi| \leq C_1(n)\, \mathbf{e}(E, C_{642}, \nu)^{\frac{1}{2(2n+1)}}, \quad \mathrm{Lip}_H(\varphi) \leq 1,$$

$$M_0 \subset M \cap \Gamma, \qquad \Gamma = \mathrm{gr}(\varphi|_{D_1}),$$

$$\mathcal{S}^{2n+1}(M \triangle \Gamma) \leq C_1(n)\, \mathbf{e}(E, C_{642}, \nu),$$
$$\int_{D_1} |\nabla^{\varphi} \varphi|^2 \, d\mathcal{L}^{2n} \leq C_1(n)\, \mathbf{e}(E, C_{642}, \nu).$$

Theorem 0.2 holds also for (Λ, r_0)-minimizers of H-perimeter, see the more general formulation of this result given in Theorem 2.2 of Chapter 2.

The proof of Theorem 0.2 is based on the ideas outlined in [**46**, Section 23.3] and goes as follows.

The first step is to prove that the natural projection $\pi\colon \mathbb{H}^n \to \mathbb{W}$ is invertible on the set $M_0 \subset \partial E$. This is a consequence of a recent result established in [**54**, Theorem 1.3], which gives a uniform control on the flatness of the boundary of the minimizer depending on the smallness of the excess. The inverse of π defines an intrinsic Lipschitz function on $\pi(M_0)$ that can be extended to the whole $\mathbb{W}$.

The second step is the approximation in measure of the boundary. This is done by estimating the terms $M \setminus \Gamma$ and $\Gamma \setminus M$ separately: the first can be controlled by a covering argument, while the second is a consequence of the area formula for intrinsic Lipschitz functions.

Finally, the third step is to prove that the intrinsic L^2-energy of the approximating function is controlled by the excess. This follows from the approximation in measure of the boundary and again from the area formula estimating the L^2-norm of the intrinsic gradient on the two sets $\pi(M \cap \Gamma)$ and $\pi(M \triangle \Gamma)$ separately.

Approximation via maximal functions. The second result is an adaptation of the ideas developed in [**24**, **25**] by De Lellis and Spadaro for area minimizing integral currents to the setting of H-perimeter minimizers in $\mathbb{H}^n$.

THEOREM 0.3. *Let $n \geq 2$ and $\alpha \in (0, \frac{1}{2})$. There exist positive constants $C_2(n)$, $\varepsilon_2(\alpha, n)$ and $k_2 = k_2(n)$ with the following property. Let $E \subset \mathbb{H}^n$ be an H-perimeter minimizer in the cylinder C_{k_2} with $0 \in \partial E$ and $\mathbf{e}(E, C_{k_2}, \nu) \leq \varepsilon_2(\alpha, n)$. Then there exist a set $K \subset D_1$ such that*

$$\mathcal{L}^{2n}(D_1 \setminus K) \leq C_2(n)\, \mathbf{e}(E, C_{k_2}, \nu)^{1-2\alpha}$$

and an intrinsic Lipschitz function $\varphi \colon \mathbb{W} \to \mathbb{R}$ with the following properties:

$$\mathrm{gr}(\varphi|_K) = \partial E \cap \Big(K * (-1, 1)\Big),$$

$$\mathrm{Lip}_H(\varphi) \leq C_2(n)\, \mathbf{e}(E, C_{k_2}, \nu)^{\alpha},$$

$$\mathcal{S}^{2n+1}\Big((\partial E \bigtriangleup \mathrm{gr}(\varphi)) \cap C_1\Big) \leq C_2(n)\, \mathbf{e}(E, C_{k_2}, \nu)^{1-2\alpha},$$

$$\int_{D_1} |\nabla^{\varphi}\varphi|^2 \, d\mathcal{L}^{2n} \leq C_2(n)\, \mathbf{e}(E, C_{k_2}, \nu).$$

Theorem 0.3 holds also for (Λ, r_0)-minimizers of H-perimeter, see the more general formulation of this result given in Corollary 3.2 of Chapter 3.

The proof of Theorem 0.3 essentially follows the scheme outlined in [**24**, **25**], although with a different starting point.

The first step in [**24**, **25**] is to establish a so-called *BV estimate* on the vertical slices of the area minimizing integral current, see [**24**, Proposition 2.1] and [**25**, Lemma A.1]. The proof of this estimate heavily uses several fundamental results of the theory of integral currents in $\mathbb{R}^n$. At the present stage of the theory, the development of a general theory for integral currents in $\mathbb{H}^n$ is a completely open problem, see [**39**], and a similar estimate for the slices of the boundary of an H-perimeter minimizer cannot be easily implemented.

However, in the special case the minimizer is the intrinsic epigraph of an intrinsic Lipschitz function, the *BV estimate* becomes an easy consequence of the Cauchy–Schwarz inequality and of the area formula. Therefore, in the general case E is an H-perimeter minimizer, we can overcome this initial problem with the following trick: first, by Theorem 0.2, we can approximate the boundary of E with the intrinsic graph of a suitable intrinsic Lipschitz function; second, up to an error which is comparable to the excess, we can replace the *BV estimate* on the slices of the boundary of E with the *BV estimate* on the slices of the approximating graph.

This idea allows us to recover De Lellis and Spadaro's approach in the setting of the Heisenberg group $\mathbb{H}^n$ with $n \geq 2$. The proof of Theorem 0.3 thus goes as follows.

The first step is to define the coincidence set K and to estimate the Lebesgue measure of $D_1 \setminus K$. This is done by a standard argument (see *Claim* #1 in the proof of [**31**, Theorem 6.12]), proving an estimate for the (local) maximal function of a suitable measure μ.

Because of our initial trick for the *BV estimate*, the measure μ depends both on the excess of E and on the excess of the graph of the approximation given by Theorem 0.2. Thus our estimate on the Lebesgue measure of $D_1 \setminus K$ is weaker than the one established in [**24**, **25**], although it catches the correct power of the excess. However, the initial trick for the *BV estimate* is not necessary when E is the epigraph of an intrinsic Lipschitz function. In fact, in this particular case, our estimate on the Lebesgue measure of $D_1 \setminus K$ is the exact counterpart of the one established in [**24**, **25**].

The second step is to estimate the intrinsic Lipschitz constant. At this point, the relevant part of φ is the one defined on the set K. Therefore, up to redefine the approximation given by Theorem 0.2 outside K, it is enough to control the Lipschitz constant only on the set K. But this is a standard fact (see *Claim* #2 in the proof of [**31**, Theorem 6.12]), up to some technicalities due to the intrinsic φ-balls appearing in the Poincaré inequality of [**16**].

The third and last step is to prove the approximation in measure of the boundary and the estimate on the intrinsic L^2-energy. This is done similarly as before, taking into account the information on the coincidence set K and on the Lipschitz constant.

Acknowledgements. With my deepest gratitude, I thank my supervisor Professor Roberto Monti for being a patient and sincere guide during the realization of this work and for the precious help he gave me during these last five years. I also thank my co-advisor Professor Luigi Ambrosio for his careful and encouraging analysis of my thesis. A special thank goes to Professor Pier Domenico Lamberti, for the opportunity he gave me to explore some “strange” features of Sobolev Spaces, and to Professors Martino Bardi, Giovanni Colombo, Valentino Magnani, Andrea Marson, Francescopaolo Montefalcone, Pietro Polesello, Franco Rampazzo and Davide Vittone, for their valuable teachings and for the enjoyable talks I had with them. I am also indebted to Sebastiano Don and Valentina Franceschi for sharing their ideas on the Heisenberg group with me. My warmest thanks to my family for supporting me always and unconditionally, giving me the opportunity to achieve this important goal. Last but not least, I wish to thank all the friends I had the chance to meet here in Padua.

CHAPTER 1

Preliminaries

1. The Heisenberg group

1.1. Group structure. Let $n \in \mathbb{N}$ and let $(\mathbb{H}^n, *)$ be the n-dimensional Heisenberg group. The group $\mathbb{H}^n$ is the set $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ with group law $*\colon \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}^n$ defined as

$$(z,t) * (w,s) = (z + w, t + s + P(z,w)) \quad \forall (z,t),(w,s) \in \mathbb{H}^n,$$

where $P\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$ is the (antisymmetric) bilinear form

$$P(z,w) = 2\,\mathrm{Im}\left(\sum_{j=1}^{n} z_j \bar{w}_j\right) \quad \forall z,w \in \mathbb{C}^n, \tag{1.1}$$

see [**60**, Chapter 12, 13] and [**13**].

The automorphisms $\delta_\lambda\colon \mathbb{H}^n \to \mathbb{H}^n$, $\lambda > 0$, of the form

$$\delta_\lambda(z,t) = (\lambda z, \lambda^2 t), \quad (z,t) \in \mathbb{H}^n,$$

are called *dilations*. We use the abbreviations $\lambda p = \delta_\lambda(p)$ and $\lambda E = \delta_\lambda(E)$ for $p \in \mathbb{H}^n$ and $E \subset \mathbb{H}^n$. We also define the left *translations* $\tau_q\colon \mathbb{H}^n \to \mathbb{H}^n$

$$\tau_q(p) = q * p, \quad p,q \in \mathbb{H}^n,$$

and the *rotations*

$$(z,t) \mapsto (Rz, t), \quad (z,t) \in \mathbb{H}^n, \quad \text{with } R \in U(n).$$

1.2. Lie algebra. We identify an element $z = x + iy \in \mathbb{C}^n$ with $(x,y) \in \mathbb{R}^{2n}$. The Lie algebra of left invariant vector fields in $\mathbb{H}^n$ is spanned by the vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}, \quad j = 1,\dots,n, \tag{1.2}$$

and the only non-trivial commutator relations are

$$[X_j, Y_j] = -4T, \quad j = 1,\dots,n.$$

We denote by H the *horizontal sub-bundle* of $T\mathbb{H}^n$. Namely, for any $p = (z,t) \in \mathbb{H}^n$, we let

$$H_p = \mathrm{span}\{X_1(p),\dots,X_n(p),Y_1(p),\dots,Y_n(p)\} \equiv \mathbb{R}^{2n}.$$

1.3. Metric structure. For any $p = (z,t) \in \mathbb{H}^n$, we let $\|p\|_\infty = \max\Big\{|z|, |t|^{1/2}\Big\}$ be the *box norm*. The box norm satisfies the triangle inequality

$$\|p * q\|_\infty \leq \|p\|_\infty + \|q\|_\infty \quad \forall p, q \in \mathbb{H}^n.$$

Moreover, the function $d_\infty \colon \mathbb{H}^n \times \mathbb{H}^n \to [0,\infty)$, $d(p,q) = \|p^{-1} * q\|$ for all $p, q \in \mathbb{H}^n$, is a left invariant distance on $\mathbb{H}^n$ equivalent to the Carnot–Carathéodory distance. In particular, left translations and rotations are isometries of $\mathbb{H}^n$ with the distance d_∞. Using the distance d_∞, we define the open ball centred at $p \in \mathbb{H}^n$ and with radius $r > 0$ the set

$$B_r(p) = \{q \in \mathbb{H}^n : d_\infty(q,p) < r\} = p * \{q \in \mathbb{H}^n : \|q\|_\infty < r\}. \tag{1.3}$$

In the case $p = 0$, we let $B_r = B_r(0)$.

For any $s \geq 0$, we denote by $\mathcal{S}^s$ the spherical Hausdorff measure in $\mathbb{H}^n$ constructed with the left invariant metric d_∞. Namely, for any $E \subset \mathbb{H}^n$ we let

$$\mathcal{S}^s(E) = \lim_{\delta \to 0} \mathcal{S}^s_\delta(E)$$

where

$$\mathcal{S}^s_\delta(E) = \inf\left\{\sum_{n\in\mathbb{N}} (\operatorname{diam} B_i)^s : E \subset \bigcup_{n\in\mathbb{N}} B_i,\ B_i \text{ balls as in (1.3)},\ \operatorname{diam} B_i < \delta\right\}$$

and diam is the diameter in the distance d_∞. By the Carathéodory's construction, $E \mapsto \mathcal{S}^s(E)$ is a Borel measure in $\mathbb{H}^n$. When $s = 2n+2$, $\mathcal{S}^{2n+2}$ turns out to be the Lebesgue measure $\mathcal{L}^{2n+1}$ up to a multiplicative constant. Thus, the correct dimension to measure hypersurfaces is $s = 2n+1$ (see also Theorem 1.2 below).

1.4. Sub-Riemmanian structure. Let g be the left invariant Riemannian metric on $\mathbb{H}^n$ that makes orthonormal the vector fields $X_1, \dots, X_n, Y_1, \dots, Y_n, T$ defined in (1.2). The metric g induces a volume form on $\mathbb{H}^n$ that is left invariant. Also the Lebesgue measure $\mathcal{L}^{2n+1} = dzdt$ on $\mathbb{H}^n$ is left invariant and thus, by the uniqueness of the Haar measure, the volume induced by g is the Lebesgue measure $\mathcal{L}^{2n+1}$ (with proportionality constant 1). For tangent vectors $V, W \in T\mathbb{H}^n$, we let

$$\langle V, W\rangle_g = g(V,W) \quad \text{and} \quad |V|_g = g(V,V)^{1/2}.$$

Let $\Omega \subset \mathbb{H}^n$ be an open set. A *horizontal section* $V \in C^1_c(\Omega; H)$ is a vector field of the form

$$V = \sum_{j=1}^n V_j X_j + V_{j+n} Y_j,$$

where $V_j \in C^1_c(\Omega)$ for any $j = 1, \dots, 2n$, that is, each coordinate V_j of the vector field V is a continuously differentiable function with compact support contained in Ω. The sup-norm with respect to g of a horizontal section $V \in C^1_c(\Omega; H)$ is

$$\|V\|_g = \max_{p\in\Omega} |V(p)|_g.$$

The *horizontal divergence* of V is

$$\operatorname{div}_H V = \sum_{j=1}^n X_j V_j + Y_j V_{j+n}.$$

2. Locally finite perimeter sets

2.1. H-perimeter, inner normal. A $\mathcal{L}^{2n+1}$-measurable set $E \subset \mathbb{H}^n$ has *locally finite H-perimeter* (or is an *H-Caccioppoli set*) in an open set $\Omega \subset \mathbb{H}^n$ if there exists a H-valued Radon measure μ_E on Ω, called *Gauss–Green measure* of E, such that

$$\int_E \mathrm{div}_H V \; d\mathcal{L}^{2n+1} = - \int_\Omega \langle V, d\mu_E \rangle_g$$

for all $V \in C^1_c(\Omega; H)$. We denote by $|\mu_E|$ the total variation of μ_E. If $|\mu_E|(\Omega) < \infty$, we say that E has *finite H-perimeter* in Ω. We also use the notation

$$P_H(E; B) = |\mu_E|(B),$$

for any Borel set $B \subset \Omega$, to denote the *H-perimeter* of E in B. When $B = \mathbb{H}^n$, we write $P_H(E) = P_H(E; \mathbb{H}^n)$. We have

$$P_H(E; \Omega) = \sup\Big\{ \int_E \mathrm{div}_H V \, d\mathcal{L}^{2n+1} : \; V \in C^1_c(\Omega; H), \; \|V\|_g \le 1 \Big\}.$$

By the Radon-Nykodim Theorem (or, equivalently, by the Riesz representation Theorem), there exists a $|\mu_E|$-measurable function $\nu_E \colon \Omega \to H$ such that $|\nu_E|_g = 1$ $|\mu_E|$-a.e. and $\mu_E = \nu_E \, |\mu_E|$. Moreover, the *Gauss–Green formula*

$$\int_E \mathrm{div}_H V \; d\mathcal{L}^{2n+1} = - \int_\Omega \langle V, \nu_E \rangle_g \; d|\mu_E|$$

holds for any $V \in C^1_c(\Omega; H)$. We call ν_E the *horizontal inner normal* of E in Ω.

2.2. Reduced boundary. The *measure theoretic boundary* of a $\mathcal{L}^{2n+1}$-measurable set $E \subset \mathbb{H}^n$ is the set

$$\partial E = \Big\{ p \in \mathbb{H}^n : \mathcal{L}^{2n+1}(E \cap B_r(p)) > 0 \text{ and } \mathcal{L}^{2n+1}(B_r(p) \setminus E) > 0 \text{ for all } r > 0 \Big\}.$$

Let E be a set with locally finite H-perimeter. Then the H-perimeter measure μ_E of E is concentrated on ∂E and, actually, on a subset $\partial^* E$ of ∂E, called the *reduced boundary* of E, see [**37**, Definition 2.17]

Definition 1.1 (Reduced boundary). The *reduced boundary* of a set $E \subset \mathbb{H}^n$ with locally finite H-perimeter is the set $\partial^* E$ of all points $p \in \mathbb{H}^n$ such that the following three conditions hold:

(1) $|\mu_E|(B_r(p)) > 0$ for all $r > 0$;
(2) we have

$$\lim_{r \to 0} \frac{1}{|\mu_E|(B_r(p))} \int_{B_r(p)} \nu_E \; d|\mu_E| = \nu_E(p);$$

(3) there holds $|\nu_E(p)| = 1$.

We always have the inclusion $\partial^* E \subset \partial E$; this follows from the Structure Theorem for sets with locally finite H-perimeter, see Theorem 1.2 below. Actually, the difference $\partial E \setminus \partial^* E$ is $\mathcal{S}^{2n+1}$-negligible, see [**50**, Lemma 2.4]. Moreover, up to modifying E on a Lebesgue negligible set, one can always assume that ∂E coincides with the topological boundary of E, see [**58**, Proposition 2.5].

2.3. Structure Theorem. The following theorem is a deep result concerning the structure of the reduced boundary of a set with locally finite H-perimeter in $\mathbb{H}^n$. It is the natural counterpart in $\mathbb{H}^n$ of the classical De Giorgi's Structure Theorem in the Euclidean setting (we refer to [**46**, Theorem 15.9], [**31**, Theorem 5.15] and [**3**, Teorema 3.7]) and was proved in [**37**]. It asserts that the reduced boundary has the structure of a 'generalized' H-regular hypersurface.

THEOREM 1.2 (Structure Theorem). *If E is a set with locally finite H-perimeter in $\mathbb{H}^n$, then $\partial^* E$ is H-rectifiable, that is, there exist countably many H-regular hypersurfaces M_h in $\mathbb{H}^n$, compact sets $K_h \subset M_h$ and a set F with $\mathcal{S}^{2n+1}(F) = 0$, such that*

$$\partial^* E = F \cup \bigcup_{h \in \mathbb{N}} K_h,$$

and, for every $p \in K_h$, $\nu_E(p)^\perp = T_p^H M_h$, the H-tangent space to M_h at p. Moreover, the Gauss-Green measure μ_E of E satisfies

$$\mu_E = \nu_E \, |\mu_E|, \qquad |\mu_E| = \frac{2\omega_{2n-1}}{\omega_{2n+1}} \, \mathcal{S}^{2n+1} \llcorner \partial^* E,$$

and the generalized Gauss–Green formula holds true:

$$\int_E \operatorname{div}_H V \, d\mathcal{L}^{2n+1} = -\frac{2\omega_{2n-1}}{\omega_{2n+1}} \int_{\partial^* E} \langle V, \nu_E \rangle_g \, d\mathcal{S}^{2n+1},$$

for any $V \in C_c^1(\mathbb{H}^n; H)$.

3. Perimeter minimizers

3.1. Minimizers, scaling, density estimates. Let $\Omega \subset \mathbb{H}^n$ be an open set and let E be a set with locally finite H-perimeter in $\mathbb{H}^n$.

DEFINITION 1.3 ((Λ, r_0)-minimizer). We say that the set E is a (Λ, r_0)-*minimizer of H-perimeter* in Ω if there exist two constants $\Lambda \in [0, \infty)$ and $r_0 \in (0, \infty]$ such that

$$P(E; B_r(p)) \le P(F; B_r(p)) + \Lambda \, \mathcal{L}^{2n+1}(E \triangle F)$$

for any measurable set $F \subset \mathbb{H}^n$, $p \in \Omega$ and $r < r_0$ such that $E \triangle F \subset\subset B_r(p) \subset\subset \Omega$.

When $\Lambda = 0$ and $r_0 = \infty$, we say that the set E is a *locally H-perimeter minimizer* in Ω, that is, there holds

$$P(E; B_r(p)) \le P(F; B_r(p))$$

for any measurable set $F \subset \mathbb{H}^n$, $p \in \Omega$ and $r > 0$ such that $E \triangle F \subset\subset B_r(p) \subset\subset \Omega$.

REMARK 1.4 (Scaling of (Λ, r_0)-minimizer). If the set E is a (Λ, r_0)-minimizer of H-perimeter in the open set $\Omega \subset \mathbb{H}^n$ then, for every $p \in \mathbb{H}^n$ and $r > 0$, the blow-up $E_{p,r} = \delta_{\frac{1}{r}}(\tau_{p^{-1}}(E))$ of E is a (Λ', r_0')-minimizer of H-perimeter in $\Omega_{p,r}$, where $\Lambda' = \Lambda r$ and $r_0' = r_0/r$. In particular, the product Λr_0 is invariant under blow-up. Thus it is convenient to assume that $\Lambda r_0 \le 1$, as we shall always do in the following.

The following result is in [**54**, Appendix A]. Its proof is a straightforward adaptation of that for (Λ, r_0)-minimizers in $\mathbb{R}^n$, see [**46**, Chapter 21].

Theorem 1.5 (Density estimates). *There are positive dimensional constants $k_1(n)$, $k_2(n)$, $k_3(n)$ and $k_4(n)$ with the following property. If $E \subset \mathbb{H}^n$ is a (Λ, r_0)-minimizer of H-perimeter in an open set $\Omega \subset \mathbb{H}^n$ with*

$$\Lambda r_0 \leq 1, \qquad p \in \partial E \cap \Omega, \qquad B_r(p) \subset \Omega, \qquad r < r_0,$$

then

$$k_1(n) \leq \frac{\mathcal{L}^{2n+1}(E \cap B_r(p))}{r^{2n+2}} \leq k_2(n), \qquad k_3(n) \leq \frac{P(E; B_r(p))}{r^{2n+1}} \leq k_4(n).$$

In particular, $\mathcal{S}^{2n+1}(\Omega \cap (\partial E \setminus \partial^* E)) = 0$.

4. Cylindrical excess

4.1. Height function, disks, cylinders. The *height function* $\mathfrak{h} \colon \mathbb{H}^n \to \mathbb{R}$ is the group homomorphism defined by $\mathfrak{h}(p) = x_1$ for every $p = (x, y, t) \in \mathbb{H}^n$. We let $\mathbb{W}$ be the (normal) subgroup of $\mathbb{H}^n$ given by the kernel of $\mathfrak{h}$,

$$\mathbb{W} := \ker \mathfrak{h} = \Big\{ p \in \mathbb{H}^n : \mathfrak{h}(p) = 0 \Big\}.$$

The *open disk* in $\mathbb{W}$ of radius $r > 0$ centred at the origin induced by the box norm is the set $D_r = \{w \in \mathbb{W} : \|w\|_\infty < r\}$. For any $p \in \mathbb{W}$, we let $D_r(p) = p * D_r \subset \mathbb{W}$. Note that, for all $p \in \mathbb{W}$ and $r > 0$,

$$\mathcal{L}^{2n}(D_r(p)) = \mathcal{L}^{2n}(D_r) = \kappa_n r^{2n+1}, \tag{1.4}$$

where we set $\kappa_n = \mathcal{L}^{2n}(D_1)$.

The *open cylinder* with central section D_r and height $2r$ is the set

$$C_r = D_r * (-r, r) := \{ w * s\mathrm{e}_1 \in \mathbb{H}^n : w \in D_r, \ s \in (-r, r) \},$$

where $s\mathrm{e}_1 = (s, 0, \dots, 0) \in \mathbb{H}^n$. For any $p \in \mathbb{H}^n$, we let $C_r(p) = p * C_r$.

We let $\pi \colon \mathbb{H}^n \to \mathbb{W}$ be the projection on $\mathbb{W}$ defined, for any $p \in \mathbb{H}^n$, by the formula

$$p = \pi(p) * \mathfrak{h}(p)\mathrm{e}_1. \tag{1.5}$$

By (1.5), for any $p \in \mathbb{H}^n$ and $r > 0$, we have

$$p \in C_r \iff \pi(p) \in D_r, \ \mathfrak{h}(p) \in (-r, r) \iff \|\pi(p)\|_\infty < r, \ |\mathfrak{h}(p)| < r.$$

We thus let $\|\cdot\|_C \colon \mathbb{H}^n \to [0, \infty)$ be the map

$$\|p\|_C := \max\Big\{ \|\pi(p)\|_\infty, |\mathfrak{h}(p)| \Big\} \tag{1.6}$$

for any $p \in \mathbb{H}^n$, so that $C_r = \{p \in \mathbb{H}^n : \|p\|_C < r\}$. The map $\|\cdot\|_C$ is a quasi norm and, by (1.5), we have

$$\|p\|_C \leq \|p\|_\infty, \quad \|p\|_\infty \leq 2\|p\|_C \quad \forall p \in \mathbb{H}^n. \tag{1.7}$$

We let $d_C \colon \mathbb{H}^n \times \mathbb{H}^n \to [0, \infty)$ be the quasi distance induced by $\|\cdot\|_C$. By (1.7), the cylinder $C_r(p)$ is comparable with the ball $B_r(p)$ induced by the box norm for any $p \in \mathbb{H}^n$. Namely, we have

$$B_r(p) \subset C_r(p) \subset B_{2r}(p) \ \text{ for all } p \in \mathbb{H}^n, \ r > 0. \tag{1.8}$$

4.2. Cylindrical excess. A concept which plays a key role in the regularity theory of (Λ, r_0)-minimizers of H-perimeter is the notion of excess.

Definition 1.6 (Cylindrical excess). Let E be a set with locally finite H-perimeter in $\mathbb{H}^n$. The *cylindrical excess* of E at the point $p \in \partial E$, at the scale $r > 0$ and with respect to the direction $\nu = -X_1$, is defined as

$$\begin{aligned}\mathbf{e}(E,p,r,\nu) &:= \frac{1}{r^{2n+1}} \int_{C_r(p)} \frac{|\nu_E - \nu|_g^2}{2}\, d|\mu_E| \\ &= \frac{\delta(n)}{r^{2n+1}} \int_{C_r(p)\cap\partial^* E} \frac{|\nu_E - \nu|_g^2}{2}\, d\mathcal{S}^{2n+1} \\ &= \frac{\delta(n)}{r^{2n+1}} \int_{C_r(p)\cap\partial^* E} \Big(1 - \langle \nu_E, \nu\rangle_g\Big)\, d\mathcal{S}^{2n+1}\end{aligned} \tag{1.9}$$

where μ_E is the Gauss-Green measure of E, ν_E is the horizontal inner normal and the multiplicative constant is $\delta(n) = \frac{2\omega_{2n-1}}{\omega_{2n+1}}$ as in Theorem 1.2.

In other words, $\mathbf{e}(E,p,r,\nu)$ is the L^2-averaged oscillation from the given direction ν of the inner unit normal to E over the cylinder $C_r(p)$. We shall need to quantify the geometric consequences of the smallness of the cylindrical excess on (Λ, r_0)-perimeter minimizers. For the sake of brevity, we will often set $\mathbf{e}(p,r) = \mathbf{e}(E,p,r,\nu)$ and, in the case $p = 0$, $\mathbf{e}(r) = \mathbf{e}(0,r)$.

We recall some basic properties of the cylindrical excess. Their proofs are easy adaptations of those for the classical excess, see [**46**, Chapter 22].

Lemma 1.7 (Elementary properties of excess). *Let E is a set with locally finite H-perimeter in $\mathbb{H}^n$ and let $p \in \partial E$. If $r > s > 0$, then*

$$\mathbf{e}(E,p,s,\nu) \le \left(\frac{r}{s}\right)^{2n+1} \mathbf{e}(E,p,r,\nu). \tag{1.10}$$

Moreover, the excess is invariant under blow-up, i.e.,

$$\mathbf{e}(E,p,r,\nu) = \mathbf{e}(E_{p,r},0,1,\nu), \tag{1.11}$$

where $E_{p,r} = \delta_{\frac{1}{r}}(\tau_{p^{-1}}(E))$.

5. Height bound

5.1. Main result. The following result is a fundamental estimate relating the height of the boundary of a (Λ, r_0)-minimizer of H-perimeter with the cylindrical excess, see [**54**, Theorem 1.3].

Theorem 1.8 (Height bound). *Given $n \ge 2$, there exist positive dimensional constants $\varepsilon_0(n)$ and $C_0(n)$ with the following property. If E is a (Λ, r_0)-minimizer of H-perimeter in the cylinder C_{16r_0} with*

$$\Lambda r_0 \le 1, \quad 0 \in \partial E, \quad \mathbf{e}(16r_0) \le \varepsilon_0(n),$$

then

$$\sup\left\{\frac{|\not{z}(p)|}{r_0} : p \in C_{r_0} \cap \partial E\right\} \le C_0(n)\, \mathbf{e}(16r_0)^{\frac{1}{2(2n+1)}}. \tag{1.12}$$

REMARK 1.9. The estimate (1.12) does not hold when $n = 1$. In fact, there are sets $E \subset \mathbb{H}^1$ such that $\mathbf{e}(E, 0, r, \nu) = 0$ but ∂E is not flat in $C_{\varepsilon r}$ for any $\varepsilon > 0$, see the conclusion of [**50**, Proposition 3.7].

5.2. Lemmata on the excess. The proof of Theorem 1.8 relies on a slicing formula for intrinsic rectifiable sets and on two lemmata on the excess.

The slicing formula is rather technical and has a non-trivial character, because the domain of integration and its slices need not to be rectifiable in the standard sense. We do not state the result here and we refer the interested reader to [**54**, Theorem 1.5].

The two lemmata on the excess are the natural reformulation of the corresponding lemmata in the Euclidean setting, see [**46**, Chapter 22].

The first lemma shows that, if the excess of a (Λ, r_0)-minimizer of H-perimeter E is sufficiently small, then its reduced boundary $\partial^* E$ lies in a strip with controlled thickness and, possibly modifying E on a $\mathcal{L}^{2n+1}$-negligible set if necessary, E is positioned under that strip.

LEMMA 1.10 (Small-excess position, [**54**, Lemma 3.3]). *Let $n \geq 2$. For any $s \in (0, 1)$, $\Lambda \in [0, \infty)$ and $r_0 \in (0, \infty]$ with $\Lambda r_0 \leq 1$, there exists a constant $\omega(n, s, \Lambda, r_0) > 0$ with the following property. If E is a (Λ, r_0)-minimizer of H-perimeter in the cylinder C_2, $0 \in \partial E$ and*

$$\mathbf{e}(2) \leq \omega(n, s, \Lambda, r_0),$$

then

$$|\mathfrak{h}(p)| < s \quad \text{for any } p \in C_1 \cap \partial E, \tag{1.13}$$

$$\mathcal{L}^{2n+1}\Big(\Big\{p \in C_1 \cap E : \mathfrak{h}(p) > s\Big\}\Big) = 0, \tag{1.14}$$

$$\mathcal{L}^{2n+1}\Big(\Big\{p \in C_1 \setminus E : \mathfrak{h}(p) < -s\Big\}\Big) = 0. \tag{1.15}$$

The second lemma combines the divergence theorem with the geometric information gathered in the previous result. To state it, we need some preliminary notation.

For any set $E \subset \mathbb{H}^n$ and for any $s \in \mathbb{R}$, we define

$$E^s = E \cap \mathfrak{h}^{-1}(s)$$

the *vertical slice of E at height $s \in \mathbb{R}$* and

$$E_s := \pi(E^s) = \{w \in \mathbb{W} : w * s\mathrm{e}_1 \in E\}.$$

the *projection* of E on $\mathbb{W}$.

LEMMA 1.11 (Excess measure, [**54**, Lemma 3.4, Corollary 3.5]). *Let $n \geq 2$. Let E be a set of locally finite H-perimeter in $\mathbb{H}^n$ with $0 \in \partial E$ and such that, for some $s_0 \in (0, 1)$, (1.13), (1.14) and (1.15) of Lemma 1.10 hold. Then, for a.e. $s \in (-1, 1)$ and for any $\varphi \in C_c(D_1)$, setting for brevity $M = C_1 \cap \partial^* E$ and $M_s = M \cap \{\mathfrak{h} > s\}$, we have*

$$\int_{E_s \cap D_1} \varphi \, d\mathcal{L}^{2n} = -\int_{M_s} \varphi \circ \pi \, \langle \nu_E, X_1 \rangle_g \, d\mathcal{S}^{2n+1}.$$

In particular, for any Borel set $G \subset D_1$, we have

$$\mathcal{L}^{2n}(G) = -\int_{M\cap\pi^{-1}(G)} \langle \nu_E, X_1\rangle_g \, d\mathcal{S}^{2n+1}, \tag{1.16}$$

$$\mathcal{L}^{2n}(G) \le \mathcal{S}^{2n+1}\left(M \cap \pi^{-1}(G)\right). \tag{1.17}$$

Moreover, for a.e. $s \in (-1,1)$, there holds

$$0 \le \mathcal{S}^{2n+1}(M_s) - \mathcal{L}^{2n}(E_s \cap D_1) \le \mathbf{e}(1), \quad \mathcal{S}^{2n+1}(M) - \mathcal{L}^{2n}(D_1) = \mathbf{e}(1).$$

6. Intrinsic Lipschitz functions

6.1. Intrinsic graphs. We identify the vertical hyperplane

$$\mathbb{W} = \mathbb{H}^{n-1} \times \mathbb{R} = \{(z,t) \in \mathbb{H}^n : x_1 = 0\}$$

with $\mathbb{R}^{2n}$ via the coordinates $w = (x_2, \dots, x_n, y_1, \dots, y_n, t)$. The line flow of the vector field X_1 starting from the point $(z,t) \in \mathbb{W}$ is the solution of the Cauchy problem

$$\begin{cases} \dot\gamma(s) = X_1(\gamma(s)), \ s \in \mathbb{R} \\ \gamma(0) = (z,t), \end{cases}$$

that is,

$$\gamma(s) = \exp(sX_1)(z,t) = (z + s\mathrm{e}_1, t + 2y_1 s), \ s \in \mathbb{R}, \tag{1.18}$$

where $\mathrm{e}_1 = (1, 0, \dots, 0) \in \mathbb{H}^n$ and $z = (x,y) \in \mathbb{C}^n \equiv \mathbb{R}^{2n}$.

Let $W \subset \mathbb{W}$ be a set and let $\varphi\colon W \to \mathbb{R}$ be a function. The set

$$E_\varphi = \{\exp(sX_1)(w) \in \mathbb{H}^n : s > \varphi(w), \ w \in W\} \tag{1.19}$$

is called *intrinsic epigraph of* φ along X_1, while the set

$$\mathrm{gr}(\varphi) = \{\exp(\varphi(w)X_1)(w) \in \mathbb{H}^n : w \in W\}$$

is called *intrinsic graph of* φ along X_1.

By (1.18), we easily find the identity

$$\exp(\varphi(w)X_1)(w) = w * \varphi(w)\mathrm{e}_1 \quad \text{for any } w \in W,$$

thus the intrinsic graph of φ is the set

$$\mathrm{gr}(\varphi) = \{w * \varphi(w)\mathrm{e}_1 \in \mathbb{H}^n : w \in W\}.$$

We will use the following notation. We let $\Phi\colon W \to \mathbb{H}^n$, $\Phi(w) = w * \varphi(w)\mathrm{e}_1$ for all $w \in W$, be the *graph map* of the function $\varphi\colon W \to \mathbb{R}$, $W \subset \mathbb{W}$. For any $A \subset W$, we let $\mathrm{gr}(\varphi|_A) = \Phi(A)$.

6.2. Intrinsic Lipschitz functions. As above, we let $\mathrm{e}_1 = (1, 0 \dots, 0) \in \mathbb{H}^n$. Recall that, for any $p \in \mathbb{H}^n$, we have $p = \pi(p) * \mathfrak{h}(p)\mathrm{e}_1$ as in (1.5). We recall the definition of intrinsic cone introduced in [**38**, Definition 3.5]. The notion of cone is relevant in the theory of H-convex sets, see [**8**].

Definition 1.12 (Intrinsic cone with axis e_1). The open cone with vertex $p \in \mathbb{H}^n$, axis $\mathrm{e}_1 \in \mathbb{H}^n$ and aperture $\alpha \in (0, \infty]$, is the set

$$C(p,\alpha) = p * C(0,\alpha) := p * \left\{q \in \mathbb{H}^n : \|\pi(q)\|_\infty < \alpha|\mathfrak{h}(q)|\right\}.$$

We can now give the definition of intrinsic Lipschitz function. The notion of intrinsic Lipschitz function was introduced in [**38**, Definition 3.1].

DEFINITION 1.13 (Intrinsic Lipschitz function). Let $W \subset \mathbb{W}$ and let $\varphi \colon W \to \mathbb{R}$ be a function. The function φ is *L-intrinsic Lipschitz*, with $L \in [0, \infty)$, if

$$\operatorname{gr}(\varphi) \cap C(p, 1/L) = \varnothing \quad \text{for any } p \in \operatorname{gr}(\varphi),$$

or, equivalently, if

$$|\varphi(\pi(p)) - \varphi(\pi(q))| \le L \|\pi(q^{-1} * p)\|_\infty \quad \text{for any } p, q \in \operatorname{gr}(\varphi). \tag{1.20}$$

We let $\operatorname{Lip}_H(W)$ and $\operatorname{Lip}_{H,loc}(W)$ be the sets of globally and locally intrinsic Lipschitz functions on the set $W \subset \mathbb{W}$ respectively. If $\varphi \in \operatorname{Lip}_H(W)$, we let $\operatorname{Lip}_H(\varphi, W)$ be the intrinsic Lipschitz constant of φ on W (we will omit the set if there is no confusion).

A detailed analysis of the set $\operatorname{Lip}_H(W)$ can be found in [**15**, **40**]. It is important to note that $\operatorname{Lip}_H(W)$ is not a vector space, see [**58**, Remark 4.2]. However, the set of the intrinsic Lipschitz functions on W is a thick class of functions, for it holds

$$\operatorname{Lip}_{loc}(W) \subsetneq \operatorname{Lip}_{H,loc}(W) \subsetneq C^{1/2}_{loc}(W),$$

where Lip_{loc} and $C^{1/2}_{loc}$ are the spaces of locally Lipschitz and $\frac{1}{2}$-Hölder functions in the classical Euclidean sense respectively, see [**40**, Propositions 4.8 and 4.11].

6.3. Extension property. An extension theorem for intrinsic Lipschitz functions was proved for the first time in [**40**, Theorem 4.25]. The following result gives an explicit estimate of the Lipschitz constant of the extension. The first part is proved in [**50**, Proposition 4.8], while the second part follows from an easy modification of the proof of the first one.

PROPOSITION 1.14. *Let $W \subset \mathbb{W}$ and let $\varphi \colon W \to \mathbb{R}$ be an L-intrinsic Lipschitz function. There exists an M-intrinsic Lipschitz function $\psi \colon \mathbb{W} \to \mathbb{R}$ with*

$$M = \left(\sqrt{1 + \frac{1}{L + 2L^2}} - 1 \right)^{-2} \tag{1.21}$$

such that $\psi(w) = \varphi(w)$ for all $w \in W$. Moreover, if φ is bounded, then we can define the extension ψ such that ψ is bounded and $\|\psi\|_{L^\infty(\mathbb{W})} = \|\varphi\|_{L^\infty(W)}$.

Note that, in (1.21), we have $M \le 2L$ for all $L \le 0,07$.

6.4. Graph distance. The notion of intrinsic Lipschitz function can be equivalently reformulated on bounded open sets introducing a suitable notion of graph distance, see [**15**, Definition 1.1] or [**16**].

DEFINITION 1.15 (Graph distance). Let $W \subset \mathbb{W}$ be set and let $\varphi \colon W \to \mathbb{R}$ be a function. The map $d_\varphi \colon W \times W \to [0, \infty)$ given by

$$d_\varphi(w, w') = \frac{1}{2} \left(\left\| \pi\Big(\Phi(w)^{-1} * \Phi(w')\Big) \right\|_\infty + \left\| \pi\Big(\Phi(w')^{-1} * \Phi(w)\Big) \right\|_\infty \right) \tag{1.22}$$

for any $w, w' \in W$, where $\Phi(w) = w * \varphi(w)\mathrm{e}_1$ for all $w \in W$, is the *graph distance* induced by φ. Explicitly, for any $w = (z,t), w' = (z',t') \in \mathbb{W}$, we have

$$d_\varphi(w,w') = \frac{1}{2}\max\Big\{|z-z'|, \sigma_\varphi(w,w')\Big\} + \frac{1}{2}\max\Big\{|z-z'|, \sigma_\varphi(w',w)\Big\},$$

where

$$\sigma_\varphi(w,w') = |t - t' + 4\varphi(w)(y_1 - y_1') + P(w,w')|^{1/2}$$

and P is as in (1.1).

Comparing (1.20) with (1.22), it is easy to see that, if $W \subset \mathbb{W}$ is a bounded open set and $\varphi\colon W \to \mathbb{R}$ is a continuous function, then φ is an intrinsic L-intrinsic Lipschitz function if and only if

$$|\varphi(w) - \varphi(w')| \le L d_\varphi(w,w') \quad \forall w, w' \in W.$$

If φ is an intrinsic L-Lipschitz function on W, then d_φ turns out to be a quasi-distance on W, that is, $d_\varphi(x,y) = 0$ if and only if $x = y$ for all $x, y \in W$, d_φ is symmetric and, for all $x, y, z \in W$,

$$d_\varphi(x,y) \le c_L(d_\varphi(x,z) + d_\varphi(z,y)), \tag{1.23}$$

where $c_L \ge 1$ depends only on L and

$$\lim_{L \to 0} c_L = 1, \tag{1.24}$$

see [**15**, Section 3].

6.5. Intrinsic gradient. We now introduce a non-linear gradient for functions $\varphi\colon W \to \mathbb{R}$ with $W \subset \mathbb{W}$ an open set. We let $\mathscr{B}\colon \mathrm{Lip}_{loc}(W) \to L^\infty_{loc}(W)$ be the Burgers' operator defined by

$$\mathscr{B}\varphi = \frac{\partial \varphi}{\partial y_1} - 4\varphi\frac{\partial \varphi}{\partial t}.$$

When $\varphi \in C(W)$ is only continuous, we say that $\mathscr{B}\varphi$ exists in the sense of distributions and is represented by a locally bounded function if there exists a function $\vartheta \in L^\infty_{loc}(W)$ such that

$$\int_W \vartheta\psi \; dw = -\int_W \left\{\varphi\frac{\partial \psi}{\partial y_1} - 2\varphi^2\frac{\partial \psi}{\partial t}\right\} \; dw$$

for any $\psi \in C^1_c(W)$. In this case, we let $\mathscr{B}\varphi = \vartheta$.

Note that the vector fields $X_2, \dots, X_n, Y_2, \dots, Y_n$ can be naturally restricted to $\mathbb{W}$ and that they are self-adjoint.

Definition 1.16 (Intrinsic gradient). Let $\varphi\colon W \to \mathbb{R}$ be a continuous function on the open set $W \subset \mathbb{W}$. We say that the intrinsic gradient $\nabla^\varphi\varphi \in L^\infty_{loc}(W;\mathbb{R}^{2n-1})$ exists in the sense of distributions if the distributional derivatives $X_i\varphi$, $\mathscr{B}\varphi$ and $Y_i\varphi$, with $i = 2, \dots, n$, are represented by locally bounded functions in W. In this case, we let

$$\nabla^\varphi\varphi = (X_2\varphi, \dots, X_n\varphi, \mathscr{B}\varphi, Y_2\varphi, \dots, Y_n\varphi), \tag{1.25}$$

and we call $\nabla^\varphi\varphi$ the *intrinsic gradient of* φ. When $n = 1$, the intrinsic gradient reduces to $\nabla^\varphi\varphi = \mathscr{B}\varphi$.

We note that the intrinsic gradient (1.25) has a strong non-linear character. This partially motivates the fact that $\mathrm{Lip}_H(W)$ is not a vector space.

The following result shows that the L^∞-norm of the intrinsic gradient is controlled by the intrinsic Lipschitz constant, see [**15**, Proposition 4.4].

PROPOSITION 1.17. *Let $W \subset \mathbb{W}$ be a bounded open set and let $\varphi\colon W \to \mathbb{R}$ be an intrinsic Lipschitz function. There exists a positive dimensional constant $C(n)$ such that*

$$\|\nabla^\varphi \varphi\|_{L^\infty(W)} \le C(n)\,\mathrm{Lip}_H(\varphi)(1 + \mathrm{Lip}_H(\varphi)).$$

6.6. Area formula for intrinsic Lipschitz functions. Let $W \subset \mathbb{W}$ be an open set and let $\varphi\colon W \to \mathbb{R}$ be a locally intrinsic Lipschitz function. Then the intrinsic epigraph E_φ of φ defined in (1.19) is a set with locally finite H-perimeter whose horizontal inner normal ν_{E_φ} depends on the intrinsic gradient $\nabla^\varphi\varphi$. Moreover, the H-perimeter of E_φ admits an area formula similar to the classical one in the Euclidean setting.

THEOREM 1.18 (Area formula). *Let $W \subset \mathbb{W}$ be an open set and let $\varphi\colon W \to \mathbb{R}$ be a locally intrinsic Lipschitz function. Then the intrinsic epigraph $E_\varphi \subset \mathbb{H}^n$ has locally finite H-perimeter in the cylinder*

$$W * \mathbb{R} = \{w * s\mathrm{e}_1 \in \mathbb{H}^n : w \in W,\ s \in \mathbb{R}\},$$

and for $\mathcal{L}^{2n}$-a.e. $w \in W$ the inner horizontal normal to ∂E_φ is given by

$$\nu_{E_\varphi}(\Phi(w)) = \left(\frac{1}{\sqrt{1 + |\nabla^\varphi\varphi(w)|^2}}, \frac{-\nabla^\varphi\varphi(w)}{\sqrt{1 + |\nabla^\varphi\varphi(w)|^2}} \right). \tag{1.26}$$

Moreover, for any $W' \subset\subset W$, the following area formula holds:

$$P_H(E_\varphi; W' * \mathbb{R}) = \int_{W'} \sqrt{1 + |\nabla^\varphi\varphi(w)|^2}\, d\mathcal{L}^{2n}. \tag{1.27}$$

Formula (1.26) for the inner horizontal normal to ∂E_φ and the area formula (1.27) are proved in [**15**], respectively in Corollary 4.2 and in Theorem 1.6. The area formula (1.27) can be improved in the following way

$$\int_{\partial E_\varphi \cap W' * \mathbb{R}} g(p)\, d|\mu_{E_\varphi}| = \int_{W'} g(\Phi(w))\sqrt{1 + |\nabla^\varphi\varphi(w)|^2}\, d\mathcal{L}^{2n}, \tag{1.28}$$

where $g\colon \partial E_\varphi \to \mathbb{R}$ is a Borel function.

To avoid long equations, in the following we will often omit the variables and the flow map Φ when we will apply the area formula (1.27) and its general version (1.28).

A result related to Theorem 1.18 can be found in [**53**, Theorem 1.1], where it is proved that if $E \subset \mathbb{H}^n$ is a set with finite H-perimeter having controlled normal ν_E, say $\langle \nu_E, X_1 \rangle_g \ge k > 0$ μ_E-a.e. for some $k \in (0, 1]$, then the reduced boundary $\partial^* E$ is an intrinsic Lipschitz graph along X_1.

CHAPTER 2

Intrinsic Lipschitz approximation

1. Main results

1.1. Monti's approximation. The starting point of De Giorgi's regularity theory for perimeter minimizers in $\mathbb{R}^n$ is a good approximation of minimizing boundaries by means of Lipschitz graphs, see [**57**].

In the Heisenberg group, the boundary of sets with finite H-perimeter is not rectifiable and, in fact, may have fractional Hausdorff dimension, [**44**]. Nevertheless, the notion of intrinsic graph in the sense of [**38**] (recall Definition 1.13) turns out to be effective in the approximation and leads to the following result, see [**50**, Theorem 5.1].

Theorem 2.1 (Monti). *Let $n \geq 1$ and let $L > 0$ be a constant suitably large when $n = 1$. There are constants $k = k(n) > 1$ and $c = c(n, L) > 0$ with the following property. For any set $E \subset \mathbb{H}^n$ that is H-perimeter minimizing in C_{kr} with $0 \in \partial E$, $r > 0$, $\nu_E(0) = -\mathrm{e}_1$, there exists an L-intrinsic Lipschitz function $\varphi\colon \mathbb{W} \to \mathbb{R}$ such that*

$$\mathcal{S}^{2n+1}\Big((\partial E \,\triangle\, \mathrm{gr}(\varphi)) \cap C_r\Big) \leq c r^{2n+1} \mathbf{e}(kr). \tag{2.1}$$

The proof of Theorem 2.1 is based on the ideas outlined in [**3**, Sections 4.3, 4.4] and goes as follows. The starting point is to analyse the pairs of points of the boundary of the minimizer E with small horizontal excess (see [**50**, Propositions 4.1, 4.2]). The set $G \subset \partial E$ of such points is compact and the projection $\pi\colon \mathbb{H}^n \to \mathbb{W}$ defined in (1.5) is injective on G and satisfies

$$|\mathscr{h}(q^{-1} * p)| \leq L\, \|\pi(q^{-1} * p)\|_\infty \quad \text{for all } p, q \in G. \tag{2.2}$$

Thus, recalling (1.20), the inverse of π restricted to G defines an intrinsic Lipschitz function on G that can be extended to the whole $\mathbb{W}$ (Proposition 1.14). The approximation (2.1) is obtained by estimating the terms $\partial E \setminus \mathrm{gr}(\varphi)$ and $\mathrm{gr}(\varphi) \setminus \partial E$ in the cylinder C_r separately: the first can be controlled by a covering argument, while the second is a consequence of the area formula (Theorem 1.18).

We remark that the case $n = 1$ is quite delicate. Indeed, as we already observed in Remark 1.9, the smallness of the excess for an H-perimeter minimizer in $\mathbb{H}^1$ in general does not ensure that its boundary is flat. This partially motivates the fact that the estimate (2.2) is proved to hold only when $L > 2$ for $n = 1$, see [**50**, Proposition 4.2].

On the other hand, examples of minimal surfaces in the first Heisenberg group $\mathbb{H}^1$ that are only Lipschitz continuous in the standard sense have been constructed, see, e.g., [**55**, **56**], but no similar examples of non-smooth minimizers are known in $\mathbb{H}^n$ with $n \geq 2$. Thus, in the following, we will restrict our attention to the case $n \geq 2$.

1.2. Improved approximation. The first step towards an improvement of Theorem 2.1 is a better control both on the intrinsic Lipschitz approximating function φ and on its intrinsic gradient $\nabla^\varphi\varphi$. In order to do so, we need to improve the estimate (2.2). The idea is to take advantage of the height bound (1.12) given in Theorem 1.8, which gives a uniform control on the flatness of the boundary of the minimizer depending on the smallness of the excess. Our result is the following and we will prove it in Section 2.

THEOREM 2.2 (Intrinsic Lipschitz approximation). *Let $n \geq 2$. There exist positive dimensional constants $C_1(n)$, $\varepsilon_1(n)$ and $\delta_1(n)$ with the following property. If $E \subset \mathbb{H}^n$ is a (Λ, r_0)-minimizer of H-perimeter in $C_{642r}(p_0)$ with*

$$\Lambda r_0 \leq 1, \qquad r_0 > 642r \qquad p_0 \in \partial E,$$

and if we set

$$M = C_r(p_0) \cap \partial E, \qquad M_0 = \left\{ q \in M : \sup_{0<s<64r} \mathbf{e}(q,s) \leq \delta_1(n) \right\},$$

then, provided $\mathbf{e}(p_0, 642r) \leq \varepsilon_1(n)$, there is an intrinsic Lipschitz function $\varphi \colon \mathbb{W} \to \mathbb{R}$ with

$$\sup_{\mathbb{W}} \frac{|\varphi|}{r} \leq C_1(n)\, \mathbf{e}(p_0, 642r)^{\frac{1}{2(2n+1)}}, \qquad \operatorname{Lip}_H(\varphi) \leq 1, \tag{2.3}$$

such that a suitable translation Γ of the graph of φ over D_r contains M_0,

$$M_0 \subset M \cap \Gamma, \qquad \Gamma = \tau_{p_0}(\operatorname{gr}(\varphi|_{D_r})),$$

and covers a large portion of M in terms of $\mathbf{e}(p_0, 642r)$,

$$\frac{\mathcal{S}^{2n+1}(M \bigtriangleup \Gamma)}{r^{2n+1}} \leq C_1(n)\, \mathbf{e}(p_0, 642r).$$

Moreover, the L^2-norm on D_r of the intrinsic gradient of φ is controlled by $\mathbf{e}(p_0, 642r)$,

$$\frac{1}{r^{2n+1}} \int_{D_r} |\nabla^\varphi \varphi|^2 \, d\mathcal{L}^{2n} \leq C_1(n)\, \mathbf{e}(p_0, 642r).$$

REMARK 2.3 (Almost harmonicity). Theorem 2.2 is the natural reformulation in $\mathbb{H}^n$ of the classical Lipschitz approximation of (Λ, r_0)-perimeter minimizers in $\mathbb{R}^n$, see [**46**, Theorem 23.7], but with a relevant difference. At the present stage of the theory, it is not clear how to prove that the intrinsic Lipschitz approximating function φ is *almost harmonic* (see (23.26) in [**46**, Theorem 23.7]), that is, it satisfies

$$\frac{1}{r^{2n+1}} \left| \int_{D_r} \langle \nabla^\varphi \varphi, \nabla^{\varphi *} \psi \rangle \, d\mathcal{L}^{2n} \right| \leq C_1(n) \sup_{D_r} |\nabla^\varphi \varphi| \Big(\mathbf{e}(p_0, 642r) + \Lambda r \Big) \tag{2.4}$$

for every $\psi \in C^1_c(D_r)$. Here $\nabla^{\varphi *}$ is the (formal) L^2-adjoint of the intrinsic gradient ∇^φ, see [**52**, Section 3]. The almost harmonicity property (2.4) is closely linked to the problem of computing the first variation of the area formula (1.27) and, more generally, of the H-perimeter; we refer the interested reader to [**36**, **52**] and to the references therein for an account on these problems.

2. Proof of Theorem 2.2

In this section, we prove Theorem 2.2. The proof follows the ideas outlined in [**46**, Section 23.3]. Up to replacing E with its blow-up $E_{p_0,r}$ and, correspondingly, φ with $\varphi_r = \frac{1}{r}\varphi \circ \delta_r$, we can simplify Theorem 2.2 to the following statement. Note that φ_r is intrinsic Lipschitz if and only if φ is intrinsic Lipschitz by (1.20) and, moreover, it can be easily verified that $\nabla^{\varphi_r}\varphi_r = \nabla^{\varphi}\varphi \circ \delta_r$.

THEOREM 2.4. *Let $n \geq 2$. There exist positive dimensional constants $C_1(n)$, $\varepsilon_1(n)$ and $\delta_1(n)$ with the following property. If $E \subset \mathbb{H}^n$ is a (Λ', r_0')-minimizer of H-perimeter in C_{642} with*

$$\Lambda' = \Lambda r, \qquad r_0' = \frac{r_0}{r}, \qquad \Lambda' r_0' \leq 1, \qquad r_0' > 642, \qquad 0 \in \partial E,$$

and if we set

$$M = C_1 \cap \partial E, \qquad M_0 = \left\{ q \in M : \sup_{0<s<64} \mathbf{e}(q,s) \leq \delta_1(n) \right\},$$

then, provided $\mathbf{e}(642) \leq \varepsilon_1(n)$, there exists an intrinsic Lipschitz function $\varphi\colon \mathbb{W} \to \mathbb{R}$ such that

$$\sup_{\mathbb{W}}|\varphi| \leq C_1(n)\,\mathbf{e}(642)^{\frac{1}{2(2n+1)}}, \qquad \mathrm{Lip}_H(\varphi) \leq 1, \tag{2.5}$$

$$M_0 \subset M \cap \Gamma, \qquad \Gamma = \mathrm{gr}(\varphi|_{D_1}), \tag{2.6}$$

$$\mathcal{S}^{2n+1}(M \triangle \Gamma) \leq C_1(n)\,\mathbf{e}(642), \tag{2.7}$$

$$\int_{D_1} |\nabla^{\varphi}\varphi|^2 \, d\mathcal{L}^{2n} \leq C_1(n)\,\mathbf{e}(642). \tag{2.8}$$

PROOF. The proof is divided in three steps.

Step 1: construction of φ. Let $\varepsilon_0(n)$ and $C_0(n)$ be the constants given in Theorem 1.8. Then, by Theorem 1.8, we have

$$\sup\Big\{ |\mathfrak{f}(p)| : p \in C_1 \cap \partial E \Big\} \leq C_0(n)\,\mathbf{e}(16)^{\frac{1}{2(2n+1)}}, \tag{2.9}$$

provided that $\mathbf{e}(16) \leq \varepsilon_0(n)$; this follows from (1.10) if $\varepsilon_1(n) \leq \varepsilon_0(n)$ is suitably small.

Let $q \in M_0$ and $p \in M$ be fixed. Then $p, q \in C_1$, so $d_C(p,q) < 4$ by (1.8), where d_C is the quasi distance induced by the quasi norm $\|\cdot\|_C$ defined in (1.6). We consider the blow-up of E at scale $d_C(p,q)$ centred in q, that is, $F = E_{q,d_C(p,q)}$. By Remark 1.4, F is a (Λ'', r_0'')-perimeter minimizer in $(C_{642})_{q,d_C(p,q)}$, with

$$\Lambda'' = \Lambda'\, d_C(p,q), \qquad r_0'' = \frac{r_0'}{d_C(p,q)} > 1.$$

Since

$$C_{16} \subset (C_{642})_{q,d_C(p,q)}, \qquad \Lambda'' r_0'' \leq 1, \qquad 0 \in \partial F$$

and, by (1.11) and by definition of M_0,

$$\mathbf{e}(F, 0, 16, \nu) = \mathbf{e}(E, q, 16 d_C(p,q), \nu) \leq \delta_1(n),$$

then, provided we assume $\delta_1(n) \le \varepsilon_0(n)$, by Theorem 1.8 we have

$$\sup\Big\{|\mathscr{f}(w)| : w \in C_1 \cap \partial F\Big\} \le C_0(n)\, \delta_1(n)^{\frac{1}{2(2n+1)}}.$$

In particular, choosing

$$w = \frac{1}{d_C(p,q)}\, q^{-1} * p \in C_1 \cap \partial F,$$

we get

$$|\mathscr{f}(q^{-1} * p)| \le C_0(n)\, \delta_1(n)^{\frac{1}{2(2n+1)}}\, d_C(p,q). \tag{2.10}$$

We now set

$$L(n) := C_0(n)\, \delta_1(n)^{\frac{1}{2(2n+1)}} \tag{2.11}$$

and we choose $\delta_1(n)$ so small that $L(n) < 1$. Then, by (2.10), we conclude that $d_C(p,q) = \|\pi(q^{-1} * p)\|_\infty$ and we get

$$|\mathscr{f}(q^{-1} * p)| \le L(n) \|\pi(q^{-1} * p)\|_\infty \quad \text{for all } p \in M,\ q \in M_0. \tag{2.12}$$

In particular, (2.12) proves that the projection π is invertible on M_0. Therefore, we can define a function $\varphi\colon \pi(M_0) \to \mathbb{R}$ setting $\varphi(\pi(p)) = \mathscr{f}(p)$ for all $p \in M_0$. From (2.12), we deduce that

$$|\varphi(\pi(p)) - \varphi(\pi(q))| \le L(n) \|\pi(q^{-1} * p)\|_\infty \quad \text{for all } p, q \in M_0,$$

so that φ is an intrinsic Lipschitz function on $\pi(M_0)$ with $\mathrm{Lip}_H(\varphi, \pi(M_0)) \le L(n) < 1$ by (1.20). Since $M_0 \subset M$, by (2.9) we also have

$$|\varphi(\pi(p))| \le C_0(n)\, \mathbf{e}(16)^{\frac{1}{2(2n+1)}} \quad \text{for all } p \in M_0.$$

Therefore, by Proposition 1.14, possibly choosing $\delta_1(n)$ smaller accordingly to (1.21), we can extend φ from $\pi(M_0)$ to the whole $\mathbb{W}$ with $\mathrm{Lip}_H(\varphi, \mathbb{W}) \le L(n) < 1$ in such a way that

$$M_0 \subset M \cap \Gamma, \quad \Gamma = \mathrm{gr}(\varphi|_{D_1}) \quad \text{and} \quad |\varphi(w)| \le C_0(n)\, \mathbf{e}(16)^{\frac{1}{2(2n+1)}} \quad \text{for all } w \in \mathbb{W}.$$

We thus proved (2.5) and (2.6) for a suitable $C_1(n) \ge C_0(n)$.

Step 2: covering argument. We now prove (2.7) via a covering argument. By definition of M_0, for every $q \in M \setminus M_0$ there exists $s = s(q) \in (0, 64)$ such that

$$\int_{C_s(q) \cap \partial E} \frac{|\nu_E - \nu|_g^2}{2}\, d\mathcal{S}^{2n+1} > \frac{\delta_1(n)}{\delta(n)}\, s^{2n+1}, \tag{2.13}$$

with $\delta(n) = \frac{2\omega_{2n-1}}{\omega_{2n+1}}$ as in (1.9) and $\nu = -X_1$ as usual. The family of balls

$$\{B_{2s}(q) : q \in M \setminus M_0,\ s = s(q)\}$$

is a covering of $M \setminus M_0$. By the $5r$-covering Lemma (see [**31**, Theorem 1.24] for example), there exist a sequence of points $q_h \in M \setminus M_0$ and a sequence of radii $s_h = s(q_h)$, $h \in \mathbb{N}$, with q_h and s_h satisfying (2.13), such that the balls $B_{2s_h}(q_h)$ are pairwise disjoint and

$$\{B_{10s_h}(q_h) : h \in \mathbb{N}\}$$

is still a covering of $M \setminus M_0$. Note that $B_{10s_h}(q_h) \subset C_{642}$, because if $p \in B_{10s_h}(q_h)$ then, by (1.7),

$$\|p\|_C \le \|p\|_\infty \le d_\infty(p, q_h) + \|q_h\|_\infty < 10s_h + 2\|q_h\|_C < 642.$$

Therefore, by Theorem 1.5, we get

$$\begin{aligned}\mathcal{S}^{2n+1}(M\setminus M_0) &\le \sum_{h\in\mathbb{N}} \mathcal{S}^{2n+1}\Big((M\setminus M_0)\cap B_{10s_h}(q_h)\Big)\\ &\le \sum_{h\in\mathbb{N}} \mathcal{S}^{2n+1}\Big(M\cap B_{10s_h}(q_h)\Big)\\ &\le C(n)\sum_{h\in\mathbb{N}} s_h^{2n+1},\end{aligned}$$

with $C(n)$ a positive dimensional constant. Since $C_{s_h}(q_h)\subset B_{2s_h}(q_h)$ by (1.8), the cylinders $C_{s_h}(q_h)$ are pairwise disjoint and contained in C_{642}, so we have

$$\mathcal{S}^{2n+1}(M\setminus M_0) \le C(n)\sum_{h\in\mathbb{N}}\int_{C_{s_h}(q_h)\cap\partial E} \frac{|\nu_E-\nu|_g^2}{2}\, d\mathcal{S}^{2n+1} \le C(n)\,\mathbf{e}(642), \tag{2.14}$$

with $C(n)$ a positive dimensional constant. Therefore, since $M\setminus\Gamma\subset M\setminus M_0$, by (2.14) it follows that

$$\mathcal{S}^{2n+1}(M\setminus\Gamma)\le C(n)\,\mathbf{e}(642), \tag{2.15}$$

which is the first half of (2.7).

We now bound the second half of (2.7). We choose $\varepsilon_1(n)$ so small that

$$\mathbf{e}(2)\le\omega(n,\tfrac{1}{2},\tfrac{1}{642},642).$$

This is possible by (1.10). Then, by (1.17) in Lemma 1.11, we have

$$\mathcal{L}^{2n}(G)\le\mathcal{S}^{2n+1}\left(M\cap\pi^{-1}(G)\right)$$

for any Borel set $G\subset D_1$. Therefore, by the area formula (1.27) in Theorem 1.18, we can estimate

$$\begin{aligned}\delta(n)\,\mathcal{S}^{2n+1}(\Gamma\setminus M) &= \int_{\pi(\Gamma\setminus M)}\sqrt{1+|\nabla^{\varphi}\varphi(w)|^2}\,d\mathcal{L}^{2n}\\ &\le \sqrt{1+\|\nabla^{\varphi}\varphi\|^2_{L^\infty(D_1)}}\,\mathcal{L}^{2n}\Big(\pi(\Gamma\setminus M)\Big)\\ &\le \sqrt{1+\|\nabla^{\varphi}\varphi\|^2_{L^\infty(D_1)}}\,\mathcal{S}^{2n+1}\left(M\cap\pi^{-1}\Big(\pi(\Gamma\setminus M)\Big)\right). \qquad (2.16)\end{aligned}$$

Since φ is intrinsic Lipschitz on D_1 with $\mathrm{Lip}_H(\varphi)<1$ by construction, by Proposition 1.17 there exists a positive dimensional constant $C(n)$ such that

$$\|\nabla^{\varphi}\varphi\|_{L^\infty(D_1)}\le C(n)\,\mathrm{Lip}_H(\varphi)\Big(\mathrm{Lip}_H(\varphi)+1\Big)<2C(n). \tag{2.17}$$

Thus, by (2.16) and (2.17), there exists a positive dimensional constant $C(n)$ such that

$$\mathcal{S}^{2n+1}(\Gamma\setminus M)\le C(n)\,\mathcal{S}^{2n+1}\left(M\cap\pi^{-1}\Big(\pi(\Gamma\setminus M)\Big)\right). \tag{2.18}$$

Since we have

$$M\cap\pi^{-1}\Big(\pi(\Gamma\setminus M)\Big)\subset M\setminus\Gamma,$$

by (2.15) and (2.18) we conclude that, for some positive dimensional constant $C'(n)$,

$$\mathcal{S}^{2n+1}(\Gamma\setminus M)\le C(n)\,\mathcal{S}^{2n+1}(M\setminus\Gamma)\le C'(n)\,\mathbf{e}(642), \tag{2.19}$$

which is the second half of (2.7). Combining (2.15) and (2.19), we prove (2.7).

Step 3: L^2-estimate. Finally, we prove (2.8). We first notice that, by Theorem 1.18, Theorem 1.2 and by [**6**, Corollary 2.6], for $\mathcal{S}^{2n+1}$-a.e. $p \in M \cap \Gamma$ there exists $\lambda(p) \in \{-1, 1\}$ such that

$$\nu_E(p) = \lambda(p) \frac{\big(1, -\nabla^\varphi \varphi(\pi(p))\big)}{\sqrt{1 + |\nabla^\varphi \varphi(\pi(p))|^2}}. \tag{2.20}$$

Taking into account that, for $\mathcal{S}^{2n+1}$-a.e. $p \in M \cap \Gamma$,

$$\frac{|\nu_E(p) - \nu(p)|_g^2}{2} = 1 - \langle \nu_E(p), \nu(p) \rangle_g \geq \frac{1 - \langle \nu_E(p), \nu(p) \rangle_g^2}{2}, \tag{2.21}$$

by (2.20) and by the general area formula (1.28) we find that

$$\begin{aligned} \mathbf{e}(1) &\geq \int_{M \cap \Gamma} \frac{1 - \langle \nu_E(p), \nu(p) \rangle_g^2}{2} \, d|\mu_E| \\ &= \frac{1}{2} \int_{M \cap \Gamma} \frac{|\nabla^\varphi \varphi(\pi(p))|^2}{1 + |\nabla^\varphi \varphi(\pi(p))|^2} \, d|\mu_E| \\ &= \frac{1}{2} \int_{\pi(M \cap \Gamma)} \frac{|\nabla^\varphi \varphi(w)|^2}{\sqrt{1 + |\nabla^\varphi \varphi(w)|^2}} \, d\mathcal{L}^{2n}. \end{aligned}$$

Recalling (2.17) and (1.10), we conclude that there exists a positive dimensional constant $C(n)$ such that

$$\int_{\pi(M \cap \Gamma)} |\nabla^\varphi \varphi(w)|^2 \, dw \leq C(n) \, \mathbf{e}(642). \tag{2.22}$$

Moreover, again by the general area formula (1.28), there exists a positive dimensional constant $C(n)$ such that

$$\begin{aligned} \int_{\pi(M \triangle \Gamma)} |\nabla^\varphi \varphi(w)|^2 \, d\mathcal{L}^{2n} &= \int_{M \triangle \Gamma} \frac{|\nabla^\varphi \varphi(\pi(p))|^2}{\sqrt{1 + |\nabla^\varphi \varphi(\pi(p))|^2}} \, d|\mu_E| \\ &\leq C(n) \|\nabla^\varphi \varphi\|_{L^\infty(D_1)}^2 \, \mathcal{S}^{2n+1}(M \triangle \Gamma). \end{aligned}$$

By (2.17) and (2.7), we find a positive dimensional constant $C(n)$ such that

$$\int_{\pi(M \triangle \Gamma)} |\nabla^\varphi \varphi(w)|^2 \, dw \leq C(n) \, \mathbf{e}(642). \tag{2.23}$$

Combining (2.22) and (2.23), we prove (2.8). □

REMARK 2.5 (σ-representative). Let $0 < \sigma \leq 1$ and $I = (-1, 1)$. We let $\mathcal{A}(\sigma)$ be the family of sets $A \subseteq D_\sigma$ such that

$$|\mathfrak{h}(q^{-1} * p)| \leq L(n) \|\pi(q^{-1} * p)\|_\infty \quad \text{for all } p \in M \cap D_\sigma * I, \ q \in M \cap A * I,$$

where $L(n)$ is the dimensional constant considered in (2.11). Note that the family $\mathcal{A}(\sigma)$ is partially ordered by inclusion and is closed under union. Thus $\mathcal{A}(\sigma)$ has a unique maximal element $A_\sigma^\star$. Then, by (2.12), we have that

$$|\mathfrak{h}(q^{-1} * p)| \leq L(n) \|\pi(q^{-1} * p)\|_\infty \quad \text{for all } p, q \in M_0 \cup (M \cap A_\sigma^\star * I).$$

Therefore, in Step 1 of the proof of Theorem 2.4, it is not restrictive to assume that the intrinsic Lipschitz approximation $\varphi\colon \mathbb{W} \to \mathbb{R}$ is defined in such a way that

$$\varphi(\pi(p)) = \mathscr{f}(p) \quad \text{for all } p \in M_0 \cup (M \cap A^{\star}_{\sigma} * I).$$

We define such an intrinsic Lipschitz function a σ-*representative* of Theorem 2.4.

A $(\sigma; r)$-*representative* of Theorem 2.2 is defined in the same way, where $r > 0$ is as in the statement of Theorem 2.2 and this time $0 < \sigma \leq r$, $I = (-r, r)$.

CHAPTER 3

Approximation via maximal functions

1. Main results

In this chapter, we develop the ideas contained in [**24**, Section 2] and in [**25**, Appendix A] to prove the following result. The proof is in Section 3. Note that Theorem 2.2 has to be applied with a suitable scaling factor.

Theorem 3.1 (α-improvement). *Let $n \geq 2$ and $\alpha \in (0, \frac{1}{2})$. There exist positive constants $C_2(n)$, $\varepsilon_2(\alpha, n)$ and $k_2 = k_2(n)$ with the following property. Let $E \subset \mathbb{H}^n$ be a (Λ, r_0)-minimizer of H-perimeter in $C_{k_2 r}(p_0)$ with*

$$\Lambda r_0 \leq 1, \qquad r_0 > k_2 r \qquad p_0 \in \partial E, \qquad \mathbf{e}(p_0, k_2 r) \leq \varepsilon_2(\alpha, n).$$

Let $\varphi\colon \mathbb{W} \to \mathbb{R}$ be a suitably chosen approximation given by Theorem 2.2. Then there exists a set $K \subset D_r$ such that

$$\mathcal{L}^{2n}(D_r \setminus K) \leq C_2(n)\, \mathbf{e}(p_0, k_2 r)^{1-2\alpha}.$$

Moreover, the function φ has the following additional properties: up to a translation, the intrinsic graph of φ coincides with ∂E over K,

$$\tau_{p_0}(\mathrm{gr}(\varphi|_K)) = \partial E \cap \tau_{p_0}(K * (-r, r)),$$

and the intrinsic Lipschitz constant of φ over K improves,

$$\mathrm{Lip}_H(\varphi, K) \leq C_2(n)\, \mathbf{e}(p_0, k_2 r)^{\alpha}.$$

Theorem 3.1 leads to the following result. The proof is in Section 4.

Corollary 3.2. *Let $n \geq 2$ and $\alpha \in (0, \frac{1}{2})$. There exist positive constants $C_3(n)$, $\varepsilon_3(\alpha, n)$ and $k_3 = k_3(n)$ with the following property. Let $E \subset \mathbb{H}^n$ be a (Λ, r_0)-minimizer of H-perimeter in $C_{k_3 r}(p_0)$ with*

$$\Lambda r_0 \leq 1, \qquad r_0 > k_3 r \qquad p_0 \in \partial E, \qquad \mathbf{e}(p_0, k_3 r) \leq \varepsilon_3(\alpha, n).$$

Then there exist a set $K \subset D_r$ and an intrinsic Lipschitz function $\varphi\colon \mathbb{W} \to \mathbb{R}$ with the following properties:

$$\mathcal{L}^{2n}(D_r \setminus K) \leq C_3(n)\, \mathbf{e}(p_0, k_3 r)^{1-2\alpha},$$

$$\tau_{p_0}(\mathrm{gr}(\varphi|_K)) = \partial E \cap \tau_{p_0}(K * (-r, r)), \qquad \mathrm{Lip}_H(\varphi) \leq C_3(n)\, \mathbf{e}(p_0, k_3 r)^{\alpha},$$

$$\frac{\mathcal{S}^{2n+1}\Big((\partial E \mathbin{\triangle} \mathrm{gr}(\varphi)) \cap C_r\Big)}{r^{2n+1}} \leq C_3(n)\, \mathbf{e}(p_0, k_3 r)^{1-2\alpha},$$

$$\frac{1}{r^{2n+1}} \int_{D_r} |\nabla^{\varphi} \varphi|^2 \, d\mathcal{L}^{2n} \leq C_3(n)\, \mathbf{e}(p_0, k_3 r).$$

2. Local maximal functions

2.1. Maximal function on disks. Given $s > 0$ and a non-negative measure μ on D_{4s}, with $D_{4s} \subset \mathbb{W}$, the *local maximal function* of μ is defined as

$$M\mu(x) := \sup_{0<r<4s-\|x\|_\infty} \frac{\mu(D_r(x))}{\kappa_n r^{2n+1}} \qquad \text{for all } x \in D_{4s}, \tag{3.1}$$

where $\kappa_n = \mathcal{L}^{2n}(D_1)$ as in (1.4).

LEMMA 3.3. *Let $s > 0$ and let $\mu\colon D_{4s} \to [0, +\infty)$ be as above. Assume that $\theta > 0$ is such that*

$$\mu(D_{4s}) \leq \frac{\theta}{5^{2n+1}} \kappa_n s^{2n+1} \tag{3.2}$$

and define

$$J_\theta = \{x \in D_{4s} : M\mu(x) > \theta\}.$$

Then

$$\mathcal{L}^{2n}(J_\theta \cap D_r) \leq \frac{5^{2n+1}}{\theta} \mu(J_{\theta/2^{2n+1}} \cap D_{r+\frac{s}{5}}) \quad \forall r \leq 3s. \tag{3.3}$$

PROOF. Let $r \leq 3s$ be fixed. Note that if $x \in J_\theta \cap D_r$, then there exists $r_x > 0$ such that

$$\mu(D_{r_x}(x)) > \theta \kappa_n r_x^{2n+1}.$$

By the 5r-covering Lemma applied to the family $\{D_{r_x}(x) : x \in J_\theta \cap D_r\}$, we find a sequence of pairwise disjoint balls $\{D_{r_i}(x_i)\}_{i\in\mathbb{N}}$, with $x_i \in J_\theta \cap D_r$ and $r_i > 0$, such that

$$J_\theta \cap D_r \subset \bigcup_{x\in J_\theta\cap D_r} D_{r_x}(x) \subset \bigcup_{i\in\mathbb{N}} D_{5r_i}(x_i), \qquad \mu(D_{r_i}(x_i)) > \theta\kappa_n r_i^{2n+1}.$$

In particular, by (3.2), we get that

$$r_i < \sqrt[2n+1]{\frac{\mu(D_{r_i}(x_i))}{\theta\kappa_n}} \leq \sqrt[2n+1]{\frac{\mu(D_{4s})}{\theta\kappa_n}} \leq \frac{s}{5}$$

and so, for any $i \in \mathbb{N}$, we have

$$D_{r_i}(x_i) \subset D_{\|x_i\|_\infty + r_i} \subset D_{r+\frac{s}{5}}.$$

We claim that

$$D_{r_i}(x_i) \subset J_{\theta/2^{2n+1}} \cap D_{r+\frac{s}{5}}$$

for any $i \in \mathbb{N}$. Indeed, on the contrary, let $y \in D_{r_i}(x_i)$ be such that $M\mu(y) \leq \frac{\theta}{2^{2n+1}}$. Then $D_{r_i}(x_i) \subset D_{2r_i}(y)$ and

$$4s - \|y\|_\infty \geq 4s - r - \frac{s}{5} \geq 4s - 3s - \frac{s}{5} = \frac{4}{5}s > 2r_i.$$

Hence

$$\frac{\theta}{2^{2n+1}} \geq M\mu(y) = \sup_{0<\delta<4s-\|y\|_\infty} \frac{\mu(D_\delta(y))}{\kappa_n \delta^{2n+1}}$$
$$\geq \sup_{2r_i<\delta<4s-\|y\|_\infty} \frac{\mu(D_\delta(y))}{\kappa_n \delta^{2n+1}}$$
$$\geq \sup_{2r_i<\delta<4s-\|y\|_\infty} \frac{\mu(D_{r_i}(x_i))}{\kappa_n \delta^{2n+1}} = \frac{\mu(D_{r_i}(x_i))}{\kappa_n (2r_i)^{2n+1}} > \frac{\theta}{2^{2n+1}},$$

a contradiction.

We can finally estimate

$$\mathcal{L}^{2n}(J_\theta \cap D_r) \leq \sum_{i\in\mathbb{N}} \mathcal{L}^{2n}(D_{5r_i}(x_i)) = 5^{2n+1}\,\kappa_n \sum_{i\in\mathbb{N}} r_i^{2n+1}$$
$$\leq 5^{2n+1}\,\kappa_n \sum_{i\in\mathbb{N}} \frac{\mu(D_{r_i}(x_i))}{\theta\kappa_n} = \frac{5^{2n+1}}{\theta} \sum_{i\in\mathbb{N}} \mu(D_{r_i}(x_i))$$
$$= \frac{5^{2n+1}}{\theta}\mu\left(\bigcup_{i\in\mathbb{N}} D_{r_i}(x_i)\right) \leq \frac{5^{2n+1}}{\theta}\mu(J_{\theta/2^{2n+1}} \cap D_{r+\frac{s}{5}})$$

and (3.3) follows. □

2.2. Maximal function on φ-balls. We need some preliminaries. In the setting of the Heisenberg group, the Poincaré inequality is the natural analogous of the Euclidean one and was established in [**16**, Theorem 1.2] for functions which belongs to an intrinsic Sobolev class, see [**16**, Definition 1.1].

To our purpose, it is enough to recall the following Poincaré inequality for Lipschitz intrinsic functions, which is a consequence of [**16**, Theorem 1.2] (see also [**16**, Corollary 1.3] for the case $p = 1$).

THEOREM 3.4 (Poincaré inequality). *Let $W \subset \mathbb{W}$ be a bounded open set, $n \geq 2$, and let $1 \leq p < \infty$. Let $\varphi\colon W \to \mathbb{R}$ be an L-intrinsic Lipschitz function. Then there exist two constants $C_1^L, C_2^L > 0$ with $C_2^L > 1$, depending on L, such that*

$$\int_{U_\varphi(x,r)} |\varphi - (\varphi)_{x,r}|^p \; d\mathcal{L}^{2n} \leq C_1^L r^p \int_{U_\varphi(x,C_2^L r)} |\nabla^\varphi \varphi|^p \; d\mathcal{L}^{2n} \tag{3.4}$$

for every $U_\varphi(x, C_2^L r) \subset W$, where

$$U_\varphi(x,r) = \{y \in W : d_\varphi(x,y) < r\} \tag{3.5}$$

and

$$(\varphi)_{x,r} = \fint_{U_\varphi(x,r)} \varphi \; d\mathcal{L}^{2n} = \frac{1}{\mathcal{L}^{2n}(U_\varphi(x,r))} \int_{U_\varphi(x,r)} \varphi \; d\mathcal{L}^{2n}.$$

The constants C_1^L, C_2^L depend continuously on L and n. For future convenience, we define

$$\gamma_2(n) = \lim_{L\to 0} C_2^L \geq 1. \tag{3.6}$$

The $\mathcal{L}^{2n}$-measure of the ball $U_\varphi(x,r)$ defined in (3.5) is comparable to r^{2n+1}, see [**16**, Section 2.3] and the references therein.

LEMMA 3.5. *Let $W \subset \mathbb{W}$ be a bounded open set, $n \geq 2$, and let $\varphi\colon W \to \mathbb{R}$ be an L-intrinsic Lipschitz function. There exist two constants $c_1^L, c_2^L > 0$, depending on L, such that, for all $U_\varphi(x,r) \subset W$, we have*

$$c_1^L \leq \frac{\mathcal{L}^{2n}(U_\varphi(x,r))}{r^{2n+1}} \leq c_2^L. \tag{3.7}$$

We can now introduce the *local φ-maximal function.* Let $n \geq 2$, $s > 0$ and let $\varphi\colon \mathbb{W} \to \mathbb{R}$ be an L-intrinsic Lipschitz function. By (1.24) and by (3.6), there exists a dimensional constant $\ell(n) > 0$ such that

$$L \in [0, \ell(n)] \implies c_L \leq 2 \text{ and } C_2^L \leq 2\gamma_2(n), \tag{3.8}$$

where c_L is as in (1.23) and C_2^L is as in Theorem 3.4. For all $L \in [0, \ell(n)]$, we define the *local φ-maximal function* of μ_φ as

$$[\mu_\varphi](x) := \sup_{0<r<r_\varphi(x,s)} \frac{\mu_\varphi(U_\varphi(x,r))}{\mathcal{L}^{2n}(U_\varphi(x,r))} \qquad \forall x \in U_\varphi(0,s), \tag{3.9}$$

where we set

$$r_\varphi(x,s) = \frac{\rho(n)}{c_L}\, s - d_\varphi(x,0) \quad \text{for all } x \in U_\varphi(0,s), \tag{3.10}$$

the dimensional constant is

$$\rho(n) = 64\gamma_2(n) + 2 \tag{3.11}$$

and the non-negative measure μ_φ on $U_\varphi(0, \rho(n)s)$ is given by

$$d\mu_\varphi = |\nabla^\varphi \varphi| \, d\mathcal{L}^{2n}.$$

The maximal function introduced in (3.9) is well-defined, since

$$x \in U_\varphi(0,s),\ r < r_\varphi(x,s) \implies U_\varphi(x,r) \subset U_\varphi(0, \rho(n)s)$$

by the quasi triangular inequality (1.23).

We use the Poincaré inequality (3.4) to prove the following result on $[\mu_\varphi]$, following the ideas of [**24**, Proposition 2.2] and [**25**, Lemma A.2].

LEMMA 3.6. *Let $n \geq 2$, $s > 0$, $\varphi\colon \mathbb{W} \to \mathbb{R}$, μ_φ, $[\mu_\varphi]$, $L \in [0, \ell(n)]$ be as above. Let $\theta > 0$ and define*

$$J_\theta^\varphi = \{x \in U_\varphi(0,s) : [\mu_\varphi](x) > \theta\}. \tag{3.12}$$

Then there exists a constant $C = C(n, L)$ such that

$$|\varphi(x) - \varphi(y)| \leq C\theta \, d_\varphi(x,y) \qquad \forall x, y \in U_\varphi(0,s) \setminus J_\theta^\varphi. \tag{3.13}$$

PROOF. Let $x \in U_\varphi(0,s) \setminus J_\theta^\varphi$ and let $C_2^L r < r_\varphi(x,s)$. Then, by Theorem 3.4 with $p = 1$, we have

$$\int_{U_\varphi(x,r)} |\varphi - (\varphi)_{x,r}| \; d\mathcal{L}^{2n} \leq C_1^L r \int_{U_\varphi(x, C_2^L r)} |\nabla^\varphi \varphi| \; d\mathcal{L}^{2n} = C_1^L r \, \mu_\varphi(U_\varphi(x, C_2^L r)).$$

By (3.9) and by (3.12), we have

$$\mu_\varphi(U_\varphi(x, C_2^L r)) \leq \theta \, \mathcal{L}^{2n}(U_\varphi(x, C_2^L r)).$$

Therefore, by (3.7), we have

$$\int_{U_\varphi(x,r)} |\varphi - (\varphi)_{x,r}| \, d\mathcal{L}^{2n} \le C_1^L \theta c_2^L (C_2^L r)^{2n+1},$$

and so, again by (3.7), we get

$$⨍_{U_\varphi(x,r)} |\varphi - (\varphi)_{x,r}| \, d\mathcal{L}^{2n} \le \frac{c_2^L}{c_1^L} C_1^L (C_2^L)^{2n+1} \theta r,$$

for all $x \in U_\varphi(0,s) \setminus J_\theta^\varphi$ and $C_2^L r < r_\varphi(x,s)$.

In particular, for all $j = 0, 1, 2, \dots$, we have

$$\begin{aligned} \left|(\varphi)_{x,\frac{r}{2^{j+1}}} - (\varphi)_{x,\frac{r}{2^j}}\right| &\le ⨍_{U_\varphi(x,\frac{r}{2^{j+1}})} \left|\varphi(u) - (\varphi)_{x,\frac{r}{2^j}}\right| \, d\mathcal{L}^{2n}(u) \\ &\le 2^{2n+1} \frac{c_2^L}{c_1^L} ⨍_{U_\varphi(x,\frac{r}{2^j})} \left|\varphi(u) - (\varphi)_{x,\frac{r}{2^j}}\right| \, d\mathcal{L}^{2n}(u) \\ &\le \frac{2^{2n+1}}{2^j} \left(\frac{c_2^L}{c_1^L}\right)^2 C_1^L (C_2^L)^{2n+1} \theta r. \end{aligned}$$

Since φ is continuous, we get

$$|\varphi(x) - (\varphi)_{x,r}| \le \sum_{j=0}^{+\infty} \left|(\varphi)_{x,\frac{r}{2^{j+1}}} - (\varphi)_{x,\frac{r}{2^j}}\right| \le 2^{2n+2} \left(\frac{c_2^L}{c_1^L}\right)^2 C_1^L (C_2^L)^{2n+1} \theta r,$$

for all $x \in U_\varphi(0,s) \setminus J_\theta^\varphi$ and $C_2^L r < r_\varphi(x,s)$.

Finally, let $x, y \in U_\varphi(0,s) \setminus J_\theta^\varphi$, $r = d_\varphi(x,y)$ and $c_3^L = 2c_L$. Then, by the quasi triangular inequality (1.23), we have

$$U_\varphi(x,r) \cup U_\varphi(y,r) \subset U_\varphi(x, c_3^L r) \cap U_\varphi(y, c_3^L r).$$

Notice that, again by (1.23), we have

$$x, y \in U_\varphi(0,s),\ r = d_\varphi(x,y) \implies U_\varphi(x, c_3^L r) \cup U_\varphi(y, c_3^L r) \subset U_\varphi(0, \rho(n)s),$$

because, by (3.8) and (3.11),

$$c_L(2c_L c_3^L + 1) = c_L(4c_L^2 + 1) \le \rho(n).$$

Therefore

$$\begin{aligned} \left|(\varphi)_{x,c_3^L r} - (\varphi)_{y,c_3^L r}\right| &\le ⨍_{U_\varphi(x,c_3^L r)\cap U_\varphi(y,c_3^L r)} \left|\varphi(u) - (\varphi)_{x,c_3^L r}\right| + \left|\varphi(u) - (\varphi)_{x,c_3^L r}\right| \, d\mathcal{L}^{2n}(u) \\ &\le \frac{c_2^L}{c_1^L} (c_3^L)^{2n+1} \Bigg(⨍_{U_\varphi(x,c_3^L r)} \left|\varphi(u) - (\varphi)_{x,c_3^L r}\right| \, d\mathcal{L}^{2n}(u) + \\ &\qquad + ⨍_{U_\varphi(y,c_3^L r)} \left|\varphi(u) - (\varphi)_{y,c_3^L r}\right| \, d\mathcal{L}^{2n}(u) \Bigg). \end{aligned}$$

Since $x, y \in U_\varphi(0,s) \setminus J_\theta^\varphi$, by (3.9) and by (3.12) we have

$$\mu_\varphi(U_\varphi(x, c_3^L C_2^L r)) \le \theta \, \mathcal{L}^{2n}(U_\varphi(x, c_3^L C_2^L r))$$

and, analogously,

$$\mu_\varphi(U_\varphi(y, c_3^L C_2^L r)) \le \theta \, \mathcal{L}^{2n}(U_\varphi(y, c_3^L C_2^L r)),$$

provided that

$$c_3^L C_2^L d_\varphi(x,y) < \min\{r_\varphi(x,s), r_\varphi(y,s)\}.$$

By (3.8), since $x, y \in U_\varphi(0,s)$, we have

$$\min\{r_\varphi(x,s), r_\varphi(y,s)\} > \frac{\rho(n)s}{c_L} - s \geq \left(\frac{\rho(n)}{2} - 1\right) s$$

and

$$c_3^L C_2^L d_\varphi(x,y) < 4c_L^2 C_2^L s \leq 32\gamma_2(n)s,$$

so it is enough to check that

$$32\gamma_2(n) \leq \frac{\rho(n)}{2} - 1,$$

but this is true thanks to the definition of $\rho(n)$ in (3.11).

We can now conclude the proof. Let $x, y \in U_\varphi(0,s) \setminus J_\theta^\varphi$ and $r = d_\varphi(x,y)$. Then

$$\begin{aligned} |\varphi(x) - \varphi(y)| &\leq |\varphi(x) - (\varphi)_{x,c_3^L r}| + |(\varphi)_{x,c_3^L r} - (\varphi)_{y,c_3^L r}| + |\varphi(y) - (\varphi)_{y,c_3^L r}| \\ &\leq \left(2(c_3^L)^{2n+2} + 2^{2n+3}c_3^L\right)\left(\frac{c_2^L}{c_1^L}\right)^2 C_1^L (C_2^L)^{2n+1}\theta r \\ &= C(n,L)\, d_\varphi(x,y) \end{aligned}$$

and (3.13) follows. □

3. Proof of Theorem 3.1

In this section, we prove Theorem 3.1 following the ideas outlined in [**24**, **25**]. As we already did for the proof of Theorem 2.2, up to replacing E with its blow-up $E_{p_0,r}$ and, correspondingly, φ with $\varphi_r = \frac{1}{r}\varphi \circ \delta_r$, we can simplify Theorem 3.1 to the following statement.

THEOREM 3.7. *Let $n \geq 2$ and $\alpha \in (0, \frac{1}{2})$. There exist positive constants $C_2(n)$, $\varepsilon_2(\alpha, n)$ and $k_2 = k_2(n)$ with the following property. Let $E \subset \mathbb{H}^n$ be a (Λ', r_0')-minimizer of H-perimeter in C_{k_2} with*

$$\Lambda' = \Lambda r, \qquad r_0' = \frac{r_0}{r} > k_2, \qquad \Lambda' r_0' \leq 1, \qquad 0 \in \partial E, \qquad \mathbf{e}(k_2) \leq \varepsilon_2(\alpha, n).$$

Let $\varphi\colon \mathbb{W} \to \mathbb{R}$ be a suitably chosen approximation given by Theorem 2.2. Then there exists a set $K \subset D_1$ such that

$$\mathcal{L}^{2n}(D_1 \setminus K) \leq C_2(n)\, \mathbf{e}(k_2)^{1-2\alpha}, \tag{3.14}$$

$$\mathrm{gr}(\varphi|_K) = \partial E \cap \Big(K * (-1,1)\Big), \tag{3.15}$$

$$\mathrm{Lip}_H(\varphi|_K) \leq C_2(n)\, \mathbf{e}(k_2)^\alpha. \tag{3.16}$$

We need some preliminaries. The following result is an easy consequence of Cauchy–Schwarz inequality, see [**25**, Lemma A.1] and [**24**, Proposition 2.1].

Lemma 3.8. *Let $W \subset \mathbb{W}$ be an open set and let $\varphi\colon W \to \mathbb{R}$ be an L-intrinsic Lipschitz function. For any Borel set $A \subset\subset W$, we have*

$$\left(\int_A |\nabla^\varphi \varphi| \, d\mathcal{L}^{2n}\right)^2 \leq \sqrt{1 + \|\nabla^\varphi\varphi\|_{L^\infty(W)}}\, \mathcal{L}^{2n}(A) \int_{\mathrm{gr}(\varphi|_A)} \frac{|\nabla^\varphi\varphi|^2}{1+|\nabla^\varphi\varphi|^2}\, d|\mu_{E_\varphi}|. \tag{3.17}$$

Proof. Let $A \subset\subset W$ be fixed. Then, by the general area formula (1.28),

$$\begin{aligned}
\int_A |\nabla^\varphi\varphi|\, d\mathcal{L}^{2n} &= \int_{\mathrm{gr}(\varphi|_A)} \frac{|\nabla^\varphi\varphi|}{\sqrt{1+|\nabla^\varphi\varphi|^2}}\, d|\mu_{E_\varphi}| \\
&\leq \left(\int_{\mathrm{gr}(\varphi|_A)} d|\mu_{E_\varphi}|\right)^{\frac12} \left(\int_{\mathrm{gr}(\varphi|_A)} \frac{|\nabla^\varphi\varphi|^2}{1+|\nabla^\varphi\varphi|^2}\, d|\mu_{E_\varphi}|\right)^{\frac12} \\
&= \left(\int_A \sqrt{1+|\nabla^\varphi\varphi|^2}\, d\mathcal{L}^{2n}\right)^{\frac12} \left(\int_{\mathrm{gr}(\varphi|_A)} \frac{|\nabla^\varphi\varphi|^2}{1+|\nabla^\varphi\varphi|^2}\, d|\mu_{E_\varphi}|\right)^{\frac12} \\
&\leq \sqrt[4]{1+\|\nabla^\varphi\varphi\|_{L^\infty(W)}}\, \mathcal{L}^{2n}(A)^{\frac12} \left(\int_{\mathrm{gr}(\varphi|_A)} \frac{|\nabla^\varphi\varphi|^2}{1+|\nabla^\varphi\varphi|^2}\, d|\mu_{E_\varphi}|\right)^{\frac12}
\end{aligned}$$

and (3.17) follows squaring both sides. □

The following lemma compares the distance d_φ with the distance of points of the graph of an intrinsic Lipschitz function φ, see [**15**, Proposition 3.6].

Lemma 3.9. *Let $W \subset \mathbb{W}$ be an open set and let $\varphi\colon W \to \mathbb{R}$ be an intrinsic Lipschitz function. Then, for all $x \in W$, $r > 0$ and $0 < C < 1/(1 + \mathrm{Lip}_H(\varphi))$, we have*

$$U_\varphi(x, Cr) \subset \pi\Big(B_r(\Phi(x)) \cap \mathrm{gr}(\varphi)\Big) \subset U_\varphi(x, r), \tag{3.18}$$

where $U_\varphi(x, r)$ is as in (3.5) *and $\Phi(x) = x * \varphi(x)\mathrm{e}_1$.*

Finally, the following result compares the distance d_φ with the distance d_∞ in W. Its proof easily follows from Definition 1.15 and is left to the reader.

Lemma 3.10. *Let $W \subset \mathbb{W}$ be an open set and let $\varphi\colon W \to \mathbb{R}$ be a bounded intrinsic Lipschitz function. Then, for all $x \in W$ and $r > 0$, we have*

$$U_\varphi(x, r) \subset D_R(x) \qquad \text{and} \qquad D_r(x) \subset U_\varphi(x, R),$$

where $R = r + 2\|\varphi\|^{1/2}_{L^\infty(W)} r^{1/2}$.

Proof of Theorem 3.7. The proof is divided in three steps.

Step 1: construction of φ, K and proof of (3.15). Let $\alpha \in (0, \frac12)$ be fixed. We assume $\varepsilon_2(n, \alpha) \leq \varepsilon_1(n)$ and $k_2 > 642$. We let $\varphi\colon \mathbb{W} \to \mathbb{R}$ be a $(1; \frac{k_2}{642})$-representative of Theorem 2.2, see Remark 2.5. Choosing $\varepsilon_2(n, \alpha)$ sufficiently small, by (2.3) we can assume that $\sup_{\mathbb{W}}|\varphi| < 1$.

Let $I=(-\frac{k_2}{642},\frac{k_2}{642})$ and let $A\subset D_{\frac{k_2}{642}}$ be a Borel set. By (2.20) and (2.21), we have

$$\begin{aligned}\int_{\mathrm{gr}(\varphi|_A)}\frac{|\nabla^\varphi\varphi|^2}{1+|\nabla^\varphi\varphi|^2}\,d|\mu_{E_\varphi}| &= \delta(n)\int_{\mathrm{gr}(\varphi|_A)}\frac{|\nabla^\varphi\varphi|^2}{1+|\nabla^\varphi\varphi|^2}\,d\mathcal{S}^{2n+1}=\\ &=\delta(n)\left(\int_{\mathrm{gr}(\varphi|_A)\cap\partial E\cap A*I}\frac{|\nabla^\varphi\varphi|^2}{1+|\nabla^\varphi\varphi|^2}\,d\mathcal{S}^{2n+1}+\int_{(\mathrm{gr}(\varphi|_A)\setminus\partial E)\cap A*I}\frac{|\nabla^\varphi\varphi|^2}{1+|\nabla^\varphi\varphi|^2}\,d\mathcal{S}^{2n+1}\right)\\ &\le 2\int_{\partial E\cap A*I}\frac{|\nu_E-\nu|_g^2}{2}\,d|\mu_E|+\int_{(\mathrm{gr}(\varphi|_A)\setminus\partial E)\cap A*I}\frac{|\nabla^\varphi\varphi|^2}{1+|\nabla^\varphi\varphi|^2}\,d|\mu_{E_\varphi}|,\end{aligned}$$

where $\delta(n)=\frac{2\omega_{2n-1}}{\omega_{2n+1}}$ as in Theorem 1.2. We let the non-negative measure μ on $D_{\frac{k_2}{642}}$ be defined as

$$\mu(A)=2\int_{\partial E\cap A*I}\frac{|\nu_E-\nu|_g^2}{2}\,d|\mu_E|+\int_{(\mathrm{gr}(\varphi|_A)\setminus\partial E)\cap A*I}\frac{|\nabla^\varphi\varphi|^2}{1+|\nabla^\varphi\varphi|^2}\,d|\mu_{E_\varphi}|, \tag{3.19}$$

for any Borel set $A\subset D_{\frac{k_2}{642}}$, where $\nu=-X_1$ as usual.

Let $0<\eta<1$ to be fixed later. We let

$$K_\eta=\Big\{x\in D_{\frac{k_2}{642}}:M\mu(x)\le\eta\Big\},$$

where $M\mu$ is the local maximal function of μ defined in (3.1) with $s=\frac{k_2}{2568}$. We assume $k_2>2568$ and we define

$$K=K_\eta\cap D_1.$$

We now prove (3.15). Since φ is a $(1;\frac{k_2}{642})$-representative of Theorem 2.2, by Remark 2.5 it is enough to prove that $K\in\mathcal{A}(1;\frac{k_2}{642})$. To this end, let us fix $p\in M\cap D_1*I$ and $q\in M\cap K*I$. We proceed as in Steps 1 of the proof of Theorem 2.4. Indeed, by (1.13) in Lemma 1.10 we have

$$|\mathpzc{h}(\xi)|<1\quad\forall\xi\in C_{\frac{k_2}{642}}\cap\partial E, \tag{3.20}$$

since E is a $(\frac{1}{k_2},k_2)$-minimizer of H-perimeter in $C_{\frac{k_2}{321}}$ and, by (1.10), we can estimate

$$\mathbf{e}(\tfrac{k_2}{321})\le 321^{2n+1}\mathbf{e}(k_2)\le 321^{2n+1}\varepsilon_2(n,\alpha)\le\omega(n,\tfrac12,\tfrac{1}{k_2},k_2),$$

provided we assume

$$\varepsilon_2(n,\alpha)\le 321^{-2n-1}\omega(n,\tfrac12,\tfrac{1}{k_2},k_2).$$

Thus we have $p,q\in C_1$ and $d_C(p,q)<4$, where d_C is the quasi distance given by the quasi norm $\|\cdot\|_C$ defined in (1.6). Moreover, $q=\pi(q)*\mathpzc{h}(q)\mathrm{e}_1$ with $\pi(q)\in K$ and $|\mathpzc{h}(q)|<1$. Since

$$C_s(\xi)\subset\pi(C_s(\xi))*(-s-\mathpzc{h}(\xi),\mathpzc{h}(\xi)+s)\subset D_{2s}(\pi(\xi))*I \tag{3.21}$$

for any $\xi \in C_1$ and $0 < s < \frac{k_2}{642} - 1$, we can estimate

$$\begin{aligned}\mathbf{e}(q,s) &= \frac{1}{s^{2n+1}} \int_{C_s(q)\cap\partial E} \frac{|\nu_E - \nu|_g^2}{2}\, d|\mu_E| \\ &\le \frac{1}{s^{2n+1}} \int_{\partial E\cap D_{2s}(\pi(q))*I} \frac{|\nu_E - \nu|_g^2}{2}\, d|\mu_E| \\ &\le 2^{2n+1}\kappa_n \sup_{0<\rho<\frac{k_2}{642}-\|\pi(q)\|_\infty} \frac{1}{\kappa_n \rho^{2n+1}} \int_{\partial E\cap D_\rho(\pi(q))*I} \frac{|\nu_E - \nu|_g^2}{2}\, d|\mu_E| \\ &\le 2^{2n}\kappa_n M\mu(\pi(q)) \le 2^{2n}\kappa_n \eta\end{aligned}$$

for any $0 < s < \frac{k_2}{1284}$, where $\kappa_n = \mathcal{L}^{2n}(D_1)$ as in (1.4).

We consider the blow-up of E at scale $d_C(p,q)$ centred at q, that is, $F = E_{q,d_C(p,q)}$. By Remark 1.4, F is a (Λ'', r_0'')-perimeter minimizer in $(C_{k_2})_{q,d_C(p,q)}$, with

$$\Lambda'' = \Lambda'\, d_C(p,q), \qquad r_0'' = \frac{r_0'}{d_C(p,q)} > 1.$$

Now

$$C_{16} \subset (C_{k_2})_{q,d_C(p,q)}, \qquad \Lambda'' r_0'' \le 1, \qquad 0 \in \partial F$$

and, by (1.11) and by definition of M_0,

$$\mathbf{e}(F,0,16,\nu) = \mathbf{e}(E,q,16 d_C(p,q),\nu) \le 2^{2n}\kappa_n\eta,$$

since we can choose $k_2 > 82176$. Therefore, provided we assume

$$2^{2n}\kappa_n\eta \le \varepsilon_0(n),$$

by Theorem 1.8 we have

$$\sup\Big\{\not{h}(\xi) : \xi \in C_1 \cap \partial F\Big\} \le C(n)\eta^{\frac{1}{2(2n+1)}},$$

where $C(n)$ is a dimensional constant. In particular, choosing

$$\xi = \frac{1}{d_C(p,q)}\, q^{-1} * p \in C_1 \cap \partial F,$$

we get

$$|\not{h}(q^{-1} * p)| \le C(n)\eta^{\frac{1}{2(2n+1)}} d_C(p,q). \tag{3.22}$$

We now set

$$L'(n,\eta) = C(n)\eta^{\frac{1}{2(2n+1)}}$$

and we choose η so small that $L'(n,\eta) \le L(n)$, where $L(n) < 1$ is as in (2.11). Then, by (3.22), we conclude that $d_C(p,q) = \|\pi(q^{-1} * p)\|_\infty$ and we get

$$|\not{h}(q^{-1} * p)| \le L(n)\|\pi(q^{-1} * p)\|_\infty \qquad \text{for all } p \in M \cap D_1 * I,\ q \in M \cap K * I, \tag{3.23}$$

so $K \in \mathcal{A}(1; \frac{k_2}{642})$. Thus, by (3.20) and (3.23), equality (3.15) follows.

Step 2: proof of (3.14). We now apply Lemma 3.3 with $s = \frac{k_2}{2568}$ and measure μ as defined in (3.19). By Theorem 2.2, we have

$$\begin{aligned}
\mu(D_{k_2/642}) &= 2\int_{\partial E\cap D_{k_2/642}*I} \frac{|\nu_E-\nu|_g^2}{2}\, d|\mu_E| + \int_{(\mathrm{gr}(\varphi)\setminus\partial E)\cap D_{k_2/642}*I} \frac{|\nabla^\varphi\varphi|^2}{1+|\nabla^\varphi\varphi|^2}\, d|\mu_{E_\varphi}| \\
&= 2\int_{\partial E\cap C_{k_2/642}} \frac{|\nu_E-\nu|_g^2}{2}\, d|\mu_E| + \int_{(\mathrm{gr}(\varphi)\setminus\partial E)\cap C_{k_2/642}} \frac{|\nabla^\varphi\varphi|^2}{1+|\nabla^\varphi\varphi|^2}\, d|\mu_{E_\varphi}| \\
&\le 2\left(\tfrac{k_2}{642}\right)^{2n+1}\mathbf{e}(\tfrac{k_2}{642}) + C(n)\mathcal{S}^{2n+1}\Big((\partial E\bigtriangleup\mathrm{gr}(\varphi))\cap C_{k_2/642}\Big) \le C'(n)\,\mathbf{e}(k_2), \qquad (3.24)
\end{aligned}$$

where $C(n)$ and $C'(n)$ are dimensional constants. We now choose $\eta = \mathbf{e}(k_2)^{2\alpha}$. In order to apply Lemma 3.3, we need to check that

$$\mu(D_{k_2/642}) \le \frac{\eta}{5^{2n+1}}\kappa_n\left(\frac{k_2}{2568}\right)^{2n+1}.$$

By (3.24), this follows if we assume that

$$\varepsilon_2(n,\alpha) \le \left(\frac{\kappa_n}{C'(n)}\left(\frac{k_2}{12840}\right)^{2n+1}\right)^{\frac{1}{1-2\alpha}}.$$

We remark that this condition on $\varepsilon_2(n,\alpha)$ is the only one that depends also on the parameter α. Thus, by (3.3) in Lemma 3.3 and by (3.24), we conclude that

$$\begin{aligned}
\mathcal{L}^{2n}(D_1\setminus K) = \mathcal{L}^{2n}(J_\eta\cap D_1) &\le \frac{5^{2n+1}}{\eta}\,\mu\left(J_{\eta/2^{2n+1}}\cap D_{1+\frac{k_2}{12840}}\right) \\
&\le \frac{5^{2n+1}}{\mathbf{e}(k_2)^{2\alpha}}\,\mu\left(D_{k_2/642}\right) \le 5^{2n+1}C'(n)\mathbf{e}(k_2)^{1-2\alpha},
\end{aligned}$$

which proves (3.14).

Step 3: proof of (3.16). By Lemma 3.8 and by Proposition 1.17, we have

$$\begin{aligned}
\mu_\varphi(A)^2 &= \left(\int_A|\nabla^\varphi\varphi|\, d\mathcal{L}^{2n}\right)^2 \\
&\le \sqrt{1+\|\nabla^\varphi\varphi\|_{L^\infty(D_{k_2/642})}}\,\mathcal{L}^{2n}(A)\int_{\mathrm{gr}(\varphi|_A)}\frac{|\nabla^\varphi\varphi|^2}{1+|\nabla^\varphi\varphi|^2}\, d|\mu_{E_\varphi}| \\
&\le C(n)\,\mathcal{L}^{2n}(A)\int_{\mathrm{gr}(\varphi|_A)}\frac{|\nabla^\varphi\varphi|^2}{1+|\nabla^\varphi\varphi|^2}\, d|\mu_{E_\varphi}|
\end{aligned}$$

for all Borel sets $A \subset D_1$, where $C(n)$ is a dimensional constant. Moreover, for any $x \in K$ and $4r < \frac{k_2}{642} - \|x\|_\infty$, by (3.18) in Lemma 3.9, by (1.8) and by (3.21), we have

$$\begin{aligned}\int_{\Phi(U_\varphi(x,r))} \frac{|\nabla^\varphi \varphi|^2}{1+|\nabla^\varphi \varphi|^2}\, d|\mu_{E_\varphi}| &\le \int_{\Gamma\cap B_{2r}(\Phi(x))} \frac{|\nabla^\varphi \varphi|^2}{1+|\nabla^\varphi \varphi|^2}\, d|\mu_{E_\varphi}| \\ &\le \int_{\Gamma\cap C_{2r}(\Phi(x))} \frac{|\nabla^\varphi \varphi|^2}{1+|\nabla^\varphi \varphi|^2}\, d|\mu_{E_\varphi}| \\ &\le 2\int_{M\cap D_{4r}(x)*I} \frac{|\nu_E-\nu|_g^2}{2}\, d|\mu_E| + \int_{(\Gamma\setminus M)\cap D_{4r}(x)*I} \frac{|\nabla^\varphi \varphi|^2}{1+|\nabla^\varphi \varphi|^2}\, d|\mu_{E_\varphi}| \\ &= \mu(D_{4r}(x)).\end{aligned}$$

Therefore, for any $x \in K$ and $4r < \frac{k_2}{642} - \|x\|_\infty$, we get

$$\mu_\varphi(U_\varphi(x,r))^2 \le C(n)\, \mathcal{L}^{2n}(U_\varphi(x,r))\, \mu(D_{4r}(x)). \tag{3.25}$$

We now apply Lemma 3.6. We choose the parameter $s > 0$ in Lemma 3.6 such that

$$D_1 \subset U_\varphi(0,s) \quad \text{and} \quad U_\varphi(0,\rho(n)s) \subset D_{k_2},$$

where $\rho(n)$ is the dimensional constant defined in (3.11). Since $\mathrm{Lip}_H(\varphi) \le L(n) < 1$, where $L(n)$ is the dimensional constant defined in (2.11), possibly choosing $\varepsilon_2(n,\alpha)$ smaller, we can directly assume that $L \le \ell(n)$ as in (3.8). In particular, the constant $c(n,L)$ appearing in (3.13) of Lemma 3.6, is controlled from above by a dimensional constant. Since $\sup_{\mathbb{W}}|\varphi| < 1$, by Lemma 3.10 we can choose $s = 3$ provided that we also choose

$$k_2(n) \ge 3\rho(n) + 2\sqrt{3\rho(n)}.$$

We then have

$$r_\varphi(x,3) = \frac{3\rho(n)}{c_L} - d_\varphi(x,0) \le 3\rho(n),$$

where $r_\varphi(x,s)$ was defined in (3.10). By (3.25) and by Lemma 3.5, for any $x \in K$ we have

$$\begin{aligned}[\mu_\varphi](x)^2 &= \sup_{0<r<r_\varphi(x,3)} \frac{\mu_\varphi(U_\varphi(x,r))^2}{\mathcal{L}^{2n}(U_\varphi(x,r))^2} \le C(n) \sup_{0<r<3\rho(n)} \frac{\mu(D_{4r}(x))}{\mathcal{L}^{2n}(U_\varphi(x,r))} \\ &\le \frac{C(n)4^{2n+1}\kappa_n}{c_1^L} \sup_{0<r<3\rho(n)} \frac{\mu(D_{4r}(x))}{\kappa_n(4r)^{2n+1}} \\ &\le C'(n) \sup_{0<\rho<12\rho(n)} \frac{\mu(D_\rho(x))}{\kappa_n \rho^{2n+1}}\end{aligned}$$

where $C'(n)$ is a dimensional constant. Now we can choose

$$k_2 > 7704\rho(n) + 642,$$

so that $12\rho(n) \le \frac{k_2}{642} - \|x\|_\infty$ for any $x \in D_1$. Therefore, for any $x \in K$, we get

$$[\mu_\varphi](x) \le \sqrt{C'(n)\,\eta} = C''(n)\, \mathbf{e}(k_2)^\alpha,$$

where $C''(n)$ is a positive dimensional constant. Thus $K \subset U_\varphi(0,3) \setminus J_\theta^\varphi$, where J_θ^φ is as in (3.12) and $\theta = C''(n)\,\mathbf{e}(k_2)^\alpha$. Therefore, by (3.13) in Lemma 3.6, we conclude that

$$|\varphi(x) - \varphi(y)| \le C(n)\,\mathbf{e}(k_2)^\alpha\, d_\varphi(x,y) \qquad \text{for all } x, y \in K.$$

This proves (3.16) and the proof of Theorem 3.7 is complete. □

4. Proof of Corollary 3.2

In this section, we prove Corollary 3.2. As we already did for the proof of Theorem 3.1, up to replacing E with its blow-up $E_{p_0,r}$ and, correspondingly, φ with $\varphi_r = \frac{1}{r}\varphi \circ \delta_r$, we can simplify Corollary 3.2 to the following statement.

COROLLARY 3.11. *Let $n \ge 2$ and $\alpha \in (0, \frac{1}{2})$. There exist positive constants $C_3(n)$, $\varepsilon_3(\alpha, n)$ and $k_3 = k_3(n)$ with the following property. Let $E \subset \mathbb{H}^n$ be a (Λ', r_0')-minimizer of H-perimeter in C_{k_3} with*

$$\Lambda' = \Lambda r, \qquad r_0' = \frac{r_0}{r} > k_3, \qquad \Lambda' r_0' \le 1, \qquad 0 \in \partial E, \qquad \mathbf{e}(k_3) \le \varepsilon_3(\alpha, n).$$

Then there exist a set $K \subset D_1$ and an intrinsic Lipschitz function $\varphi\colon \mathbb{W} \to \mathbb{R}$ with the following properties:

$$\mathcal{L}^{2n}(D_1 \setminus K) \le C_3(n)\,\mathbf{e}(k_3)^{1-2\alpha},$$

$$\operatorname{gr}(\varphi|_K) = \partial E \cap K * (-1,1), \tag{3.26}$$

$$\mathcal{S}^{2n+1}\Big((\partial E \mathbin{\triangle} \operatorname{gr}(\varphi)) \cap C_1\Big) \le C_3(n)\,\mathbf{e}(k_3)^{1-2\alpha}, \tag{3.27}$$

$$\operatorname{Lip}_H(\varphi) \le C_3(n)\,\mathbf{e}(k_3)^\alpha,$$

$$\int_{D_1} |\nabla^\varphi \varphi|^2 \, d\mathcal{L}^{2n} \le C_3(n)\,\mathbf{e}(k_3). \tag{3.28}$$

PROOF. Let $\alpha \in (0, \frac{1}{2})$ be fixed and assume that $\varepsilon_3(n,\alpha) \le \varepsilon_2(n,\alpha)$ and $k_3 = k_2$. Let K and φ be as in Theorem 3.7. Recall that, by construction, $\operatorname{Lip}_H(\varphi) < 1$ and $\sup_{\mathbb{W}}|\varphi| < 1$. Moreover, by (3.16), we have

$$\operatorname{Lip}_H(\varphi|_K) \le C_2(n)\,\mathbf{e}(k_2)^\alpha.$$

Thus, according to Proposition 1.14, choosing $\varepsilon_3(n,\alpha) \le \varepsilon_2(n,\alpha)$ sufficiently small, we can extend φ outside K to the whole $\mathbb{W}$ in such a way that $\sup_{\mathbb{W}}|\varphi| < 1$ and

$$\operatorname{Lip}_H(\varphi) \le C(n)\,\mathbf{e}(k_3)^\alpha,$$

where $C(n)$ is a dimensional constant. Thus we only need to prove (3.27) and (3.28).

We prove (3.27). Let $J = D_1 \setminus K$, $I = (-1,1)$ and note that, by (3.26), we have

$$\begin{aligned}
\mathcal{S}^{2n+1}\Big((\partial E \mathbin{\triangle} \operatorname{gr}(\varphi)) \cap C_1\Big) &= \mathcal{S}^{2n+1}\Big((\partial E \mathbin{\triangle} \operatorname{gr}(\varphi)) \cap J * I\Big) \\
&= \mathcal{S}^{2n+1}\Big((\partial E \setminus \operatorname{gr}(\varphi)) \cap J * I\Big) + \mathcal{S}^{2n+1}\Big((\operatorname{gr}(\varphi) \setminus \partial E) \cap J * I\Big) \\
&\le \mathcal{S}^{2n+1}(\partial E \cap J * I) + \mathcal{S}^{2n+1}(\operatorname{gr}(\varphi) \cap J * I).
\end{aligned}$$

On the one hand, by definition of excess 1.9 and by equality (1.16) in Lemma 1.11, we have

$$\begin{aligned}\mathcal{S}^{2n+1}(\partial E\cap J*I) &= \int_{\partial E\cap J*I} 1+\langle \nu_E, X_1\rangle \, d\mathcal{S}^{2n+1} - \int_{\partial E\cap J*I}\langle \nu_E, X_1\rangle \, d\mathcal{S}^{2n+1} = \\ &= \delta(n)^{-1}\int_{\partial E\cap J*I}\frac{|\nu_E-\nu|_g^2}{2}\, d|\mu_E| + \mathcal{L}^{2n}(J)\\ &\le \delta(n)^{-1}\mathbf{e}(1)+\mathcal{L}^{2n}(J), \end{aligned} \tag{3.29}$$

thus, by (1.10) and by (3.14), we can estimate

$$\mathcal{S}^{2n+1}(\partial E\cap J*I)\le \delta(n)^{-1}\, k_3^{2n+1}\,\mathbf{e}(k_3)+C_2(n)\,\mathbf{e}(k_3)^{1-2\alpha}\le C(n)\,\mathbf{e}(k_3)^{1-2\alpha}, \tag{3.30}$$

where $C(n)$ is a dimensional constant. On the other hand, by the area formula (1.27), we have

$$\begin{aligned}\mathcal{S}^{2n+1}(\mathrm{gr}(\varphi)\cap J*I) &= \delta(n)^{-1}\int_J\sqrt{1+|\nabla^\varphi\varphi|^2}\, d\mathcal{L}^{2n}\\ &\le \delta(n)^{-1}\sqrt{1+\|\nabla^\varphi\varphi\|^2_{L^\infty(D_1)}}\,\mathcal{L}^{2n}(J),\end{aligned} \tag{3.31}$$

thus, by Proposition 1.17 and again by (3.14), we can estimate

$$\mathcal{S}^{2n+1}(\mathrm{gr}(\varphi)\cap J*I)\le C(n)\,\mathbf{e}(k_3)^{1-2\alpha},$$

where $C(n)$ is a dimensional constant. Combining (3.29) with (3.30) and (3.31), we prove (3.27).

We prove (3.28). Since $D_1 = K\cup J$ with disjoint union, we can split

$$\int_{D_1}|\nabla^\varphi\varphi|^2\, d\mathcal{L}^{2n} = \int_K|\nabla^\varphi\varphi|^2\, d\mathcal{L}^{2n}+\int_J|\nabla^\varphi\varphi|^2\, d\mathcal{L}^{2n}. \tag{3.32}$$

On the one hand, by Proposition 1.17 and by (3.15), we have

$$\begin{aligned}\int_K|\nabla^\varphi\varphi|^2\, d\mathcal{L}^{2n} &= \int_{\mathrm{gr}(\varphi|_K)}\frac{|\nabla^\varphi\varphi|^2}{\sqrt{1+|\nabla^\varphi\varphi|^2}}\, d|\mu_{E_\varphi}|\\ &\le \sqrt{1+\|\nabla^\varphi\varphi\|^2_{L^\infty(D_1)}}\int_{\mathrm{gr}(\varphi|_K)}\frac{|\nabla^\varphi\varphi|^2}{1+|\nabla^\varphi\varphi|^2}\, d|\mu_{E_\varphi}|\\ &\le C(n)\int_{M\cap K*I}\frac{|\nu_E-\nu|_g^2}{2}\, d|\mu_E|\le C(n)\,\mathbf{e}(1)\le C'(n)\,\mathbf{e}(k_3)\end{aligned} \tag{3.33}$$

where $C(n)$ and $C'(n)$ are dimensional constants. On the other hand, again by Proposition 1.17 and by (3.16), we have

$$\begin{aligned}\int_J|\nabla^\varphi\varphi|^2\, d\mathcal{L}^{2n} &\le \|\nabla^\varphi\varphi\|^2_{L^\infty(D_1)}\,\mathcal{L}^{2n}(J)\\ &\le C(n)\,\mathrm{Lip}_H(\varphi)^2\mathcal{L}^{2n}(J)\le C'(n)\,\mathbf{e}(k_3).\end{aligned} \tag{3.34}$$

Combining (3.32) with (3.33) and (3.34), we prove (3.28). □

Bibliography

[1] W. K. Allard, *On boundary regularity for Plateau's problem*, Bull. Amer. Math. Soc. **75** (1969), 522–523.

[2] F. J. Almgren Jr., *Almgren's big regularity paper*, World Scientific Monograph Series in Mathematics, vol. 1, World Scientific Publishing Co., Inc., River Edge, NJ, 2000. Q-valued functions minimizing Dirichlet's integral and the regularity of area-minimizing rectifiable currents up to codimension 2; With a preface by Jean E. Taylor and Vladimir Scheffer.

[3] L. Ambrosio, *Corso introduttivo alla teoria geometrica della misura ed alle superfici minime*, Appunti dei Corsi Tenuti da Docenti della Scuola. [Notes of Courses Given by Teachers at the School], Scuola Normale Superiore, Pisa, 1997 (Italian).

[4] L. Ambrosio and B. Kirchheim, *Currents in metric spaces*, Acta Math. **185** (2000), no. 1, 1–80.

[5] L. Ambrosio, C. De Lellis, and T. Schmidt, *Partial regularity for area-minimizing currents in Hilbert spaces*, J. Reine Angew. Math., to appear, available at http://cvgmt.sns.it/paper/2112/.

[6] L. Ambrosio and M. Scienza, *Locality of the perimeter in Carnot groups and chain rule*, Ann. Mat. Pura Appl. (4) **189** (2010), no. 4, 661–678.

[7] L. Ambrosio, F. Serra Cassano, and D. Vittone, *Intrinsic regular hypersurfaces in Heisenberg groups*, J. Geom. Anal. **16** (2006), no. 2, 187–232.

[8] G. Arena, A. O. Caruso, and R. Monti, *Regularity properties of H-convex sets*, J. Geom. Anal. **22** (2012), no. 2, 583–602.

[9] E. Bombieri, *Regularity theory for almost minimal currents*, Arch. Rational Mech. Anal. **78** (1982), no. 2, 99–130.

[10] F. Bigolin, L. Caravenna, and F. Serra Cassano, *Intrinsic Lipschitz graphs in Heisenberg groups and continuous solutions of a balance equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **32** (2015), no. 5, 925–963.

[11] L. Capogna, G. Citti, and M. Manfredini, *Regularity of non-characteristic minimal graphs in the Heisenberg group $\mathbb{H}^1$*, Indiana Univ. Math. J. **58** (2009), no. 5, 2115–2160.

[12] ———, *Smoothness of Lipschitz minimal intrinsic graphs in Heisenberg groups $\mathbb{H}^n$, $n > 1$*, J. Reine Angew. Math. **648** (2010), 75–110.

[13] L. Capogna, D. Danielli, S. D. Pauls, and J. T. Tyson, *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, Progress in Mathematics, vol. 259, Birkhäuser Verlag, Basel, 2007.

[14] J.-H. Cheng, J.-F. Hwang, and P. Yang, *Regularity of C^1 smooth surfaces with prescribed p-mean curvature in the Heisenberg group*, Math. Ann. **344** (2009), no. 1, 1–35.

[15] G. Citti, M. Manfredini, A. Pinamonti, and F. Serra Cassano, *Smooth approximation for intrinsic Lipschitz functions in the Heisenberg group*, Calc. Var. Partial Differential Equations **49** (2014), no. 3-4, 1279–1308.

[16] ———, *Poincaré-type inequality for Lipschitz continuous vector fields*, J. Math. Pures Appl. (9) **105** (2016), no. 3, 265–292.

[17] D. Danielli, N. Garofalo, and D. M. Nhieu, *Sub-Riemannian calculus and monotonicity of the perimeter for graphical strips*, Math. Z. **265** (2010), no. 3, 617–637.

[18] E. De Giorgi, *Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad r dimensioni*, Ann. Mat. Pura Appl. (4) **36** (1954), 191–213 (Italian).

[19] ———, *Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio ad r dimensioni*, Ricerche Mat. **4** (1955), 95–113 (Italian).

[20] ———, *Sulla proprietà isoperimetrica dell'ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita*, Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I (8) **5** (1958), 33–44 (Italian).

[21] ———, *Frontiere orientate di misura minima*, Seminario di Matematica della Scuola Normale Superiore di Pisa, 1960-61, Editrice Tecnico Scientifica, Pisa, 1961 (Italian).

[22] E. De Giorgi, F. Colombini, and L. C. Piccinini, *Frontiere orientate di misura minima e questioni collegate*, Scuola Normale Superiore, Pisa, 1972 (Italian). MR0493669

[23] C. De Lellis, *Almgren's Q-valued functions revisited*, Proceedings of the International Congress of Mathematicians. Volume III, Hindustan Book Agency, New Delhi, 2010, pp. 1910–1933.

[24] C. De Lellis and E. Spadaro, *Higher integrability and approximation of minimal currents* (2009), preprint, available at http://arxiv.org/abs/0910.5878.

[25] ———, *Center manifold: a case study*, Discrete Contin. Dyn. Syst. **31** (2011), no. 4, 1249–1272.

[26] C. De Lellis and E. N. Spadaro, *Q-valued functions revisited*, Mem. Amer. Math. Soc. **211** (2011), no. 991, vi+79.

[27] C. De Lellis and E. Spadaro, *Regularity of area minimizing currents II: center manifold* (2013), preprint, available at http://arxiv.org/abs/1306.1191.

[28] ———, *Regularity of area minimizing currents III: blow up* (2013), preprint, available at http://arxiv.org/abs/1306.1194.

[29] ———, *Multiple valued functions and integral currents*, Ann. SC. Norm- Super. Pisa Cl. Sci. (5) (2014), to appear, available at http://arxiv.org/abs/1306.1188.

[30] ———, *Regularity of area minimizing currents I: gradient L^p estimates*, Geom. Funct. Anal. **24** (2014), no. 6, 1831–1884.

[31] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Revised edition, Textbooks in Mathematics, CRC Press, Boca Raton, FL, 2015.

[32] H. Federer, *Geometric Measure Theory*, Reprint of the 1969 Edition (Die Grundlehren der mathematischen Wissenschaften, Band 153), Classics in Mathematics, Springer-Verlag New York Inc., New York, 1996.

[33] ———, *The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension*, Bull. Amer. Math. Soc. **76** (1970), 767–771.

[34] H. Federer and W. H. Fleming, *Normal and integral currents*, Ann. of Math. (2) **72** (1960), 458–520.

[35] W. H. Fleming, *On the oriented Plateau problem*, Rend. Circ. Mat. Palermo (2) **11** (1962), 69–90.

[36] M. Fogagnolo, R. Monti, and D. Vittone, *Variation formulas for H-rectifiable sets* (2012), preprint, available at http://cvgmt.sns.it/paper/2871/.

[37] B. Franchi, R. Serapioni, and F. Serra Cassano, *Rectifiability and perimeter in the Heisenberg group*, Math. Ann. **321** (2001), no. 3, 479–531.

[38] ———, *Intrinsic Lipschitz graphs in Heisenberg groups*, J. Nonlinear Convex Anal. **7** (2006), no. 3, 423–441.

[39] ———, *Regular submanifolds, graphs and area formula in Heisenberg groups*, Adv. Math. **211** (2007), no. 1, 152–203.

[40] ———, *Differentiability of intrinsic Lipschitz functions within Heisenberg groups*, J. Geom. Anal. **21** (2011), no. 4, 1044–1084.

[41] M. Giaquinta and L. Martinazzi, *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, 2nd ed., Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)], vol. 11, Edizioni della Normale, Pisa, 2012.

[42] E. Giusti, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics, vol. 80, Birkhäuser Verlag, Basel, 1984.

[43] P. Hajłasz and P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **145** (2000), no. 688, x+101.

[44] B. Kirchheim and F. Serra Cassano, *Rectifiability and parameterization of intrinsic regular surfaces in the Heisenberg group*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **3** (2004), no. 4, 871–896.

[45] A. Korányi and H. M. Reimann, *Foundations for the theory of quasiconformal mappings on the Heisenberg group*, Adv. Math. **111** (1995), no. 1, 1–87.

[46] F. Maggi, *Sets of finite perimeter and geometric variational problems*, Cambridge Studies in Advanced Mathematics, vol. 135, Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.

[47] P. Mattila, R. Serapioni, and F. Serra Cassano, *Characterizations of intrinsic rectifiability in Heisenberg groups*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **9** (2010), no. 4, 687–723.

[48] R. Minio, *An interview with Michael Atiyah*, Math. Intelligencer **6** (1984), no. 1, 9–19.

[49] R. Monti, *Distances, Boundaries and Surface Measures in Carnot-Carathéodory Spaces* (2001), Ph.D. Thesis, available at http://www.math.unipd.it/~monti/PAPERS/TesiFinale.pdf.

[50] ———, *Lipschitz approximation of $\mathbb{H}$-perimeter minimizing boundaries*, Calc. Var. Partial Differential Equations **50** (2014), no. 1-2, 171–198.

[51] ———, *Minimal surfaces and harmonic functions in the Heisenberg group*, Nonlinear Anal. **126** (2015), 378–393.

[52] R. Monti, F. Serra Cassano, and D. Vittone, *A negative answer to the Bernstein problem for intrinsic graphs in the Heisenberg group*, Boll. Unione Mat. Ital. (9) **1** (2008), no. 3, 709–727.

[53] R. Monti and D. Vittone, *Sets with finite $\mathbb{H}$-perimeter and controlled normal*, Math. Z. **270** (2012), no. 1-2, 351–367.

[54] ———, *Height estimate and slicing formulas in the Heisenberg group*, Anal. PDE **8** (2015), no. 6, 1421–1454.

[55] S. D. Pauls, *H-minimal graphs of low regularity in $\mathbb{H}^1$*, Comment. Math. Helv. **81** (2006), no. 2, 337–381.

[56] M. Ritoré, *Examples of area-minimizing surfaces in the sub-Riemannian Heisenberg group $\mathbb{H}^1$ with low regularity*, Calc. Var. Partial Differential Equations **34** (2009), no. 2, 179–192.

[57] R. Schoen and L. Simon, *A new proof of the regularity theorem for rectifiable currents which minimize parametric elliptic functionals*, Indiana Univ. Math. J. **31** (1982), no. 3, 415–434.

[58] F. Serra Cassano and D. Vittone, *Graphs of bounded variation, existence and local boundedness of non-parametric minimal surfaces in Heisenberg groups*, Adv. Calc. Var. **7** (2014), no. 4, 409–492.

[59] E. N. Spadaro, *Complex varieties and higher integrability of Dir-minimizing Q-valued functions*, Manuscripta Math. **132** (2010), no. 3-4, 415–429.

[60] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy; Monographs in Harmonic Analysis, III.