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Rearrangements in metric spaces

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Contents

Introduction

In the Euclidean space $\mathbb{R}^d$, the Steiner rearrangement of functions or sets is defined as follows. Fix a direction $v \in \mathbb{R}^d$ and let $\Gamma = \{\tau_t\}_{t\in\mathbb{R}}$ be the group of translations of direction v. Namely, for any $x \in \mathbb{R}^d$, $\tau_t(x) = x + tv$. Clearly, the Lebesgue measure is Γ-invariant. We identify the quotient $\mathbb{R}^d/\Gamma$ with the hyperplane $H \subset \mathbb{R}^d$ through the origin and orthogonal to v. Moreover, for each $x \in H$, we can define the orbit Γ_x of x, i.e. $\Gamma_x = \{x + tv : \ t \in \mathbb{R}\}$. Then we define the rearrangement E^* of a set $E \subset \mathbb{R}^d$ as the set whose sections $E^*_x = E^* \cap \Gamma_x$, $x \in H$, are symmetric intervals with respect to H and have the same measure of the sections of E. The rearrangement f^* of a function $f : \mathbb{R}^d \to \mathbb{R}$, is then defined rearranging the upper-level sets of f.

The Steiner rearrangement produces functions and sets which more symmetric than the starting ones. In [Gio58] De Giorgi proved that the perimeter of sets does not increase under Steiner rearrangement.

The purpose of this thesis is to develop a general theory of rearrangements in the context of metric measure spaces. The first question is to give a reasonable definition of quantities as the perimeter or the L^p-norm of the gradient of a function without using any differentiable structure. Then we identify the structure required to define a rearrangement, thus defining the two-points rearrangement and the Steiner and Schwarz rearrangements. Finally, for these rearrangements, we prove several results.

There is an increasing literature on the subject of Sobolev and bounded variations functions in metric measure spaces. At least two counterparts of Sobolev spaces in metric measure spaces are studied: the Hajłasz spaces ([Haj96] and [Hei01]) and the Newtonian spaces ([Sha00]), while a definition of functions of bounded variation (and hence of perimeter) is given in [Mir03].

In this thesis, however, we choose a different approach which is motivated by the characterization of the Euclidean Sobolev and BV norms in the works [BBM01] and [Dáv02]. In this spirit, we prove Theorems 1.2.4 and 1.3.14. Here, we prove the equivalence between some norm of the gradient of functions $f : \mathbb{R}^d \to \mathbb{R}$, or its total variation, and the limit of a sort of "averaged incremental quotient". This quantity can be defined in any metric measure space. Here we state the result obtained in the case of the Euclidean Sobolev and BV norms.

Theorem 1. *Let $f \in L^p(\mathbb{R}^d)$, for $1 \le p < \infty$, then it holds:*

(i) *If $p > 1$, then $f \in W^{1,p}(\mathbb{R}^d)$ if and only if*

$$\liminf_{r \downarrow 0} \frac{1}{r^p} \int_{\mathbb{R}^d} \fint_{B_r(x)} |f(x) - f(y)|^p \, dy \, dx < +\infty. \tag{1}$$

Moreover, in this case the $\liminf$ *is in fact a* $\lim$ *and it holds*

$$\lim_{r \downarrow 0} \frac{1}{r^p} \int_{\mathbb{R}^d} \fint_{B_r(x)} |f(x) - f(y)|^p \, dy \, dx = K_{p,d} \|\nabla f\|_{L^p(\mathbb{R}^d)}. \tag{2}$$

(ii) *If $p = 1$, then $f \in BV(\mathbb{R}^d)$ if and only if*

$$\liminf_{r \downarrow 0} \frac{1}{r} \int_{\mathbb{R}^d} \fint_{B_r(x)} |f(x) - f(y)| \, dy \, dx < +\infty. \tag{3}$$

Moreover, in this case the $\liminf$ *is in fact a* $\lim$ *and it holds*

$$\lim_{r \downarrow 0} \frac{1}{r} \int_{\mathbb{R}^d} \fint_{B_r(x)} |f(x) - f(y)| \, dy \, dx = K_{1,d} |\nabla f|(\mathbb{R}^d). \tag{4}$$

Here, $K_{p,d}$ is a geometric constant depending only on the dimension d and the exponent p, defined in (1.3).

We use this Theorem as a blueprint to define the notion of "length of the gradient" in a metric measure space. Indeed, in the first part of chapter 3 we define Sobolev and BV functions in a metric measure space (X, d, μ) in the following way. Let $f \in L^1_{\text{loc}}(X, \mu)$ and $1 \le p < \infty$, then we let

$$\|\nabla f\|^-_{L^p(X,\mu)} = \liminf_{r \downarrow 0} \frac{1}{r^p} \int_X \fint_{B_r(x)} |f(x) - f(y)|^p \, d\mu(y) \, d\mu(x). \tag{5}$$

We point out that in the above definition the l.h.s. is a number, possibly $+\infty$. Then, we say that a function f is a Sobolev function with exponent $p > 1$ if $f \in L^p(X, \mu)$ and $\|\nabla f\|^-_{L^p(X,\mu)} < +\infty$ and that f is a function of bounded variation if $f \in L^1(X, \mu)$ and $\|\nabla f\|^-_{L^1(X,\mu)} < +\infty$. We also let the lower perimeter of a set E to be $P^-(E) = \|\nabla \chi_E\|^-_{L^1(X,\mu)}$, where χ_E denotes the characteristic function of E.

To justify definition (5), the first two chapters of this thesis are devoted to prove results analogous to Theorem 1 in specific non-Euclidean spaces, where it is possible to define naturally Sobolev and BV spaces. In particular, in chapter 1 we study the case of a finite dimensional Banach space, where we can make use of the differential structure of $\mathbb{R}^d$, while in chapter 2 we study the case of the Heisenberg group, where the horizontal Sobolev and BV spaces can be easily defined. We do not consider the case of an infinite dimensional Banach space.

More in detail, the core of the first chapter are Theorems 1.2.4 and 1.3.12. In these Theorems the r.h.s. of (2) and (4) is expressed in terms of what we call the p-mean norm associated to the norm of the Banach space. In the Euclidean case, the p-mean norm reduces to $K_{p,d}|\cdot|$, where $K_{p,d}$ is the geometric constant defined in (1.3). The proof of Theorem 1.2.4 uses some ideas from [Bré02], while the one of Theorem 1.3.12 some techniques of [Dáv02] and some results regarding the total variation with respect to non-Euclidean norms, found in [AB94].

Chapter 2 deals with the Heisenberg group $\mathbf{H}^d$. The first part is devoted to a short introduction to this space and to the definition of the horizontal Sobolev and BV spaces. For this part we refer to [CDPT07]. In the second part we prove Theorem 2.1.9 for horizontal Sobolev functions. Theorem 2.1.11 deals with horizontal bounded variation functions. The latter is fairly weaker than statement (ii) in Theorem 1. In fact, for (4) to hold we need to assume that the function is both of horizontal bounded variation and of bounded variation in the sense of $\mathbb{R}^{2d+1}$.

In chapter 3 we study the *two-points rearrangement*, or polarization, with respect to a reflection system $\mathcal{R}$, consisting of a partition $\{H^-, H, H^+\}$ of the metric space X and of a reflection with respect to H. This technique is central in the proofs of results regarding more general rearrangements. The two-points rearrangement of a function f is a function $f_{\mathcal{R}}$ "polarized" such that on each couple of points x and ϱx the function attains the maximum value in H^+ and minimum value in H^-. For a set we simply rearrange its characteristic function. This procedure does not increase $\|\nabla f\|^-_{L^p(X,\mu)}$ and the lower perimeter. This is proved in Theorem 3.3.18.

Next we define a *rearrangement system*, the minimal structure needed to rearrange functions or sets. As in the Euclidean case, we need a group of isometries Γ, such that μ is Γ-invariant, identifying a quotient space $X/\Gamma \subset X$, and orbits Γ_x, $x \in X/\Gamma$. An essential condition is the existence of a so-called *disintegration of* μ *along* Γ. Namely, we need a sort of non-orthogonal Fubini theorem, in the sense that there exist measures $(\mu_x)_{x\in X/\Gamma}$ over Γ_x and a measure $\bar{\mu}$ over X/Γ such that

$$\mu(E) = \int_{X/\Gamma} \mu_x(E\cap\Gamma_x)\,d\bar{\mu}(x), \qquad \text{for any Borel set } E\subset X.$$

We call a 3-tuple $(\Gamma, (\mu_x)_{x\in X/\Gamma}, \bar{\mu})$ consisting of a group of isometries and a disintegration of μ along Γ satisfying some structural assumptions, a *rearrangement system* (see Definition 3.4.24).

Section 3.5 is devoted to the problem of the existence of a disintegration. Here, we prove that such a disintegration is always possible in separable metric spaces, if the measure μ is finite. This proof is an adaption of the arguments in [AFP00] and [DM78]. In Proposition 3.5.29 we extend this result to all Γ-invariant measures, if Γ is a 1-parameter group.

In the last part of chapter 3, section 3.6, we prove the main Theorems on the rearrangement. Given a reflection system $\mathcal{R}$ that behaves coherently with respect to a 1-parameter group of isometries T, we will call the tuple $(\mathcal{R}, T)$ a *Steiner system*. Introducing a compact group of isometries G, we can consider the group $\Gamma = \Gamma(T, G)$, generated by T and G. If the reflection system $\mathcal{R}$ behaves coherently with respect to the group Γ, the 3-tuple $(\mathcal{R}, T, G)$ is called a *Schwarz system.* We remark that a Steiner system is just a Schwarz system where the group G consists only of the identity.

Coupling a Schwarz system with a rearrangement system we are able to prove the following Theorem.

Theorem 2. *Let (X, d) be a proper metric space endowed with a Schwarz system $(\mathcal{R}, T, G)$. Let μ be a non-degenerate and diffuse Borel measure, in the sense of* (3.4) *and* (3.5)*, that is invariant with respect to the Schwarz system and let $(\Gamma, (\mu_x)_{x\in X/\Gamma}, \bar{\mu})$ be a regular rearrangement system of (X, μ), where $\Gamma = \Gamma(T, G)$. Finally, let the metric measure space (X, d, μ) have the Lebesgue property* (3.6)*. Then the rearrangement f^* of any compactly supported and non-negative function $f \in L^p(X, \mu)$, $1 < p < \infty$, satisfies*

$$\|f^*\|_{L^p(X,\mu)} = \|f\|_{L^p(X,\mu)} \quad \text{and} \quad \|\nabla f^*\|^-_{L^p(X,\mu)} \leq \|\nabla f\|^-_{L^p(X,\mu)}. \tag{6}$$

This is Theorem 3.6.37 and is the main result of the thesis. For the proof, we use some ideas introduced in [Bae94].

The last chapter of the thesis is devoted to develop a theory of rearrangements in the Heisenberg group. Indeed, in $\mathbf{H}^d$ endowed with the Carnot-Carathéodory metric, the natural way to define a reflection system does not yield a reflection system in the sense of the previous chapter. It yields, however, a reflection system for $\mathbf{H}^d$ with respect to the Euclidean metric. Given the particular structure of the Heisenberg group, it is then possible to bypass this problem working only with functions and sets with certain symmetries. In particular we define the horizontal and vertical reflection systems with symmetry σ (see Definitions 4.1.1 and 4.1.2). Exploiting the characterization of the norm of the horizontal gradient given in chapter 2 and using the approximation result in [FSC96], we then prove Theorem 4.1.4, a result on the monotonicity of the norm of the horizontal gradient analogous to Theorem 3.3.18. Then we define the *Steiner rearrangement* and the *cap rearrangement*, related to the horizontal and the vertical reflection systems with symmetry, respectively (see sections 4.2 and 4.3). We conclude the chapter proving, in Theorems 4.2.6 and 4.3.9, the following result on the monotonicity of the norm of the horizontal gradient.

Theorem 3. *Let $f \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$, $1 < p < \infty$, be a non-negative, σ-symmetric function and let f^* be the Steiner or the cap rearrangement of f. Then $f^* \in W^{1,p}_{\mathbf{H}}$ and*

$$\|f^*\|_{L^p(\mathbf{H}^d)} = \|f\|_{L^p(\mathbf{H}^d)} \quad \text{and} \quad \|\nabla_{\mathbf{H}} f^*\|_{L^p(\mathbf{H}^d)} \leq \|\nabla_{\mathbf{H}} f\|_{L^p(\mathbf{H}^d)}. \tag{7}$$

We did not investigate Theorem 1 in the Riemannian case, nor the connection between our definition of Sobolev and BV functions in metric measure spaces and the Hajłasz or the Newtonian spaces. We did not study the equality case in (6) and in (7), either. For a discussion of the Euclidean case we refer to [CCF05]. Finally we did not study applications of this theory. Whenever possible, however, we enriched the discussion with examples.

The topics of this thesis are part of a forthcoming paper, [MP]. Chapters 1 and 2 are the result of the work of the author under the supervision of the thesis advisor R.

Monti. In the last two chapters the contribution of the candidate was mainly a revision work, with the exception of sections 3.5 and 4.3 that contain original work of the author.

Chapter 1

The case of a finite dimensional Banach space

In this chapter we give a characterization of the length of the gradient of $W^{1,p}(\mathbb{R}^d)$ functions, $1 < p < +\infty$, and of functions of bounded variation, for finite dimensional Banach spaces.

Let $(V, \|\cdot\|)$ be a real Banach space of dimension d. For a basis $\{v_i\}_{i=1}^d$ of V, we can define the standard isomorphism $\Phi : V \to \mathbb{R}^d$ as $\Phi(v) = \Phi(\sum_{i=1}^d \alpha_i v_i) = \sum_{i=1}^d \alpha_i e_i$. Here, $\{e_i\}_{i=1}^d$ is the standard orthonormal basis of $\mathbb{R}^d$. With abuse of notation we let $\|x\| = \|\Phi^{-1}(x)\|$. All the results obtained for $(\mathbb{R}^d, \|\cdot\|)$ will be valid for $(V, \|\cdot\|)$.

From now on, let $x \cdot y$ denote the standard inner product on $\mathbb{R}^d$, $|x| = \sqrt{x \cdot y}$ the Euclidean norm, dx or dw the Lebesgue measure $\mathcal{L}^d$, and let $B_r = \{y \in \mathbb{R}^d : \|y\| < r\}$, $B_r(x) = \{y \in \mathbb{R}^d : \|y - x\| < r\}$ be the open balls of radius r with center in the origin and in $x \in \mathbb{R}^d$, respectively. For $v \in \mathbb{R}^d$, $v \neq 0$ and $r > 0$, the half balls $B_r^{\pm}(v)$ with respect to v of radius r and centered at $x \in \mathbb{R}^d$ are defined as

$$\begin{aligned} B_r^+(x; v) &= \{y \in B_r(x) : (y - x) \cdot v \geq 0\}, \\ B_r^-(x; v) &= \{y \in B_r(x) : (y - x) \cdot v \leq 0\}. \end{aligned} \tag{1.1}$$

Obviously, $B_r(x) = B_r^+(x; v) \cup B_r^-(x; v)$ for any $x, v \in \mathbb{R}^d$ and $r > 0$. As above, if the center is the origin, we let $B_r^{\pm}(v) = B_r^{\pm}(0; v)$. We recall also the notation for the averaged integral

$$⨍_A f(x)dx = \frac{1}{|A|} \int_A f(x)dx,$$

where $|A| = \mathcal{L}^d(A)$ denotes the Lebegue measure of A.

For the sake of simplicity, we work in $W^{1,p}(\mathbb{R}^d)$ and $BV(\mathbb{R}^d)$, but similar results hold in the case of $W^{1,p}(\Omega)$ and $BV(\Omega)$ for a smooth bounded domain Ω of $\mathbb{R}^n$. Our arguments are an adaption of the ones used in [Bré02, BBM01] in the case of $W^{1,p}(\mathbb{R}^d)$ and in [Dáv02] in the case of $BV(\mathbb{R}^d)$.

1.1 The p-mean norm

Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$. We define the *p-mean norm* $\|\cdot\|_{*,p}$, $1 \le p < \infty$, as

$$\|v\|_{*,p} = \left(\fint_{B_1} |v \cdot w|^p dw \right)^{1/p}, \qquad v \in \mathbb{R}^d. \tag{1.2}$$

Let $v \in \mathbb{R}^d$ and define $g_v : w \mapsto |v \cdot w|$, $w \in \mathbb{R}^d$. Then we have $\|v\|_{*,p} = |B_1|^{-1/p} \|g_v\|_{L^p(B_1)}$. Since $g_{v+u} \le g_v + g_u$, this proves that $\|\cdot\|_{*,p}$ is a norm on $\mathbb{R}^d$.

Any vector space of dimension d is isomorphic to $(\mathbb{R}^d, |\cdot|)$. Hence, for any $p \ge 1$ there exist two constants $C_{1,p}$, $C_{2,p}$ such that for all $x \in \mathbb{R}^d$

$$C_{1,p}\|x\|_{*,p} \le |x| \le C_{2,p}\|x\|_{*,p}.$$

Remark 1.1.1. Using the Coarea Formula it is easy to show that $|\cdot|_{*,p} = K_{p,d}|\cdot|$, where $K_{p,d}$ is a constant depending only on the dimension d and the exponent p. Namely, if e is any unit vector of $\mathbb{R}^d$, it holds

$$K_{p,d} = \left(\frac{d}{d+p} \fint_{|w|=1} |e \cdot w|^p \, dw \right)^{1/p}. \tag{1.3}$$

We write $\|\cdot\|_{*,1} = \|\cdot\|_*$, as the case $p = 1$ is special. In fact, using the symmetry with respect to the origin of the norm, we can compute,

$$\begin{aligned} \|v\|_* &= \frac{1}{2|B_1^+(v)|} v \cdot \left(\int_{B_1^+(v)} w \, dw - \int_{B_1^-(v)} w \, dw \right) \\ &= v \cdot \left(\frac{1}{|B_1^+(v)|} \int_{B_1^+(v)} w \, dw \right) \\ &= v \cdot \fint_{B_1^+(v)} w \, dw, \end{aligned}$$

Let $E : \mathbb{R}^d \to \mathbb{R}^d$ be defined as

$$E(v) = \fint_{B_1^+(v)} w \, dw. \tag{1.4}$$

It is trivial that:

i. $E(cv) = E(v)$ for any $c > 0$;

ii. $E(-v) = -E(v)$;

iii. $\|v\|_* = v \cdot E(v)$.

If $\|\cdot\| = |\cdot|$, the computations in Remark 1.1.1 yields $E(v) = K_{1,d}\, v$.

Lemma 1.1.2. *Let $\varphi \in C_c^1(\mathbb{R}^d)$ and $v \in \mathbb{R}^d$, then for all $x \in \mathbb{R}^d$ it holds*

$$\lim_{r\downarrow 0} \frac{1}{r} \fint_{B_r^+(x;v)} \big(\varphi(y) - \varphi(x)\big)\, dy = \nabla\varphi(x)\cdot E(v). \tag{1.5}$$

Proof. Since $\varphi \in C_c^1(\mathbb{R}^d)$ we can write

$$\varphi(y) - \varphi(x) = \nabla\varphi(x)\cdot(y-x) + R(x,y)$$

where $\frac{R(x,y)}{|x-y|} \to 0$ uniformly as $y \to x$. Hence we get

$$\begin{aligned}\frac{1}{r}\fint_{B_r^+(x;v)} \big(\varphi(y) - \varphi(x)\big)\, dy &= \frac{1}{r}\fint_{B_r^+(x;v)} \big(\nabla\varphi(x)\cdot(y-x) + R(x,y)\big)\, dy\\ &= \frac{1}{r}\fint_{B_r^+(v)} \big(\nabla\varphi(x)\cdot w\big)\, dw + \frac{1}{r}\fint_{B_r^+(x;v)} R(x,y)\, dy \\ &= \nabla\varphi(x)\cdot E(v) + \frac{1}{r}\fint_{B_r^+(x;v)} R(x,y)\, dy.\end{aligned} \tag{1.6}$$

From

$$\left| \frac{1}{r}\fint_{B_r^+(x;v)} R(x,y)\, dy \right| \le \sup_{y\in B_r(x)} \frac{1}{r}|R(x,y)| \le \sup_{y\in B_r(x)} \frac{|R(x,y)|}{|y-x|},$$

we get

$$\lim_{r\downarrow 0} \fint_{B_r^+(x;v)} \frac{R(x,y)}{r}\, dy = 0.$$

Hence, letting $r \downarrow 0$ in (1.6) yields (1.5). □

1.2 Sobolev spaces

Definition 1.2.3. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, we say that $g_i \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $i = 1, \dots, d$, is a *weak partial derivative* of f with respect to x_i if

$$\int_{\mathbb{R}^d} f \frac{\partial \varphi}{\partial x_i}\, dx = -\int_{\mathbb{R}^d} g_i\, \varphi\, dx, \qquad \text{for all } \varphi \in C_c^1(\mathbb{R}^d).$$

The weak partial derivative, if it exists, is uniquely defined $\mathcal{L}^d$-a.e. . We denote the weak partial derivative with respect to x_i of f as $\partial f/\partial x_i$. If f admits a weak partial derivative for any $i = 1, \dots, d$ we write

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_d}\right).$$

The Sobolev space $W^{1,p}(\mathbb{R}^d)$ for $1 \le p \le \infty$ is the space of all $f \in L^p(\mathbb{R}^d)$ such that the weak partial derivative $\partial f/\partial x_i$ exists and belongs to $L^p(\mathbb{R}^d)$ for any $i = 1, \dots, d$. It is a Banach space, when endowed with the norm

$$\|f\|^p_{W^{1,p}(\mathbb{R}^d)} = \|f\|^p_{L^p(\mathbb{R}^d)} + \int_{\mathbb{R}^d} |\nabla f(x)|^p\, dx.$$

1.2.1 Length of the gradient of $W^{1,p}(\mathbb{R}^d)$ functions, $1 < p < \infty$

Theorem 1.2.4. *Let $\|\cdot\|$ be a norm in $\mathbb{R}^d$ with balls $B_r(x)$ and let $\|\cdot\|_{*,p}$, $1 < p < \infty$, be the associated p-mean norm defined in* (1.2)*. Let $f \in L^p(\mathbb{R}^d)$. Then $f \in W^{1,p}(\mathbb{R}^d)$ if and only if*

$$\liminf_{r\downarrow 0} \frac{1}{r^p} \int_{\mathbb{R}^d} \fint_{B_r(x)} |f(x) - f(y)|^p \, dy \, dx < +\infty. \tag{1.7}$$

Moreover, in such case the limit inferior is in fact a limit and it results

$$\lim_{r\downarrow 0} \frac{1}{r^p} \int_{\mathbb{R}^d} \fint_{B_r(x)} |f(x) - f(y)|^p \, dy \, dx = \int_{\mathbb{R}^d} \|\nabla f(x)\|_{*,p}^p \, dx. \tag{1.8}$$

Proof. Let $f \in W^{1,p}(\mathbb{R}^d)$. We claim that there exists a constant $C > 0$ independent of f, such that for any $r > 0$

$$\frac{1}{r^p} \int_{\mathbb{R}^d} \fint_{B_r(x)} |f(x) - f(y)|^p \, dy \, dx \le C \int_{\mathbb{R}^d} \|\nabla f(x)\|_{*,p}^p \, dx. \tag{1.9}$$

This will imply (1.7). In order to prove (1.9), we recall that it is a well established fact (see Proposition IX.3 in [Bré83]) that, for any $f \in W^{1,p}(\mathbb{R}^d)$ and any $h \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} |f(x+h) - f(x)|^p \, dh \le |h|^p \int_{\mathbb{R}^d} |\nabla f(x)|^p \, dx.$$

From this we get

$$\int_{\mathbb{R}^d} |f(x+h) - f(x)|^p \, dh \le c^p |h|^p \int_{\mathbb{R}^d} \|\nabla f(x)\|_{*,p}^p \, dx. \tag{1.10}$$

Here $c > 0$ is a constant such that $|x| \le c\|x\|_{*,p}$ for any $x \in \mathbb{R}^d$. Integrating inequality (1.10) in the ball with radius $r > 0$ centered in the origin and dividing by $|B_r|$, yields

$$\begin{aligned} \fint_{B_r} \int_{\mathbb{R}^d} |f(x+h) - f(x)|^p \, dx \, dh &\le c^p \fint_{B_r} |h|^p \, dh \int_{\mathbb{R}^d} \|\nabla f(x)\|_{*,p}^p \, dx \\ &\le c^p r^p \int_{\mathbb{R}^d} \|\nabla f(x)\|_{*,p}^p \, dx. \end{aligned} \tag{1.11}$$

Hence, claim (1.9) is proved.

We now want to prove that if $f \in W^{1,p}(\mathbb{R}^d)$ then (1.8) holds. This is equivalent to prove that

$$\lim_{r\downarrow 0} \|T_r[f]\|_{L^p(\mathbb{R}^d \times \mathbb{R}^d)}^p = \int_{\mathbb{R}^d} \|\nabla f(x)\|_{*,p}^p \, dx, \tag{1.12}$$

where

$$T_r[f](x,y) = \frac{|f(x) - f(y)|}{r} \left(\frac{\chi_{B_r}(y-x)}{|B_r|} \right)^{1/p}.$$

The triangle inequality implies that the operator T_r is subadditive. This, together with (1.9), implies that, for any $r > 0$ and $f, g \in W^{1,p}(\mathbb{R}^d)$, it holds

$$\big| \|T_r[f]\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)} - \|T_r[g]\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)} \big| \le \|T_r[f-g]\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)} \le C \int_{\mathbb{R}^d} \|D(f-g)(x)\|_{*,p}^p \, dx.$$

Therefore we only need to establish (1.8) in some dense subset of $W^{1,p}(\mathbb{R}^d)$, e.g. in $C^2_c(\mathbb{R}^d)$.

In order to prove that (1.8) holds for $g \in C^2_c(\mathbb{R}^d)$, it suffices to prove it pointwise. Namely we claim that for any $x \in \mathbb{R}^d$ it holds

$$\lim_{r\downarrow 0} \frac{1}{r^p} \fint_{B_r(x)} |g(x) - g(y)|^p \, dy = \|\nabla g(x)\|_{*,p}^p. \tag{1.13}$$

Identity (1.8) then follows from (1.13) by dominated convergence. In fact, if L is a Lipschitz constant for g (i.e. $|g(x) - g(y)| \le L|x - y|$ for any $x, y \in \mathbb{R}^d$), then

$$\frac{1}{r^p} \fint_{B_r(x)} |g(x) - g(y)|^p \, dx \le L^p.$$

To prove (1.13), we fix $r > 0$ and $x \in \mathbb{R}^d$. Then, for any $y \in B_r(x)$, we have

$$g(x) - g(y) = \nabla g(x) \cdot (y - x) + R(x, y), \quad |R(x, y)| \le c|x - y|^2. \tag{1.14}$$

Here, c is independent of x and y. Thus we have that

$$|g(x) - g(y)|^p = |\nabla g(x) \cdot (y - x)|^p + L(x, y),$$

where we let

$$L(x, y) = |\nabla g(x) \cdot (y - x) + R(x, y)|^p - |\nabla g(x) \cdot (y - x)|^p.$$

Therefore we can write

$$\begin{aligned} \frac{1}{r^p} \fint_{B_r(x)} |g(x) - g(y)|^p \, dy &= \frac{1}{r^p} \fint_{B_r(x)} |\nabla g(x) \cdot (y - x)|^p \, dy + \fint_{B_r(x)} \frac{L(x, y)}{r^p} \, dy \\ &= \|\nabla g(x)\|_{*,p}^p + \fint_{B_r(x)} \frac{L(x, y)}{r^p} \, dy. \end{aligned} \tag{1.15}$$

Now we show that

$$\lim_{r\downarrow 0} \fint_{B_r(x)} \frac{L(x, y)}{r^p} \, dy = 0. \tag{1.16}$$

In fact, letting $\phi(t) = t^p$, by the mean value theorem we get that for any $0 < s < t$ it holds $\phi(t) - \phi(s) = \phi'(s^*)(t - s)$, where $s^* \in [s, t]$. Hence we have that

$$|L(x, y)| \le \big(|\nabla g(x) \cdot (y - x)| + |R(x, y)|\big)^p - |\nabla g(x) \cdot (y - x)|^p = p(s^*)^{p-1}|R(x, y)|,$$

where $s^* \in \big[|\nabla g(x)\cdot(y-x)|, |\nabla g(x)\cdot(y-x)| + |R(x,y)|\big]$. By the fact that $\alpha \mapsto \alpha^{p-1}$, $\alpha > 0$, is non-decreasing and by (1.14), we get

$$\begin{aligned}|L(x,y)| &\le cp(s^*)^{p-1}|x-y|^2\\ &\le cp\big(|\nabla g(x)\cdot(y-x)| + |R(x,y)|\big)^{p-1}|x-y|^2\\ &\le cp\big(|\nabla g(x)| + c|x-y|\big)^{p-1}|x-y|^{p+1}.\end{aligned}$$

Here we used the Cauchy-Schwarz inequality. This implies that, for $y \in B_r(x)$ and $r<1$, there exists a constant $C>0$ independent of y, such that $|L(x,y)| \le Cr^{p+1}$. Thus (1.16) follows from

$$\left| \fint_{B_r(x)} \frac{L(x,y)}{r^p}\,dy \right| \le Cr.$$

By (1.16), letting $r \downarrow 0$ in (1.15) we get (1.13). This proves the claim and hence that (1.8) holds for any $g \in C_c^2(\mathbb{R}^d)$. By the previous considerations, this implies that (1.8) holds for any $f \in W^{1,p}(\mathbb{R}^d)$.

In order to complete the proof of the Theorem, it remains to prove that if $f \in L^p(\mathbb{R}^d)$ satisfies (1.7), then $f \in W^{1,p}(\mathbb{R}^d)$. For such an f, by the definition of lim inf, there exists a sequence $(r_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} r_n = 0$, such that

$$\lim_{n\to\infty} A_n[f] = \liminf_{r\downarrow 0} \frac{1}{r^p}\int_{\mathbb{R}^d}\fint_{B_r(x)} |f(x)-f(y)|^p\,dy\,dx < +\infty.$$

Here we let

$$A_n[f] = \frac{1}{r_n^p}\int_{\mathbb{R}^d}\fint_{B_{r_n}(x)} |f(x)-f(y)|^p\,dy\,dx.$$

Then there exists a constant $M>0$, such that,

$$A_n[f] \le M, \text{ for any } n \in \mathbb{N}. \tag{1.17}$$

Let $f_\delta \in C_c^\infty(\mathbb{R}^d)$ be a smooth approximation of f, where $(\gamma_\delta)_{\delta>0}$ is a family of mollifiers. Namely we let

$$f_\delta(x) = f * \gamma_\delta(x) = \int_{\mathbb{R}^d} f(h)\gamma_\delta(x-h)\,dh.$$

We recall that the convolution is associative and that it commutes with the translation operator $\tau_w f(x) = f(x+w)$, $w \in \mathbb{R}^d$, in the sense that $\tau_w(f*g) = \tau_w f * g = f * \tau_w g$. Therefore

$$|f_\delta(x) - \tau_w f_\delta(x)| = |f * \gamma_\delta(x) - (\tau_w f * \gamma_\delta)(x)| = |(f - \tau_w f) * \gamma_\delta(x)|. \tag{1.18}$$

Moreover for any $f \in L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, and $g \in L^1(\mathbb{R}^d)$ it holds that $\|f*g\|_{L^p(\mathbb{R}^d)} \le$

$\|g\|_{L^1(\mathbb{R}^d)}\|f\|_{L^p(\mathbb{R}^d)}$ (see [Bré83, Theorem IV.15]). This and (1.18) imply that

$$\begin{aligned}
A_n[f_\delta] &= \frac{1}{r_n^p}\int_{\mathbb{R}^d} \fint_{B_{r_n}(x)} |f_\delta(x)-f_\delta(y)|^p\,dy\,dx \\
&= \frac{1}{r_n^p}\int_{\mathbb{R}^d} \fint_{B_{r_n}} |f_\delta(x)-\tau_w f_\delta(x)|^p\,dw\,dx \\
&= \frac{1}{r_n^p}\fint_{B_{r_n}} \int_{\mathbb{R}^d} |(f-\tau_w f)*\gamma_\delta(x)|^p\,dx\,dw \\
&= \frac{1}{r_n^p}\fint_{B_{r_n}} \|(f-\tau_w f)*\gamma_\delta\|^p_{L^p(\mathbb{R}^d)}\,dw \\
&\le \frac{1}{r_n^p}\fint_{B_{r_n}} \|\gamma_\delta\|^p_{L^1(\mathbb{R}^d)}\|f-\tau_w f\|^p_{L^p(\mathbb{R}^d)}\,dw \\
&= \frac{1}{r_n^p}\int_{\mathbb{R}^d} \fint_{B_{r_n}} |f(x)-f(x+w)|^p\,dw\,dx \\
&= A_n[f].
\end{aligned} \tag{1.19}$$

Here, we used the Fubini theorem and the fact that $\|\gamma_\delta\|_{L^1(\mathbb{R}^d)} = 1$ for any $\delta > 0$. Therefore, inequality (1.17) is satisfied exchanging f with f_δ, with the same constant M. Since $C_c^\infty(\mathbb{R}^d) \subset W^{1,p}(\mathbb{R}^d)$, by (1.8) and (1.17), we get

$$\lim_{n\to\infty} A_n[f_\delta] = \int_{\mathbb{R}^d} \|\nabla f_\delta\|^p_{*,p}\,dx \le M. \tag{1.20}$$

Finally we claim that (1.20) implies that $\nabla f \in L^p(\mathbb{R}^d;\mathbb{R}^d)$. This, by the definition of Sobolev space, proves that $f \in W^{1,p}(\mathbb{R}^d)$. Since $f \in L^p(\mathbb{R}^d) \subset L^1_{loc}(\mathbb{R}^d)$, we can define ∇f as a distribution, i.e. $\nabla f \in \big(C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)\big)'$, defining, for $\varphi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$,

$$\langle \nabla f, \varphi\rangle = -\langle f, \nabla\varphi\rangle.$$

Moreover $\nabla f_\delta = \nabla(f*\gamma_\delta) = (\nabla f)*\gamma_\delta$, and hence $\nabla f_\delta \to \nabla f$ in the sense of distributions. Namely,

$$\lim_{\delta\downarrow 0}\int_{\mathbb{R}^d} \nabla f_\delta(x)\varphi(x)\,dx = \langle \nabla f, \varphi\rangle \quad \text{for any } \varphi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d). \tag{1.21}$$

On the other hand, by (1.20), using the equivalence of $\|\cdot\|_{*,p}$ and $|\cdot|$ we get that $\{\nabla f_\delta \in L^p(\mathbb{R}^d;\mathbb{R}^d) : \delta > 0\}$ is bounded in $L^p(\mathbb{R}^d;\mathbb{R}^d)$. Hence, a well known result on the weak compactness of L^p spaces, $1 < p < \infty$, (see for example Theorem 1.36 in [AFP00]) states that, up to subsequences, $\nabla f_\delta \rightharpoonup g$ for some $g \in L^p(\mathbb{R}^d;\mathbb{R}^d)$. Equivalently,

$$\lim_{\delta\downarrow 0}\int_{\mathbb{R}^d} \nabla f_\delta(x)\psi(x)\,dx = \int_{\mathbb{R}^d} g(x)\psi(x)\,dx = \langle g,\psi\rangle \quad \text{for any } \psi \in L^{p'}(\mathbb{R}^d;\mathbb{R}^d). \tag{1.22}$$

Here $1/p + 1/p' = 1$. Since $C_c^\infty(\mathbb{R}^d;\mathbb{R}^d) \subset L^{p'}(\mathbb{R}^d;\mathbb{R}^d)$, combining (1.21) and (1.22) we get that $\langle \nabla f, \varphi\rangle = \langle g, \varphi\rangle$ for any $\varphi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$. Hence ∇f can be represented as a function of $L^p(\mathbb{R}^d;\mathbb{R}^d)$. This proves the claim and the Theorem. □

Remark 1.2.5. Using the same arguments as above, we can prove that for $f \in L^p(\mathbb{R}^d)$, $1 < p < \infty$, it holds

$$\lim_{r\downarrow 0} \int_{\mathbb{R}^d} \fint_{B_r(x)} \frac{|f(x)-f(y)|^p}{\|x-y\|^p}\, dy\, dx = \int_{\mathbb{R}^d} \|\nabla f(x)\|_{\dagger,p}^p\, dx. \tag{1.23}$$

Where

$$\|v\|_{\dagger,p} = \left(\fint_{B_1} \left| v \cdot \frac{w}{\|w\|} \right|^p dw \right)^{1/p}.$$

In the case of the Euclidean norm $|\cdot|$, we have $\|\cdot\|_{\dagger,p} = K'_{d,p}|\cdot|$, where, for any unit vector e of $\mathbb{R}^d$,

$$K'_{d,p} = \fint_{|w|=1} |e\cdot w|^p\, dw.$$

We observe that (1.23) is a special case of the results in [BBM01, Bré02]. In these papers the authors consider a sequence $(\rho_n)_{n\in\mathbb{N}}$ of radial mollifiers such that

$$\rho_n(x) = \rho_n(|x|), \quad \rho_n \geq 0, \quad \int_{\mathbb{R}^d} \rho_n(x)\, dx = 1, \tag{1.24}$$

and

$$\lim_{n\to\infty} \int_\delta^\infty \rho_n(r) r^{d-1}\, dr = 0 \quad \text{for any } \delta > 0. \tag{1.25}$$

Then they prove(see [BBM01], Theorem 2) the following theorem:

Theorem 1.2.6. *Let Ω be a smooth bounded domain of $\mathbb{R}^d$ and let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of radial mollifiers satisfying* (1.24) *and* (1.25). *Assume $f \in L^p(\Omega)$, $1 < p < \infty$. Then*

$$\lim_{n\to\infty} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p} \rho_n(y-x)\, dy\, dx = K'_{d,p} \int_\Omega |\nabla f(x)|^p\, dx,$$

with the convention that $\int_\Omega |\nabla f(x)|^p\, dx = \infty$ if $f \notin W^{1,p}(\Omega)$.

Our result (1.23) follows by choosing $\rho_n(x) = |B_{1/n}|^{-1} \chi_{B_{1/n}}(x)$.

1.3 Functions of bounded variation

Let Ω be an open subset of $\mathbb{R}^d$. A measure μ in $\Omega \subset \mathbb{R}^d$ is said to be a *Radon measure* if $\mu(K) < +\infty$ for any $K \subset \Omega$ compact. A vector measure $\mu = (\mu_1, \ldots, \mu_k)$ is said to be a *vector Radon measure* if μ_i is a Radon measure for $i = 1, \ldots, k$.

Definition 1.3.7. Let μ be a vector Radon measure in Ω taking values in $\mathbb{R}^d$. We define the *total variation measure* of μ with respect to the norm $\|\cdot\|$ of a Borel set $A \subset \Omega$ as

$$\|\mu\|(A) := \sup\left\{ \sum_{i\in I} \|\mu(A_i)\| \;:\; \{A_i\}_{i\in I} \text{ is a finite Borel partition of } A \right\}.$$

We denote with $|\mu|$ the total variation measure of μ with respect to the euclidean norm.

Since all norms on $\mathbb{R}^d$ are equivalent, two total variation measures of μ with respect to different norms are always mutually absolutely continuous. It is also clear that $\mu \ll |\mu|$, hence by the differentiation theorem for Radon measures we get the existance for $|\mu|$-a.e. $x \in \Omega$ of the Radon-Nykodim derivatives

$$\frac{d\mu}{d|\mu|}(x) = \lim_{r\downarrow 0} \frac{\mu(B_r(x))}{|\mu|(B_r(x))}, \qquad \frac{d\,\|\mu\|}{d|\mu|}(x) = \lim_{r\downarrow 0} \frac{\|\mu\|(B_r(x))}{|\mu|(B_r(x))}. \tag{1.26}$$

The following Theorem intertwines these two functions.

Theorem 1.3.8. *Let μ a $\mathbb{R}^d$-valued Radon measure on Ω, then*

$$\|\mu\|(B) = \int_B \left\| \frac{d\mu}{d|\mu|} \right\| d|\mu| \quad \textit{for any Borel set } B \subset \Omega.$$

Proof. By the differentiation theorem for Radon measures (see [EG92]) we get that

$$\|\mu\|(B) = \int_B \frac{d\,\|\mu\|}{d|\mu|} d|\mu|.$$

To prove the assertion, by (1.26), it is enough to show that for $|\mu|$-a.e. $x \in \Omega$ it holds

$$\lim_{r\downarrow 0} \frac{\|\mu(B_r(x))\|}{|\mu|(B_r(x))} = \lim_{r\downarrow 0} \frac{\|\mu\|(B_r(x)}{|\mu|(B_r(x))},$$

which is equivalent to

$$\lim_{r\downarrow 0} \frac{\|\mu(B_r(x))\|}{\|\mu\|(B_r(x))} = \left\| \frac{d\mu}{d\|\mu\|}(x) \right\| = 1.$$

By the definition of $\|\mu\|$ it is obvious that $\|\mu(B_r(x))\| \leq \|\mu\|(B_r(x))$. To prove the other inequality we first fix a Borel set $A \subset \Omega$ and a finite Borel partition $\{A_i\}_{i\in I}$ of A and compute

$$\begin{aligned} \sum_{i\in I} \|\mu(A_i)\| &= \sum_{i\in I} \left\| \int_{A_i} \frac{d\mu}{d\|\mu\|}\, d\|\mu\| \right\| \\ &\leq \sum_{i\in I} \int_{A_i} \left\| \frac{d\mu}{d\|\mu\|} \right\| d\|\mu\| \\ &= \int_A \left\| \frac{d\mu}{d\|\mu\|} \right\| d\|\mu\|. \end{aligned}$$

Taking the supremum over all such partitions yields

$$\|\mu\|(A) = \int_A d\|\mu\| \leq \int_A \left\| \frac{d\mu}{d\|\mu\|} \right\| d\|\mu\|.$$

Since the previous inequality holds for all Borel set of Ω, we get that for $\|\mu\|$-a.e. $x \in \Omega$ holds

$$\left\| \frac{d\mu}{d\|\mu\|}(x) \right\| \geq 1,$$

that, since $|\mu| \ll \|\mu\|$, proves the claim. □

We recall that we write $\|\cdot\|_*$ for the 1-mean norm $\|\cdot\|_{*,1}$, defined in (1.2).

Definition 1.3.9. Let $f \in L^1(\Omega)$, we say that f is a *function of bounded variation in* Ω if the distributional derivative ∇f of f is representable by a finite $\mathbb{R}^d$-valued Radon measure in Ω, i.e. if

$$\int_\Omega f \cdot \nabla\varphi \, dx = -\int_\Omega \varphi \, d(\nabla f), \qquad \forall \varphi \in C_c^\infty(\Omega; \mathbb{R}^d).$$

In such case we will call $\|\nabla f\|_*$ the total variation measure of the function f with respect to $\|\cdot\|_*$.

It is possible to show that the operator $f \mapsto \|\nabla f\|_*$ is lower semicontinuous, in the sense that if $(f_n)_{n\in\mathbb{N}}$ is a sequence in $BV(\Omega)$ such that $f_n \to f$ in $L^1_{\text{loc}}(\Omega)$, then

$$\|\nabla f\|_*(\Omega) \le \liminf_{n\to\infty} \|\nabla f_n\|_*(\Omega). \tag{1.27}$$

In [AB94] it is shown that $\|\nabla f\|_*(A)$ for $A \subset \Omega$ open can also be characterized in the following variational way:

$$\|\nabla f\|_*(A) := \sup\left\{ \int_A f \ \operatorname{div}\varphi \, dx \,:\, \varphi \in C_c^1(A; \mathbb{R}^n), \|\varphi\|_*^\circ \le 1 \right\}, \tag{1.28}$$

where $\|\cdot\|_*^\circ$ denotes the polar norm of $\|\cdot\|_*$, defined as

$$\|v\|_*^\circ = \sup_{\|w\|_* \le 1} |v \cdot w|. \tag{1.29}$$

The vector space of functions of bounded variation in Ω will be denoted as $BV(\Omega)$. Endowed with the norm

$$\|f\|_{BV(\Omega)} = \|f\|_{L^1(\Omega)} + \|\nabla f\|_*(\Omega),$$

it is a Banach space. However the topology induced by the norm is too strong for our purposes (in fact it can be shown that $C^1(\Omega)$ is not dense in $BV(\Omega)$, see [AFP00]). We will use the following convergence result, an adaption of Theorem 2 Section 5.2.2 in [EG92].

Theorem 1.3.10 (Local approximation by smooth functions)**.** *Let $\|\cdot\|_*$ be a norm on $\mathbb{R}^d$, Ω an open subset of $\mathbb{R}^d$ and $f \in BV(\Omega)$. Then there exist functions $(f_n)_{n\in\mathbb{N}} \subset BV(\Omega) \cap C^\infty(\Omega)$, such that*

1. *$f_n \to f$ in $L^1(\Omega)$ and*
2. *$\|\nabla f_n\|_*(A) \to \|\nabla f\|_*(A)$ as $n \to \infty$ for any $A \subset \Omega$ open.*

Proof. Fix $\varepsilon > 0$. Given $k \in \mathbb{N}$, define the open sets

$$\Omega_k = \left\{ x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \frac{1}{m+k} \right\} \cap B_{k+m}.$$

Here m is a fixed integer such that

$$\|\nabla f\|_*(\Omega \setminus \Omega_1) < \varepsilon. \tag{1.30}$$

Set $\Omega_0 = \varnothing$ and define, for any $k \in \mathbb{N}$,

$$V_k = \Omega_{k+1} \setminus \overline{\Omega}_{k-1}.$$

Let then $(\zeta_k)_{k\in\mathbb{N}}$ be a partition of the unity subordinated to the open cover $\{V_k\}_{k\in\mathbb{N}}$ of Ω, i.e. a sequence of smooth functions such that $\zeta_k \in C_c^\infty(V_k)$, $0 \le \zeta_k \le 1$ for any $k \in \mathbb{N}$ and

$$\sum_{k=1}^{\infty} \zeta_k \equiv 1 \quad \text{on } \Omega,$$

where only a finite number of terms of the sum is non-zero at any given point. Let $(\eta_\varepsilon)_{\varepsilon>0}$ be the family of radial mollifiers defined as $\eta_\varepsilon(x) = \varepsilon^{-n}\eta(x/\varepsilon)$, with η a standard convolution kernel (see for example [EG92, p. 122]). Then, for each $k \in \mathbb{N}$ select ε_k so small that

$$\operatorname{supp}(\eta_{\varepsilon_k} * (f\zeta_k)) \subset V_k, \tag{1.31}$$

$$\int_\Omega \left|\eta_{\varepsilon_k} * (f\zeta_k)(x) - f\zeta_k(x)\right| dx < \frac{\varepsilon}{2^k}, \tag{1.32}$$

$$\int_\Omega \left|\eta_{\varepsilon_k} * (fD\zeta_k)(x) - fD\zeta_k(x)\right| dx < \frac{\varepsilon}{2^k}. \tag{1.33}$$

Define

$$f_\varepsilon = \sum_{k=1}^{\infty} \eta_{\varepsilon_k} * (f\zeta_k). \tag{1.34}$$

The function f_ε is of class C^∞ on Ω. In fact for any $x \in \Omega$ there exists a neighborhood in which only finitely many terms of the sum in (1.34) are non-zero.

We claim that $f_\varepsilon \to f$ in $L^1(\Omega)$. In fact by (1.32) and the fact that

$$f = \sum_{k=1}^{\infty} f\zeta_k,$$

we get

$$\|f_\varepsilon - f\|_{L^1(\Omega)} \le \sum_{k=1}^{\infty} \int_\Omega \left|\eta_{\varepsilon_k} * (f\zeta_k)(x) - f\zeta_k(x)\right| dx < \varepsilon.$$

Now we claim that $\|\nabla f_\varepsilon\|_*(A) \to \|\nabla f\|_*(A)$, as $\varepsilon \downarrow 0$. We will use the variational characterization (1.28). Fix $\varphi \in C_c^1(A;\mathbb{R}^d)$, $\|\varphi\|_*^\circ \le 1$. For any $g : \Omega \to \mathbb{R}$ such that $\operatorname{supp} g \subset A$ and for any $j = 1, \dots, d$ we get

$$\begin{aligned}
\int_A (\eta_{\varepsilon_k} * g) \frac{\partial \varphi}{\partial x_j}\, dx &= \int_A \int_A g(h)\, \eta_{\varepsilon_k}(x+h) \frac{\partial \varphi}{\partial x_j}(x)\, dh\, dx \\
&= \int_A g(h) \left(\eta_{\varepsilon_k} * \frac{\partial \varphi}{\partial x_j} \right)(h)\, dh \\
&= \int_A g(x) \frac{\partial}{\partial x_j} (\eta_{\varepsilon_k} * \varphi)(x)\, dx.
\end{aligned}$$

Hence we can compute

$$\begin{aligned}
\int_A f_\varepsilon \operatorname{div} \varphi\, dx &= \sum_{k=1}^\infty \int_A \eta_{\varepsilon_k} * (f\zeta_k) \operatorname{div} \varphi\, dx \\
&= \sum_{k=1}^\infty \int_A f\zeta_k \operatorname{div}(\eta_{\varepsilon_k} * \varphi)\, dx \\
&= \sum_{k=1}^\infty \left[\int_A f \operatorname{div}\big(\zeta_k(\eta_{\varepsilon_k} * \varphi)\big)\, dx - \int_A f\, D\zeta_k \cdot (\eta_{\varepsilon_k} * \varphi)\, dx \right] \\
&= \sum_{k=1}^\infty \int_A f \operatorname{div}\big(\zeta_k(\eta_{\varepsilon_k} * \varphi)\big)\, dx - \sum_{k=1}^\infty \int_A \varphi \cdot \big[\eta_{\varepsilon_k} * (fD\zeta_k) - fD\zeta_k\big]\, dx \\
&= \mathcal{I}_1^\varepsilon + \mathcal{I}_2^\varepsilon,
\end{aligned}$$

where $\mathcal{I}_1^\varepsilon, \mathcal{I}_1^\varepsilon$ are defined in the last equality. Here we used the fact that $\sum_{k=1}^\infty D\zeta_k = 0$ in A.

If $w \in \mathbb{R}^d$ is such that $\|w\|_* \le 1$ we get

$$|w \cdot (\eta_{\varepsilon_k} * \varphi)| \le \int_A \eta_{\varepsilon_k}(x+h) \big| w \cdot \varphi(h) \big|\, dx \le \int_A \eta_{\varepsilon_k}(x+h)\, dh = 1. \tag{1.35}$$

Here we used the definition of polar norm (1.29), and the fact that $\|\varphi\|_*^\circ \le 1$. Inequality (1.35) proves that $\|\zeta_k(\eta_{\varepsilon_k} * \varphi)\|_*^\circ \le 1$. Since each point of A belongs to at most three of the sets $(V_k)_{k\in\mathbb{N}}$ and using (1.28), we estimate

$$\begin{aligned}
|\mathcal{I}_1^\varepsilon| &= \left| \int_A f \operatorname{div}\big(\zeta_1(\eta_{\varepsilon_1} * \varphi)\big)\, dx + \sum_{k=2}^\infty \int_A f \operatorname{div}\big(\zeta_k(\eta_{\varepsilon_k} * \varphi)\big)\, dx \right| \\
&\le \|\nabla f\|_*(A) + \sum_{k=2}^\infty \|\nabla f\|(V_k \cap A) \\
&\le \|\nabla f\|_*(A) + 3\|\nabla f\|_*\big((\Omega \setminus \Omega_1) \cap A\big) \\
&\le \|\nabla f\|_*(A) + 3\varepsilon.
\end{aligned}$$

On the other hand by (1.33) we get that $|\mathcal{I}_2^\varepsilon| \le \varepsilon$. Therefore we have proven that

$$\int_A f_\varepsilon \operatorname{div}\varphi\, dx \le \|\nabla f\|_*(A) + 4\varepsilon,$$

and so, by (1.28), that

$$\|\nabla f_\varepsilon\|_*(A) \le \|\nabla f\|_*(A) + 4\varepsilon. \tag{1.36}$$

The claim, and hence the Theorem, follows by (1.36) and the lower semicontinuity of the total variation as in (1.27). □

Proposition 1.3.11. *If $f \in W^{1,1}(\Omega)$ then, for any measurable set A,*

$$\|\nabla f\|_*(A) = \int_A \|\nabla f(x)\|_*\, dx. \tag{1.37}$$

Proof. Integrating by parts in (1.28) we get

$$\|\nabla f\|_*(A) = \sup\left\{\int_A \nabla f \cdot \varphi\, dx \,:\, \varphi \in C_c^1(A;\mathbb{R}^n), \|\varphi\|_*^\circ \le 1\right\}. \tag{1.38}$$

Since $\sup_{\|\varphi\|_*^\circ \le 1} |\nabla f \cdot \varphi| = \|\nabla f\|_*^{\circ\circ} = \|\nabla f\|_*$, from (1.38) it follows that

$$\|\nabla f\|_*(A) \le \int_A \|\nabla f(x)\|_*\, dx. \tag{1.39}$$

Let $f \in C_c^\infty(A)$ and choose

$$\varphi(x) := \begin{cases} \|\nabla f(x)\|_* \frac{\nabla f(x)}{|\nabla f(x)|^2} & \text{if } \nabla f(x) \ne 0 \\ 0 & \text{otherwise} \end{cases}.$$

If c is a constant such that $\|v\|_* \le c|v|$ for any $v \in \mathbb{R}^d$, we get

$$\int_\Omega |\varphi(x)|\, dx = \int_{\Omega\cap \operatorname{supp} f} |\varphi(x)|\, dx = \int_{\Omega\cap \operatorname{supp} f} \frac{\|\nabla f(x)\|_*}{|\nabla f(x)|}\, dx \le c|\operatorname{supp} f| < \infty.$$

This implies that $\varphi \in L^1(\Omega)$. Moreover it holds also that $\|\varphi\|_*^\circ \le 1$. In fact if $\nabla f(x) \ne 0$ we get

$$\|\varphi(x)\|_*^\circ = \sup_{\|v\|_* \le 1} \left| v \cdot \|\nabla f(x)\|_* \frac{\nabla f(x)}{|\nabla f(x)|^2}\right| = 1.$$

Hence, calling $(\varphi_n) \subset C_c^1(A)$ a sequence such that $\varphi_n \to \varphi$ in $L^1(A)$, $\|\varphi_n\|_*^\circ \le 1$ and $|\varphi_n(x)| \le |\varphi(x)|$, it holds

$$\|\nabla f\|_*(A) \ge \lim_{n\to\infty} \int_A \nabla f \cdot \varphi_n\, dx = \int_A \nabla f \cdot \varphi\, dx = \int_A \|\nabla f(x)\|_*\, dx,$$

which together with (1.39) proves (1.37).

If $f \in W^{1,1}(A)$ there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(A)$ such that $f_n \to f$ in $W^{1,1}(A)$. Since (1.37) holds for each f_n and

$$\int_A \|\nabla f_n(x)\|_* \, dx \to \int_A \|\nabla f(x)\|_* \, dx,$$

in order to prove that (1.37) holds for f, it is enough to prove that $\|\nabla f_n\|_* \rightharpoonup \|\nabla f\|_*$ in the sense of Radon measures. This claim follows from the obvious fact that

$$\int_A f_n \operatorname{div} \varphi \, dx \to \int_A f \operatorname{div} \varphi \, dx.$$

To conclude the proof we need only to observe that if $f \in W^{1,1}(\Omega)$ then $f \in W^{1,1}(A)$. □

We recall that if $E \subset \mathbb{R}^d$ is a Borel set, we say that E is a *set of finite perimeter in* Ω if $\chi_E \in BV(\Omega)$. Here χ_E denotes the characteristic function of the set E. In such case we define the perimeter of E in Ω with respect to $\|\cdot\|_*$ as $P(E;\Omega) = \|D\chi_E\|_*(\Omega)$.

1.3.1 Total variation of functions in $BV(\mathbb{R}^d)$

The following Theorem is the core of Theorems 1.3.14 and 1.3.15.

Theorem 1.3.12. *Let $\Omega \subset \mathbb{R}^d$ be an open set, $f \in BV(\Omega)$ and let μ_r be defined as*

$$\mu_r(A) = \frac{1}{r} \int_A \left(\fint_{B_r(x)\cap\Omega} |f(x) - f(y)| \, dy \right) dx,$$

for any $A \subset \mathbb{R}^d$ Borel. Then $\mu_r \rightharpoonup \|\nabla f\|_*$ as $r \downarrow 0$ weakly in the sense of Radon measures in Ω.

The first step of the proof of Theorem 1.3.12 is the following

Lemma 1.3.13. *Under the assumptions of Theorem 1.3.12, let E be a Borel subset of Ω and $E_r = E + B_r(x)$. If $E_r \subset \Omega$ then*

$$\mu_r(E) \le \|\nabla f\|_*(E_r). \tag{1.40}$$

Proof. We start by proving the inequality for $f \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$.

Since, in this case, for $y \in E$ and $x \in B_r(y)$ it holds that

$$f(x) - f(y) = \int_0^1 \nabla f(tx + (1-t)y) \cdot (x - y) \, dt,$$

we can apply Fubini theorem and get

$$\begin{aligned}
\mu_r(E) &= \int_E d\mu_r \\
&\leq \frac{1}{r}\int_E \fint_{B_r(y)} \int_0^1 |\nabla f(tx+(1-t)y)\cdot(x-y)|\, dt\, dx\, dy \\
&= \frac{1}{r}\fint_{B_r}\int_0^1\int_E |\nabla f(y+tw)\cdot w|\, dy\, dt\, dw \\
&\leq \frac{1}{r}\fint_{\|w\|<r}\int_{E_r} |\nabla f(z)\cdot w|\, dz\, dw \\
&= \int_{E_r}\fint_{\|w\|<1} |\nabla f(z)\cdot w|\, dw\, dz \\
&= \int_{E_r} \|\nabla f(z)\|_*\, dz = \|\nabla f\|_*(E_r),
\end{aligned}$$

where in the last equality we used Proposition 1.3.11.

If $f \in BV(\Omega)$ by Theorem 1.3.10 there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset BV(\Omega)\cap C^\infty(\Omega)$ such that $f_n \to f$ in $L^1(\Omega)$ and for all $A \subset \Omega$ open it holds

$$\|\nabla f_n\|_*(A) \to \|\nabla f\|_*(A).$$

Therefore, to complete the proof it is enough to apply the results obtained above to such a sequence, observe that E_r is open and let $n \to \infty$. □

Proof of Theorem 1.3.12. We claim that for any sequence $(r_n)_{n\in\mathbb{N}}$ such that $r_n \searrow 0$ when $n \to \infty$ we have $\mu_{r_n} \rightharpoonup \|\nabla f\|_*$. Let us choose any such $(r_n)_{n\in\mathbb{N}}$.

A well known compactness result for Radon measures (see for example [AFP00, Theorem 1.59]) states that if given a sequence $(r_n)_n$ is such that for each compact set $K \subset \Omega$ it holds that

$$\sup_n \mu_{r_n}(K) < \infty,$$

then there exists a subsequence $(r_{n_i})_i$ and a Radon measure μ such that (defining $\mu_i := \mu_{r_{n_i}}$)

$$\mu_i \rightharpoonup \mu.$$

This follows from (1.40), in fact the compactness of K ensures the existance of n_0 such that $K_{r_{n_0}} \subset \Omega$ and this, together with the facts that for all n we have $K_{r_{n+1}} \subset K_{r_n}$ and that $\|\nabla f\|_*$ is a Radon measure over Ω , leads to

$$\sup_{n\geq n_0} \mu_{r_n}(K) \leq \sup_{n\geq n_0} \|\nabla f\|_*(K_{r_n}) = \|\nabla f\|_*(K_{r_{n_0}}) < \infty,$$

from which follows $\sup_{n\in\mathbb{N}} \mu_{r_n}(K) < \infty$.

Now we show that, for all Borel subsets B of Ω, this holds:

$$\mu(B) \leq \|\nabla f\|_*(B). \tag{1.41}$$

In fact, if K is a compact subset of Ω and $R > 0$ is small enough, there exists n_0 such that $\forall n \geq n_0$ holds $K_{R+r_n} \subset \Omega$. Then, by (1.40),

$$\begin{aligned}\mu(K) &\leq \mu(K_R) \leq \liminf_{n\to\infty} \mu_{r_n}(K_R) \\ &\leq \lim_{n\to\infty} \|\nabla f\|_* (K_{R+r_n}) = \|\nabla f\|_* \left(\bigcap_{n=1}^{\infty} K_{R+r_n}\right) \\ &= \|\nabla f\|_* \left(\overline{K_R}\right).\end{aligned}$$

Here we used the continuity from above of measures and the lower semicontinuity on open subsets of the weak convergence of Radon measures (see, for example, [EG92]). Letting $R \downarrow 0$, since $K = \bigcap_{R>0} \overline{K_R}$, we obtain

$$\|\nabla f\|_*(\overline{K_R}) \searrow \|\nabla f\|_*(K).$$

Thus our claim (1.41) holds for compact subsets of Ω and therefore for all Borel sets.

Finally, to complete the proof, it suffices to show that, for all Borel subsets B of Ω, there holds

$$\mu(B) \geq \|\nabla f\|_*(B). \tag{1.42}$$

This, together with (1.41), proves that $\mu = \|\nabla f\|_*$. Because of the uniqueness of the limit, the whole family $(\mu_r)_{r>0}$ converges to μ as $r \downarrow 0$.

To prove (1.42) let us fix $v \in \mathbb{R}^d$ and $\varphi \in C_c^\infty(\Omega)$ such that $\varphi \geq 0$. We extend f with 0 outside of Ω and define

$$\mathcal{I}_i(\varphi; v) = \frac{1}{|B_{r_{n_i}}|} \left| \frac{1}{r_{n_i}} \int_{\mathbb{R}^d} \int_{B^+_{r_{n_i}}(x;v)} f(x)(\varphi(y) - \varphi(x))\, dy\, dx \right|.$$

It is clear that $\varrho(x, y) = \chi_{B_{r_{n_i}}}(y - x)$ satisfies the hypotheses of Lemma 1.3.18, proven at the end of the chapter. By such Lemma and the Minkowsky inequality, we get

$$\mathcal{I}_i(\varphi; v) + \mathcal{I}_i(\varphi; -v) \leq \frac{1}{|B_{r_{n_i}}|} \frac{1}{r_{n_i}} \int_{\mathbb{R}^d} \int_{B_{r_{n_i}}(y)} |f(x) - f(y)|\, \varphi(y)\, dx\, dy. \tag{1.43}$$

Choosing $i \geq i_0$, where i_0 is such that $r_{n_{i_0}} < \mathrm{dist}(\operatorname{supp}\varphi, \partial\Omega)$, we can rewrite the right hand side of (1.43) as

$$\begin{aligned}\frac{1}{r_{n_i}} \int_{\mathbb{R}^d} \fint_{B_{r_{n_i}}(y)} |f(x) - f(y)|\, \varphi(y)\, dx\, dy &= \frac{1}{r_{n_i}} \int_{\operatorname{supp}\varphi} \varphi(y) \fint_{B_{r_{n_i}}(y)} |f(x) - f(y)|\, dx\, dy \\ &= \int_\Omega \varphi\, d\mu_{n_i}.\end{aligned}$$

By the weak convergence in the sense of Radon measures of $(\mu_{r_{n_i}})_i$, this implies

$$\lim_{i\to\infty} \frac{1}{r_{n_i}} \int_{\mathbb{R}^d} \fint_{B_{r_{n_i}}(y)} |f(x) - f(y)|\, \varphi(y)\, dx\, dy = \int_\Omega \varphi\, d\mu. \tag{1.44}$$

As for the left hand side of (1.43), passing to the limit for $i \to \infty$ and observing that $|B_r| = 2\,|B_r \cap \{w : \ w \cdot v \geq 0\}|$ leads to

$$\lim_{i\to\infty} \mathcal{I}_i(\varphi; v) = \lim_{i\to\infty} \frac{1}{2} \left| \frac{1}{r_{n_i}} \int_{\mathbb{R}^d} ⨍_{B_{r_{n_i}}(x;v)} f(x)(\varphi(y) - \varphi(x))\, dy\, dx \right|. \tag{1.45}$$

Observe now that by the Dominated Convergence Theorem we can take the limit into the integral in (1.45). In fact, since there exists $c > 0$ such that for any $v \in \mathbb{R}^d$ it holds that $\|v\|_* \leq c|v|$, for i large we have

$$\begin{aligned} \left| ⨍_{B_{r_{n_i}}(x;v)} f(x) \frac{\varphi(y) - \varphi(x)}{r_{n_i}}\, dy \right| &\leq \frac{2|f(x)|}{|B_{r_{n_i}}|} \frac{1}{r_{n_i}} \int_{B_{r_{n_i}}(x)} |\varphi(y) - \varphi(x)|\, dy \\ &= 2|f(x)| \frac{1}{r_{n_i}} ⨍_{B_{r_{n_i}}(x)} (|\nabla\varphi(x) \cdot (y - x)| + |R(x, y)|)\, dy \\ &= 2|f(x)| \left(⨍_{B_1} |\nabla\varphi(x) \cdot w|\, dw + ⨍_{B_{r_{n_i}}(x)} \frac{|R(x, y)|}{r_{n_i}}\, dy \right) \\ &= 2|f(x)|\, (\|\nabla\varphi(x)\|_* + 1) \leq 2|f(x)|(c|\nabla\varphi(x)| + 1), \end{aligned}$$

which is in $L^1(\mathbb{R}^d)$. By Lemma 1.1.2 we obtain

$$\begin{aligned} \lim_{i\to\infty} \mathcal{I}_i(\varphi; v) &= \frac{1}{2} \left| \int_{\mathbb{R}^d} f(x) \lim_{i\to\infty} \left(\frac{1}{r_{n_i}} ⨍_{B_{r_{n_i}}(x;v)} (\varphi(y) - \varphi(x))\, dy \right) dx \right| \\ &= \frac{1}{2} \left| \int_{\mathbb{R}^d} f(x)\, \nabla\varphi(x) \cdot E(v)\, dx \right| \\ &= \frac{1}{2} \left| \int_\Omega \varphi\, d\,(\nabla f \cdot E(v)) \right|, \end{aligned} \tag{1.46}$$

where we applied the integration by parts formula for BV functions. By the fact that $E(-v) = -E(v)$ it follows that

$$\lim_{i\to\infty} \mathcal{I}_i(\varphi; -v) = \frac{1}{2} \left| \int_\Omega \varphi\, d\,(\nabla f \cdot E(v)) \right|. \tag{1.47}$$

Passing to the limit as $i \to \infty$ in (1.43) yields, by (1.44), (1.46) and (1.47),

$$\left| \int_\Omega \varphi\, d\,(\nabla f \cdot E(v)) \right| \leq \int_\Omega \varphi\, d\mu$$

for all vectors $v \in \mathbb{R}^d$ and for all $\varphi \in C_c^\infty(\mathbb{R}^d)$, or equivalently

$$|\nabla f \cdot E(v)(A)| \leq \mu(A) \tag{1.48}$$

for all $v \in \mathbb{R}^d$ and A open.

As observed before, the total variation measures with respect to different norms are always mutually absolutely continuous. Then, by the differentiation theorem for Radon measures we have

$$g(x) = \lim_{R\to 0} \frac{\mu(B_R(x))}{|\nabla f|(B_R(x))} \quad \text{and} \quad \sigma(x) = \lim_{R\to 0} \frac{\nabla f(B_R(x))}{|\nabla f|(B_R(x))},$$

for $|\nabla f|$-a.e. $x \in \Omega$ and $|\sigma| = 1$ a.e. .

Taking such an x, for any $R > 0$ small and vector v we have, by (1.48),

$$\frac{\nabla f(B_R(x)) \cdot E(v)}{|\nabla f|(B_R(x))} \leq \frac{\mu(B_R(x))}{|\nabla f|(B_R(x))}.$$

Taking the supremum for $v \in \mathbb{R}^d$ and applying $\|v\|_* = v \cdot E(v)$, this leads to

$$\left\| \frac{\nabla f(B_R(x))}{|\nabla f|(B_R(x))} \right\|_* \leq \sup_{v\in\mathbb{R}^d} \frac{\nabla f(B_R(x)) \cdot E(v)}{|\nabla f|(B_R(x))} \leq \frac{\mu(B_R(x))}{|\nabla f|(B_R(x))}.$$

By the continuity of the norm, letting $R \downarrow 0$ we obtain $g(x) \geq \|\sigma(x)\|_*$. Then, by Theorem 1.3.8, we get

$$\mu(B) = \int_B g\, d|\nabla f| \geq \int_B \|\sigma\|_*\, d|\nabla f| = \|\nabla f\|_*(B).$$

This proves the inequality $\mu \geq \|\nabla f\|_*$ and so Theorem 1.3.12. □

We state now the two consequences of Theorem 1.3.12 giving the characterization we were looking for the total variation of functions in $BV(\mathbb{R}^d)$ and for sets of finite perimeter in $\mathbb{R}^d$.

Theorem 1.3.14. *Let $\|\cdot\|$ be a norm in $\mathbb{R}^d$ with balls $B_r(x)$ and let $\|\cdot\|_*$ be the associated 1-mean norm defined in (1.2). Let $f \in L^1(\mathbb{R}^d)$. Then $f \in BV(\mathbb{R}^d)$ if and only if*

$$\liminf_{r\downarrow 0} \frac{1}{r} \int_{\mathbb{R}^d} \fint_{B_r(x)} |f(x) - f(y)|\, dy\, dx < +\infty. \tag{1.49}$$

Moreover in such a case the limit inferior is in fact a limit and results

$$\lim_{r\downarrow 0} \frac{1}{r} \int_{\mathbb{R}^d} \fint_{B_r(x)} |f(x) - f(y)|\, dy\, dx = \|\nabla f\|_*(\mathbb{R}^d). \tag{1.50}$$

Proof. Thanks to Theorem 1.3.12 it suffices to prove that if $f \in L^1(\mathbb{R}^d)$ satisfies (1.49), then $f \in BV(\mathbb{R}^d)$. We will proceed as in the proof of Theorem 1.2.4. In fact for such an f, by the definition of lim inf, there exists a sequence $(r_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} r_n = 0$, such that

$$\lim_{n\to\infty} A_n[f] = \liminf_{r\downarrow 0} \frac{1}{r} \int_{\mathbb{R}^d} \fint_{B_r(x)} |f(x) - f(y)|\, dy\, dx < +\infty.$$

Here we let

$$A_n[f] = \frac{1}{r_n}\int_{\mathbb{R}^d} \fint_{B_{r_n}(x)} |f(x) - f(y)|\, dy\, dx.$$

Then there exists a constant $M > 0$, such that

$$A_n[f] \le M, \text{ for any } n \in \mathbb{N}. \tag{1.51}$$

Let $f_\delta \in C_c^\infty(\mathbb{R}^d)$ be a smooth approximation of f, where $(\gamma_\delta)_{\delta>0}$ is a family of mollifiers such that $0 \le \gamma_\delta \le 1$. With the same computations as in (1.19) we can prove that $A_n[f_\delta] \le A_n[f]$. Therefore, inequality (1.51) yields $A_n[f_\delta] \le M$. Since $C_c^\infty(\mathbb{R}^d) \subset BV(\mathbb{R}^d)$, by Theorem 1.3.12 and (1.51), we get

$$\lim_{n\to\infty} A_n[f_\delta] = \int_{\mathbb{R}^d} \|\nabla f_\delta\|_*\, dx \le M. \tag{1.52}$$

Finally we claim that (1.52) implies that ∇f is a finite vector Radon measure on $\mathbb{R}^d$. This will prove that $f \in BV(\mathbb{R}^d)$. Since $f \in L^1(\mathbb{R}^d) \subset L^1_{loc}(\mathbb{R}^d)$, we can define ∇f as a distribution, i.e. $\nabla f \in \big(C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)\big)'$, defining, for $\varphi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$,

$$\langle \nabla f, \varphi\rangle = -\langle f, \nabla\varphi\rangle.$$

Moreover $\nabla f_\delta = \nabla(f * \gamma_\delta) = (\nabla f) * \gamma_\delta$, and hence $\nabla f_\delta \to \nabla f$ in the sense of distributions. Namely

$$\lim_{\delta\downarrow 0} \int_{\mathbb{R}^d} \nabla f_\delta(x)\varphi(x)\, dx = \langle \nabla f, \varphi\rangle \quad \text{for any } \varphi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d). \tag{1.53}$$

On the other hand we can associate to every ∇f_δ a finite vector Radon measure by

$$\nabla f_\delta(A) = \int_A \nabla f_\delta(x)\, dx.$$

Using the equivalence of $\|\cdot\|_{*,p}$ and $|\cdot|$ and (1.52), we get that $\sup_{\delta>0} |\nabla f_\delta|(\mathbb{R}^d) < +\infty$. Hence, a well known results on the weak compactness of Radon measures (see for example [AFP00, Theorem 1.59]) states that, up to subsequences, $\nabla f_\delta \rightharpoonup \mu$ weakly in the sense of Radon measures, for some finite vector Radon measure μ on $\mathbb{R}^d$. Equivalently,

$$\lim_{\delta\downarrow 0} \int_{\mathbb{R}^d} \psi\, d\nabla f_\delta = \int_{\mathbb{R}^d} \psi\, d\mu = \langle \mu, \psi\rangle \quad \text{for any } \psi \in C_c(\mathbb{R}^d;\mathbb{R}^d). \tag{1.54}$$

Since $C_c^\infty(\mathbb{R}^d;\mathbb{R}^d) \subset C_c(\mathbb{R}^d;\mathbb{R}^d)$, combining (1.53) and (1.54) we get that $\langle \nabla f, \varphi\rangle = \langle \mu, \varphi\rangle$ for any $\varphi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$. Hence ∇f can be represented as a finite vector Radon measure on $\mathbb{R}^d$. This proves the claim and the Theorem. □

Theorem 1.3.15. *Let E be a Borel subset of $\mathbb{R}^d$. Then E is a set of finite perimeter if and only if*

$$\liminf_{r\downarrow 0} \frac{1}{r} \int_{\mathbb{R}^d} \fint_{B_r(x)} |\chi_E(x) - \chi_E(y)|\, dy\, dx < +\infty.$$

Proof. It follows directly by the definition of set of finite perimeter and Theorem 1.3.14. □

Remark 1.3.16. Let ν_r be defined as

$$\nu_r(A) = \int_A \left(\fint_{B_r(x)\cap\Omega} \frac{|f(x) - f(y)|}{\|x - y\|} \, dx \right) dy.$$

Using the same arguments used in the proof of Theorem 1.3.12 it is possible to show that $\nu_r \rightharpoonup \|\nabla f\|_\dagger$ as $r \downarrow 0$ weakly in the sense of Radon measures in Ω. Here we defined

$$\|v\|_\dagger := \fint_{\|w\|\leq 1} \left| v \cdot \frac{w}{\|w\|} \right| dw.$$

In the case of the Euclidean norm $|\cdot|$, we have $\|\cdot\|_\dagger = K'_{d,1}|\cdot|$, where, for e any unit vector of $\mathbb{R}^d$,

$$K'_{d,1} = \fint_{|w|=1} |e \cdot w| \, dw.$$

We observe that (1.50) is a special case of a result in [Dáv02]. In his work, Dàvila, following [BBM01], considers a sequence $(\rho_n)_{n\in\mathbb{N}}$ of radial mollifiers satisfying the following assumptions:

$$\rho_n(x) = \rho_n(|x|), \quad \rho_n \geq 0, \quad \int_{\mathbb{R}^d} \rho_n(x) \, dx = 1, \tag{1.55}$$

and

$$\lim_{n\to\infty} \int_\delta^\infty \rho_n(r) r^{d-1} \, dr = 0 \quad \text{for any } \delta > 0. \tag{1.56}$$

Then it is proved (see [Dáv02, Theorem 1]) the following

Theorem 1.3.17. *Let $\Omega \subset \mathbb{R}^d$ be open, bounded with a Lipschitz boundary, and let $f \in BV(\Omega)$. Consider a sequence $(\rho_n)_{n\in\mathbb{N}}$ satisfying* (1.24) *and* (1.25)*. Then*

$$\lim_{n\to\infty} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{|x - y|} \rho_n(y - x) \, dy \, dx = K'_{d,1} |\nabla f|(\Omega).$$

Our result (1.50) follows by choosing $\rho_n(x) = |B_{1/n}|^{-1} \chi_{B_{1/n}}(x)$ and $\Omega = \mathbb{R}^d$.

Lemma 1.3.18. *Let $v \in \mathbb{R}^d$ with $v \neq 0$, $\varphi \in C_c(\mathbb{R}^d)$, $f \in L^1(\mathbb{R}^d)$, and let $\varrho : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a function such that:*

i) $\varrho(x, y) = \varrho(y, x)$ for all $x, y \in \mathbb{R}^d$ the following integral exist and

$$\int_{(x-y)\cdot w \geq 0} \varrho(x, y) \, dx = \int_{(x-y)\cdot w \leq 0} \varrho(x, y) \, dx \tag{1.57}$$

for all $y \in \mathbb{R}^d$;

ii) $\varrho \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ *and* $x \mapsto \int_{\mathbb{R}^d} \varrho(x,y)\,dy$ *is locally bounded.*

Then we have

$$\int_{\mathbb{R}^d} \int_{(y-x)\cdot v \geq 0} f(x)(\varphi(y) - \varphi(x))\varrho(x,y)\,dy\,dx = \int_{\mathbb{R}^d} \int_{(y-x)\cdot v \geq 0} (f(x) - f(y))\varphi(y)\varrho(x,y)\,dx\,dy. \quad (1.58)$$

Proof. Let $\mathcal{I}$ denote the left hand side of (1.58) and define $\mathcal{I}_1$ and $\mathcal{I}_2$ such that

$$\mathcal{I} = \int_{\mathbb{R}^d} \int_{(y-x)\cdot v \geq 0} f(x)\varphi(y)\varrho(x,y)\,dy\,dx - \int_{\mathbb{R}^d} \int_{(y-x)\cdot v \geq 0} f(x)\varphi(x)\varrho(x,y)\,dy\,dx = \mathcal{I}_1 - \mathcal{I}_2.$$

By the change of variable $(x,y) \mapsto (y,x)$ in $\mathcal{I}_2$, $\varrho(x,y) = \varrho(y,x)$ and (1.57), we obtain

$$\begin{aligned} \mathcal{I}_2 &= \int_{\mathbb{R}^d} \int_{(x-y)\cdot v \geq 0} f(y)\varphi(y)\varrho(y,x)\,dy\,dx \\ &= \int_{\mathbb{R}^d} f(y)\varphi(y) \int_{(x-y)\cdot v \leq 0} \varrho(x,y)\,dy\,dx. \end{aligned}$$

Summing up we get identity (1.58). □

Chapter 2

The case of the Heisenberg Group

Let $\mathbf{H}^d = \mathbb{C}^d \times \mathbb{R}$ be endowed with the non-commutative group law

$$(z,t) * (w,s) = (z + w, t + s + 2\mathrm{Im}(z \cdot \bar{w})).$$

$\mathbf{H}^d$ with this group law is a Lie group known as the Heisenberg group. The identity of the group is the origin 0, while $(z,t)^{-1} = (-z,-t)$. The Heisenberg Lie algebra can be realized as a $(2d+1)$-dimensional algebra of left invariant differential operator, namely, letting $z = x + iy$,

$$T = \frac{\partial}{\partial t}, \quad X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t} \quad \text{and} \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t} \qquad j = 1, \ldots, d.$$

Let $\triangle$ be the $2d$-dimensional left invariant distribution spanned by X_j, Y_j $j = 1, \ldots, d$. $\triangle$ is called *horizontal distribution.* Using the horizontal distribution it is possible, via Lie bracket, to generate all of $T\mathbf{H}^d$, in fact $[X_i, Y_j] = \delta_{i,j} 4T$. Let $\pi : \mathbf{H}^d \to \mathbb{C}^d$ be the projection $\pi(z,t) = z$.

A Lipschitz curve $\gamma : [0,1] \to \mathbf{H}^n$ is *horizontal* if $\dot{\gamma}(s) \in \triangle(\gamma(s))$ for all $s \in [0,1]$, i.e. if there exist $\alpha_j, \beta_j : [0,1] \to \mathbb{R}$ such that

$$\dot{\gamma}(s) = \sum_{j=1}^{d} \big(\alpha_j(s) X_j(\gamma(s)) + \beta_j(s) Y_j(\gamma(s))\big).$$

We denote $|\dot{\gamma}(s)|$ the length of $\dot{\gamma}(s)$ with respect to the left invariant metric on $\triangle$ that makes $X_1, \ldots, X_d, Y_1, \ldots, Y_d$ orthonormal, namely

$$|\dot{\gamma}(s)| = \left(\sum_{j=1}^{d} \big(\alpha_j(s)^2 + \beta_j(s)^2\big) \right)^{1/2}.$$

The lenght of γ is then defined as

$$L(\gamma) = \int_0^1 |\dot{\gamma}(s)| \, ds. \tag{2.1}$$

We set the distance between two points $p, q \in \mathbf{H}^d$ to be

$$d(p,q) = \inf \big\{ L(\gamma) : \gamma \in \mathrm{Lip}([0,1]; \mathbf{H}^n) \text{ is horizontal and } \gamma : p \mapsto q \big\}. \tag{2.2}$$

Here $\gamma : p \mapsto q$ means $\gamma(0) = p$ and $\gamma(1) = q$. The function d turns out to be a left-invariant metric on $\mathbf{H}^d$, usually called the Carnot-Carathéodory metric. The Carnot-Carathéodory metric is rich of isometries (although not as rich as the Euclidean one), in particular any translation $\tau_\alpha : (z,t) \mapsto (z, t+\alpha)$ is an isometry. The metric d is homogenous of order 1 with respect to the non-isotropic dilations $\delta_\lambda(z,t) = (\lambda z, \lambda^2 t)$, $\lambda > 0$. Namely, $d(\delta_\lambda p, \delta_\lambda q) = \lambda d(p,q)$ for any $p, q \in \mathbf{H}^d$.

Let $p \in \mathbf{H}^d$ and $r > 0$, in the following we will denote by $B_r(p) = \{q \in \mathbf{H}^d : d(p,q) < r\}$ the open ball of radius r with center in p and with $B_r = \{q \in \mathbf{H}^d : d(0,q) < r\}$ the one centered in 0. Let $v \in \mathbb{C}^d$ and $r > 0$, the half balls $B_r^{\pm}(v)$ with respect to v of radius r and centered in $x \in \mathbf{H}^d$ are defined as

$$\begin{aligned} B_r^+(x; v) &= \{y \in B_r(x) : \pi(x^{-1} * y) \cdot v \geq 0\}, \\ B_r^-(x; v) &= \{y \in B_r(x) : \pi(x^{-1} * y) \cdot v \leq 0\}. \end{aligned} \tag{2.3}$$

Here, for $u, v \in \mathbb{C}^d$, we let $u \cdot v$ be the usual scalar product in $\mathbb{C}^d = \mathbb{R}^{2d}$. Obviously $B_r(x) = B_r^+(x; v) \cup B_r^-(x; v)$ for any $x \in \mathbf{H}^d$, $v \in \mathbb{C}^d$ and $r > 0$. As above, we let $B_r^{\pm}(v) = B_r^{\pm}(0; v)$.

Proposition 2.0.1. *Let*

$$\big\| (z,t) \big\|_{\mathbf{H}} = |z| + \sqrt{t}, \quad (z,t) \in \mathbf{H}^d.$$

Then the metric $d_{\mathbf{H}}(p,q) = \|p^{-1} * q\|_{\mathbf{H}}$, $p, q \in \mathbf{H}^d$, is equivalent to the Carnot-Carathéodory metric.

Proof. By the left invariance of both $\|\cdot\|_{\mathbf{H}}$ and the Carnot-Carathéodory metric, we have to prove that there exist two constants $C_1, C_2 > 0$ such that for any $p \in \mathbf{H}^d$ it holds

$$C_1 \|p\|_{\mathbf{H}} \leq d(0,p) \leq C_2 \|p\|_{\mathbf{H}}. \tag{2.4}$$

Both d and $\|\cdot\|_{\mathbf{H}}$ are homogeneous of order 1 with respect to the dilations δ_λ. Hence to prove (2.4) it suffices to show that, for $C_1, C_2 > 0$ and for any $p \in \{q \in \mathbf{H}^d : \|q\|_{\mathbf{H}} = 1\}$, it holds

$$C_1 \leq d(0,p) \leq C_2.$$

This follows from the Weierstrass theorem, since d is continuous, positive and locally finite and the set $\{q \in \mathbf{H}^d : \|q\|_{\mathbf{H}} = 1\}$ is compact. $\square$

Proposition 2.0.1 implies that a ball B_r centered in the origin of radius $r > 0$, behaves like the box

$$\mathrm{Box}(0,r) = \{(z,t) \in \mathbb{C}^d \times \mathbb{R} : |z| \leq r, \ |t| \leq r^2\}.$$

It is possible to prove that the infimum in (2.2) is in fact a minimum. The curves realizing such minimum are called *geodesics*. Thanks to the left-invariance of the metric

d, the geodesics joining two points $p, q \in \mathbf{H}^d$ can be written as left translations of geodesics joining the origin with $p^{-1} * q$. For any $p \in \mathbf{H}^d$ we will denote by γ_p the geodesic joining 0 to p. Such geodesic is not unique if p lies on the t axis, but our arguments will not be affected by this fact. The curve γ_p is of class C^∞ and we may assume it is parametrized with constant speed:

$$|\pi(\dot\gamma_p(s))| = d(p,0) \text{ for all } s \in [0,1]. \tag{2.5}$$

The left invariant Haar measure on $\mathbf{H}^d$ is the Lebesgue measure $\mathcal{L}^{2d+1}$. With such measure we construct the usual $L^p(\mathbf{H}^d)$ spaces of p-integrable functions.

Definition 2.0.2. Let $\Omega \subset \mathbf{H}^d$ be an open set and let $\phi \in C^1(\Omega)$. Then, for any $p \in \Omega$, we let the *horizontal gradient* of ϕ at p to be

$$\nabla_{\mathbf{H}}\phi(p) = \big(X_1\,\phi(p), Y_1\,\phi(p), \ldots, X_d\,\phi(p), Y_d\,\phi(p)\big) \subset \mathbb{C}^d. \tag{2.6}$$

In the following Lemma we prove a Taylor development formula in the Heisenberg group.

Proposition 2.0.3. *Let Ω be an open subset of $\mathbf{H}^d$, such that for any $p, q \in \Omega$, the geodesic $p * \gamma_q$ is entirely contained in Ω. If $\phi \in C^2_c(\Omega)$, then there exist a constant $C > 0$ such that for any $p, q \in \Omega$ it holds*

$$\phi(p*q) = \phi(p) + \nabla_{\mathbf{H}}\phi(p)\cdot\pi(q) + R(p,q), \quad |R(p,q)| \le Cd(q,0)^2. \tag{2.7}$$

Proof. By the fundamental theorem of calculus we get

$$\phi(p*q) - \phi(p) = \int_0^1 \frac{d}{ds}\phi(p*\gamma_q(s))\,ds = \int_0^1 \nabla_{\mathbf{H}}\phi(p*\gamma_q(s))\cdot\pi(\dot\gamma_q(s))\,ds. \tag{2.8}$$

Adding and subtracting $\nabla_{\mathbf{H}}\phi(p)$ inside the integral in (2.8) yields

$$\phi(p*q) - \phi(p) = \nabla_{\mathbf{H}}\phi(p)\cdot\int_0^1 \pi(\dot\gamma_q(s))\,ds + \int_0^1 \big(\nabla_{\mathbf{H}}\phi(p*\gamma_q(s)) - \nabla_{\mathbf{H}}\phi(p)\big)\cdot\pi(\dot\gamma_q(s))\,ds. \tag{2.9}$$

The projection π is linear and continuous, hence

$$\frac{d}{ds}\pi(\gamma_q(s)) = \lim_{h\to 0}\frac{\pi(\gamma_q(s)) - \pi(\gamma_q(s+h))}{h} = \lim_{h\to 0}\pi\left(\frac{\gamma_q(s) - \gamma_q(s+h)}{h}\right) = \pi(\dot\gamma_q(s)).$$

Using again the fundamental theorem of calculus, this implies that

$$\int_0^1 \pi(\dot\gamma_q(s))\,ds = \int_0^1 \frac{d}{ds}\pi(\gamma_q(s))\,ds = \pi(q). \tag{2.10}$$

Combining (2.9) and (2.10) and defining

$$R(p,q) = \int_0^1 \big(\nabla_{\mathbf{H}}\phi(p*\gamma_q(s)) - \nabla_{\mathbf{H}}\phi(p)\big)\cdot\pi(\dot\gamma_q(s))\,ds,$$

we get

$$\phi(p * q) - \phi(p) = \nabla_{\mathbf{H}}\phi(p) \cdot \pi(q) + R(p,q).$$

To complete the proof, it suffices to show that there exists a constant $C > 0$ such that for any $p, q \in \Omega$ it holds that $|R(p,q)| \leq Cd(q,0)^2$. By $\phi \in C^2_c(\Omega)$ follows that $\nabla_{\mathbf{H}}\phi$ is a Lipschitz function. Let L be a Lipschitz constant for $\nabla_{\mathbf{H}}\phi$. Since there exists a constant $M > 0$ such that for any $x, y \in \operatorname{supp}\phi$ there holds $|x - y| \leq Md(x, y)$, by the Cauchy-Schwarz inequality and (2.5) we get

$$\begin{aligned} |R(p,q)| &\leq \sup_{s\in[0,1]} \big|\nabla_{\mathbf{H}}\phi(p * \gamma_q(s)) - \nabla_{\mathbf{H}}\phi(p)\big||\pi(\dot{\gamma}_q(s))| \\ &\leq \sup_{s\in[0,1]} LMd(p * \gamma_q(s), p)d(q,0) \\ &\leq LMd(q,0)^2. \end{aligned}$$

This proves the claim with $C = LM$, and hence the Lemma. □

In the following sections, in analogy with the Euclidean case studied in the previous chapter, we use the geometric constant

$$C_{p,d} = \fint_{B_1} |v \cdot \pi(w)|^p \, dw, \tag{2.11}$$

where $v \in \mathbb{C}^d$ is any vector with $|v| = 1$. We remark that, due to the dilation invariance, for any $r > 0$ we have

$$C_{p,d} = \frac{1}{r^p} \fint_{B_r} |v \cdot \pi(w)|^p \, dw. \tag{2.12}$$

The following Lemma is an adaption to $\mathbf{H}^d$ of Lemma 1.1.2.

Lemma 2.0.4. *Let $\varphi \in C^2_c(\mathbf{H}^d)$ and $v \in \mathbb{C}^d$, $|v| = 1$. Then for all $p \in \mathbf{H}^d$ we have*

$$\lim_{r\downarrow 0} \frac{1}{r} \fint_{B_r^+(p;v)} (\varphi(q) - \varphi(p)) \, dq = C_{1,d} \, \nabla_{\mathbf{H}}\varphi(p) \cdot v, \tag{2.13}$$

where $B_r^+(p;v)$ is the half ball of radius $r > 0$ and centered at p defined in (2.3).

Proof. Making the change of variables $q \mapsto p * w$ and using the Taylor formula (2.7) we get

$$\begin{aligned} \frac{1}{r} \fint_{B_r^+(p;v)} (\varphi(q) - \varphi(p)) \, dq &= \frac{1}{r} \fint_{B_r^+(v)} (\varphi(p * w) - \varphi(p)) \, dw \\ &= \nabla_{\mathbf{H}}\varphi(p) \cdot \frac{1}{r} \fint_{B_r^+(v)} \pi(w) \, dw + \fint_{B_r^+(v)} \frac{R(p,w)}{r} \, dw. \end{aligned} \tag{2.14}$$

Here the last term tends to 0 as $r \downarrow 0$.

We claim that

$$\frac{1}{r} \fint_{B_r^+(v)} \pi(w) \, dw = C_{1,d} \, v. \tag{2.15}$$

The Lemma will then be proved letting $r \downarrow 0$ in (2.14). Let $\{v, e_2, \dots, e_d\}$ be a orthonormal basis of $\mathbb{C}^d$. Hence we can write

$$\frac{1}{r} \fint_{B_r^+(v)} \pi(w)\, dw = \fint_{B_1^+(v)} \pi(w)\, dw = v \fint_{B_1^+(v)} v \cdot \pi(w)\, dw + \sum_{i=2}^{d} e_i \fint_{B_1^+(v)} e_i \cdot \pi(w)\, dw.$$

For $i = 2, \dots, d$ we get

$$\begin{aligned} \int_{B_1^+(v)} e_i \cdot \pi(w)\, dw &= \int_{B_1^+(v) \cap B_1^+(e_i)} e_i \cdot \pi(w)\, dw + \int_{B_1^+(v) \cap B_1^-(e_i)} e_i \cdot \pi(w)\, dw \\ &= \int_{B_1^+(v) \cap B_1^+(e_i)} e_i \cdot \pi(w)\, dw - \int_{B_1^+(v) \cap B_1^+(e_i)} e_i \cdot \pi(w)\, dw \\ &= 0, \end{aligned}$$

where we made the change of variables $w \mapsto h = w - 2(e_i \cdot \pi(w))w$.

Validity of (2.13) follows from

$$\fint_{B_1^+(v)} v \cdot \pi(w)\, dw = \fint_{B_1} |v \cdot \pi(w)|\, dw = C_{1,d}.$$

□

The arguments of the following section are adaptions of the ones used in [BBM01] for the Sobolev case, and in [Dáv02] for the bounded variation case.

2.1 The horizontal Sobolev and BV spaces

Let $f \in L^1_{\text{loc}}(\Omega)$ we say that $g \in \left(C_c^\infty(\Omega)\right)'$ is a distributional derivative of f with respect to X_j if

$$\int_\Omega f X_j \varphi \, dz\, dt = -\langle g, \varphi \rangle, \quad \forall \varphi \in C_c^\infty(\Omega).$$

In such case we write $g = X_j f$. The distributional derivatives $Y_j f$ are defined in the same way. If f admits distributional derivatives with respect to any X_j and Y_j, $j = 1, \dots, d$, we define the distributional horizontal gradient $\nabla_{\mathbf{H}} f$ as in (2.6).

Definition 2.1.5. The *horizontal Sobolev space* $W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$ is the set of all functions $f \in L^p(\mathbf{H}^d)$ such that all the horizontal distributional derivatives $X_1 f, \dots, X_d f, Y_1 f, \dots, Y_d f$ are in $L^p(\mathbf{H}^d)$.

We set

$$\|\nabla_{\mathbf{H}} f\|^p_{L^p(\mathbf{H}^d)} = \begin{cases} \int_{\mathbf{H}^d} |\nabla_{\mathbf{H}} f|^p \; dz\, dt & \text{if } f \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^d) \\ +\infty & \text{otherwise} \end{cases},$$

where we used

$$\int_{\mathbf{H}^d} |\nabla_{\mathbf{H}} f|^p \; dz\, dt = \int_{\mathbf{H}^d} \sum_{j=1}^{d} \left[(X_j f(z,t))^2 + (Y_j f(z,t))^2 \right]^{p/2} \; dz\, dt.$$

The space $W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$ is a Banach space, when it is endowed with the norm

$$\|f\|_{W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)} = \|f\|_{L^p(\mathbf{H}^d)} + \left(\int_{\mathbf{H}^d} |\nabla_{\mathbf{H}} f|^p \, dz\, dt\right)^{1/p}.$$

Definition 2.1.6. Let $f \in L^1(\mathbf{H}^d)$. We say that f is a *function of bounded horizontal variation* in $\mathbf{H}^d$, if the distributional horizontal gradient $\nabla_{\mathbf{H}} f$ of f is representable by a finite $\mathbb{R}^{2d}$-valued measure in $\mathbf{H}^d$.

In such case we will call $|\nabla_{\mathbf{H}} f|$ the horizontal total variation measure of the function f and denote with $BV_{\mathbf{H}}(\mathbf{H}^d)$ the vector space of the functions with bounded horizontal variation.

We recall the following characterization of $W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$ for $1 < p \leq \infty$ and $BV_{\mathbf{H}}(\mathbf{H}^d)$.

Theorem 2.1.7. *Let $f \in L^p(\mathbf{H}^d)$ with $1 \leq p \leq \infty$ and let p' such that $1/p + 1/p' = 1$. The following are equivalent:*

i. $p > 1$ and $f \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$ (resp. $p = 1$ and $f \in BV_{\mathbf{H}}(\mathbf{H}^d)$);

ii. there exist a constant $C > 0$ such that for all $\varphi \in C^\infty_c(\mathbf{H}^d)$ it holds

$$\left|\int_{\mathbf{H}^d} f\,(X_j\varphi)\,dx\right| \leq C\|\varphi\|_{L^{p'}(\mathbf{H}^d)}, \quad \left|\int_{\mathbf{H}^d} f\,(Y_j\varphi)\,dx\right| \leq C\|\varphi\|_{L^{p'}(\mathbf{H}^d)}, \quad j = 1,\dots,d. \tag{2.16}$$

Proof. We follow closely the proof of Proposition VIII.3 in [Bré83].

We start by proving that i⇒ii. If $f \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$, $1 < p \leq \infty$, (2.16) follows by Hölder inequality, with $C = \sup\{\|X_j f\|_{L^p(\mathbf{H}^d)}, \|Y_j f\|_{(\mathbf{H}^d)} : j = 1,\dots,d\}$. On the other hand, if $f \in BV_{\mathbf{H}}(\mathbf{H}^d)$ it clear that

$$\left|\int_{\mathbf{H}^d} f\,(X_j\varphi)\,dx\right| \leq X_j f\big(\mathbf{H}^d\big)\|\varphi\|_{L^\infty(\mathbf{H}^d)}, \quad \left|\int_{\mathbf{H}^d} f\,(Y_j\varphi)\,dx\right| \leq Y_j f\big(\mathbf{H}^d\big)\|\varphi\|_{L^\infty(\mathbf{H}^d)}.$$

Hence (2.16) holds with $C = |\nabla_{\mathbf{H}} f|(\mathbf{H}^d)$.

To prove the other implication, we claim that if $f \in L^p(\mathbf{H}^d)$ is such that (2.16) holds, then $X_j f, Y_j f \in \big(L^{p'}(\mathbf{H}^d)\big)'$, $j = 1,\dots,d$. We prove the claim for $X_j f$, $j = 1,\dots,d$, but the same argument applies to Y_j. Let $\varphi \in C^\infty_c(\mathbf{H}^d)$ and define

$$T_f(\varphi) = -\int_{\mathbf{H}^d} f\, X_j\varphi\, dx. \tag{2.17}$$

The linear functional T_f is defined on a dense subset of $L^{p'}(\mathbf{H}^d)$ and by (2.16) is continuous. Therefore, by the Hahn-Banach theorem, T_f can be extended to a continuous functional on $L^{p'}(\mathbf{H}^d)$, i.e. $T_f \in \big(L^{p'}(\mathbf{H}^d)\big)'$.

Recall that $\big(L^{p'}(\mathbf{H}^d)\big)' \subset \big(C^\infty_c(\mathbf{H}^d)\big)'$, since $C^\infty_c(\mathbf{H}^d) \subset L^{p'}(\mathbf{H}^d)$. Hence $T_f \in \big(C^\infty_c(\mathbf{H}^d)\big)'$ and we can write

$$T_f(\varphi) = \langle T_f, \varphi\rangle, \quad \text{for any } \varphi \in C^\infty_c(\mathbf{H}^d).$$

Confronting this identity with (2.17), we get that T_f is a representation of the distributional derivative $X_j f$. Hence $X_j f \in \big(L^{p'}(\mathbf{H}^d)\big)'$ and the claim is proved.

If $1 < p \leq \infty$, by the Riesz representation theorem (see [Bré83, Theorems IV.11 and IV.14]) we get that $\big(L^{p'}(\mathbf{H}^d)\big)' = L^p(\mathbf{H}^d)$. Hence, by the previous claim, $X_j f, Y_j f \in L^p(\mathbf{H}^d)$, $j = 1, \dots, d$, and then $f \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$.

If $p = 1$, one can prove (see [Yos80, p. 118]) that $\big(L^{\infty}(\mathbf{H}^d)\big)'$ is the set of all Radon measures on $\mathbf{H}^d$, $\mathcal{L}^{2d+1}$-absolutely continuous and of bounded total variation. Therefore $X_j f, Y_j f$, $j = 1, \dots, d$, are Radon measures of bounded total variation. Hence $\nabla_{\mathbf{H}} f$ is a vector Radon measure of bounded total variation and so $f \in BV_{\mathbf{H}}(\mathbf{H}^d)$.

□

Finally we state a density theorem, proved in [FSC96] in a more general framework. In particular we refer to [FSC96, Theorem 1.2.3] in the horizontal Sobolev case and to [FSC96, Theorem 2.2.2] in the $BV_{\mathbf{H}}$ case.

Theorem 2.1.8. *Let $f \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$, $1 \leq p < \infty$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset C^1(\mathbf{H}^d) \cap W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$, such that*

$$\lim_{n\to\infty} \|f_n - f\|_{L^p(\mathbf{H}^d)} = \lim_{n\to\infty} \|\nabla_{\mathbf{H}} f_n - \nabla_{\mathbf{H}} f\|_{L^p(\mathbf{H}^d)} = 0.$$

On the other hand, if $f \in BV_{\mathbf{H}}(\mathbf{H}^d)$, then there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset C^1(\mathbf{H}^d) \cap BV_{\mathbf{H}}(\mathbf{H}^d)$, such that

$$\lim_{n\to\infty} \|f_n - f\|_{L^1(\mathbf{H}^d)} = 0 \quad \textit{and} \quad \lim_{n\to\infty} |\nabla_{\mathbf{H}} f_n|(\mathbf{H}^d) = |\nabla_{\mathbf{H}} f|(\mathbf{H}^d).$$

2.1.1 Length of the gradient of $W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$ functions, $1 < p < \infty$

Theorem 2.1.9. *Let $f \in L^p(\mathbf{H}^d)$, $1 < p < \infty$. Then $f \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$ if and only if*

$$\liminf_{r\downarrow 0} \frac{1}{r^p} \int_{\mathbf{H}^d} \fint_{B_r(x)} |f(x) - f(y)|^p \, dy \, dx < +\infty. \tag{2.18}$$

Moreover, in such case the limit inferior is in fact a limit and it results

$$\lim_{r\downarrow 0} \frac{1}{r^p} \int_{\mathbf{H}^d} \fint_{B_r(x)} |f(x) - f(y)|^p \, dy \, dx = C_{p,d} \|\nabla_{\mathbf{H}} f\|^p_{L^p(\mathbf{H}^d)}, \tag{2.19}$$

where $C_{p,d}$ is defined in (2.12).

First of all we need the following

Lemma 2.1.10. *Let $f \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$, $1 \leq p < \infty$. Then it holds*

$$\frac{1}{r^p} \int_{\mathbf{H}^d} \fint_{B_r(x)} |f(x) - f(y)|^p \, dy \, dx \leq \|\nabla_{\mathbf{H}} f\|^p_{L^p(\mathbf{H}^d)}. \tag{2.20}$$

Proof. Let $g : \mathbf{H}^d \to \mathbb{R}$. Thanks to the left translation of the distance, it holds

$$\fint_{B_r(x)} g(y)\,dy = \frac{1}{|B_r|}\int_{\mathbf{H}^d} g(y)\chi_{B_r}(x^{-1} * y)\,dy.$$

Hence a change of variables in (2.20) yields

$$\begin{aligned}\frac{1}{r^p}\int_{\mathbf{H}^d}\fint_{B_r(x)} |f(x)-f(y)|^p\,dy\,dx &= \frac{1}{r^p|B_r|}\int_{\mathbf{H}^d}\int_{\mathbf{H}^d} |f(x)-f(x*h)|^p\chi_{B_r}(h)\,dh\,dx\\ &= \frac{1}{r^p}\fint_{B_r}\int_{\mathbf{H}^d} |f(x)-f(x*h)|^p\,dx\,dh.\end{aligned}$$

Since $\fint_{B_r} d(h,0)^p\,dh \le r^p$ to complete the proof it suffices to show that

$$\fint_{\mathbf{H}^d} |f(x)-f(x*h)|^p\,dx \le d(h,0)^p\|\nabla_{\mathbf{H}} f\|^p_{L^p(\mathbf{H}^d)}. \tag{2.21}$$

If $f \in W^{1,p}(\mathbf{H}^d)\cap C^1(\mathbf{H}^d)$, we can use the integral Minkowski inequality and the Cauchy-Schwarz inequality to get

$$\begin{aligned}\int_{\mathbf{H}^d} |f(x)-f(x*h)|^p\,dx &= \int_{\mathbf{H}^d}\left|\int_0^1 \frac{d}{dt}(f\circ(x*\gamma_h))(s)\,ds\right|^p dx\\ &= \int_{\mathbf{H}^d}\left|\int_0^1 \nabla_{\mathbf{H}} f(x*\gamma_h(s))\cdot\pi(\dot\gamma_h(s))\,ds\right|^p dx\\ &\le \int_0^1\int_{\mathbf{H}^d} |\nabla_{\mathbf{H}} f(x*\gamma_h(s))\cdot\pi(\dot\gamma_h(s))|^p\,dx\,ds\\ &\le \int_0^1 |\pi(\dot\gamma_h(s))|^p\int_{\mathbf{H}^d} |\nabla_{\mathbf{H}} f(x*\gamma_h(s))|^p\,dx\,ds\\ &= d(h,0)^p\|\nabla_{\mathbf{H}} f\|^p_{L^p(\mathbf{H}^d)}.\end{aligned}$$

In the last equality, we used (2.5) and the right invariance of the Lebesgue measure.

The general case follows by a standard approximation argument using Theorem 2.1.8. □

Proof of Theorem 2.1.9. Let $f \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$, we need to prove that

$$\lim_{r\downarrow 0}\|T_r[f]\|^p_{L^p(\mathbf{H}^d\times\mathbf{H}^d)} = C_{p,d}\|\nabla_{\mathbf{H}} f\|^p_{L^p(\mathbf{H}^d)}, \tag{2.22}$$

where

$$T_r[f](x,y) = \frac{|f(x)-f(y)|}{r}\left(\frac{\chi_{B_r}(x^{-1}*y)}{|B_r|}\right)^{1/p}.$$

The triangle inequality implies that the operator T_r is subadditive. This, together with (2.20) in Lemma 2.1.10, implies that, for any $r > 0$ and $f, g \in W^{1,p}(\mathbf{H}^d)$, it holds

$$\big|\|T_r[f]\|_{L^p(\mathbf{H}^d\times\mathbf{H}^d)} - \|T_r[g]\|_{L^p(\mathbf{H}^d\times\mathbf{H}^d)}\big| \le \|T_r[f-g]\|_{L^p(\mathbf{H}^d\times\mathbf{H}^d)} \le C_{p,d}\|\nabla_{\mathbf{H}}(f-g)\|_{L^p(\mathbf{H}^d)}.$$

Therefore we only need to establish (2.19) in some dense subset of $W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$, e.g. in $C^2_c(\mathbf{H}^d)$.

In order to prove that (1.8) holds for $g \in C^2_c(\mathbf{H}^d)$, it suffices to prove it pointwise. Namely we claim that for any $x \in \mathbb{R}^d$ it holds

$$\lim_{r\downarrow 0} \frac{1}{r^p} \fint_{B_r(x)} |g(x) - g(y)|^p \, dy = C_{p,d} |\nabla_{\mathbf{H}} g(x)|^p, \tag{2.23}$$

where $C_{p,d}$ is defined in (2.12). Identity (2.19) then follows from (2.23) by dominated convergence. In fact, if L is a Lipschitz constant for g (i.e. $|g(x) - g(y)| \le L|x - y|$ for any $x, y \in \mathbf{H}^d$) and if $M > 0$ is a constant such that for any $x, y \in \operatorname{supp} g$ there holds $|x - y| \le M d(x, y)$ then

$$\frac{1}{r^p} \fint_{B_r(x)} |g(x) - g(y)|^p \, dx \le L^p M^p.$$

To prove (2.23), we fix $r > 0$ and $x \in \mathbf{H}^d$. Then, for any $w \in B_r$, by (2.7) we have

$$g(x * w) - g(x) = \nabla_{\mathbf{H}} g(x) \cdot \pi(w) + R(x, w), \quad |R(x, w)| \le C d(w, 0)^2. \tag{2.24}$$

Here, C is independent of x and w. Thus we have that

$$|g(x) - g(x * w)|^p = |\nabla_{\mathbf{H}} g(x) \cdot \pi(w)|^p + L(x, w),$$

where we let

$$L(x, w) = |\nabla_{\mathbf{H}} g(x) \cdot \pi(w) + R(x, w)|^p - |\nabla_{\mathbf{H}} g(x) \cdot \pi(w)|^p.$$

Therefore we can write

$$\begin{aligned} \frac{1}{r^p} \fint_{B_r(x)} |g(x) - g(y)|^p \, dy &= \frac{1}{r^p} \fint_{B_r} |g(x) - g(x * w)|^p \, dw \\ &= \frac{1}{r^p} \fint_{B_r(x)} |\nabla_{\mathbf{H}} g(x) \cdot \pi(w)|^p \, dw + \fint_{B_r(x)} \frac{L(x, w)}{r^p} \, dw \\ &= C_{p,d} |\nabla_{\mathbf{H}} g(x)|^p + \fint_{B_r} \frac{L(x, w)}{r^p} \, dw. \end{aligned} \tag{2.25}$$

Now we show that

$$\lim_{r\downarrow 0} \fint_{B_r} \frac{L(x, w)}{r^p} \, dw = 0. \tag{2.26}$$

In fact, letting $\phi(t) = t^p$, by the mean value theorem we get that for any $0 < s < t$ it holds $\phi(t) - \phi(s) = \phi'(s^*)(t - s)$, where $s^* \in [s, t]$. Hence we have that

$$|L(x, w)| \le \big(|\nabla_{\mathbf{H}} g(x) \cdot \pi(w)| + |R(x, w)|\big)^p - |\nabla_{\mathbf{H}} g(x) \cdot \pi(w)|^p = p(s^*)^{p-1} |R(x, w)|,$$

where $s^* \in \big[|\nabla_{\mathbf{H}} g(x) \cdot w)|, |\nabla_{\mathbf{H}} g(x) \cdot w| + |R(x,w)|\big]$. By the fact that $\alpha \mapsto \alpha^{p-1}$, $\alpha > 0$, is non-decreasing and by (2.24), we get

$$\begin{aligned} |L(x,y)| &\leq Cp(s^*)^{p-1} d(w,0)^2 \\ &\leq Cp\big(|\nabla_{\mathbf{H}} g(x) \cdot w| + |R(x,w)|\big)^{p-1} d(w,0)^2 \\ &\leq Cp\big(|\nabla_{\mathbf{H}} g(x)| + Cd(w,0)\big)^{p-1} d(w,0)^{p+1}. \end{aligned}$$

Here we used the Cauchy-Schwarz inequality. This implies that, for $w \in B_r$ and $r < 1$, there exists a constant $C' > 0$ independent of w, such that $|L(x,w)| \leq C' r^{p+1}$. Thus (2.26) follows from

$$\left| \fint_{B_r} \frac{L(x,w)}{r^p} \, dw \right| \leq C' r.$$

By (2.26), letting $r \downarrow 0$ in (2.25) we get (2.23). This proves the claim and hence that (2.19) holds for any $g \in C_c^2(\mathbf{H}^d)$. By the previous considerations, this implies that (2.19) holds for any $f \in W^{1,p}(\mathbf{H}^d)$. This completes the first part of the proof.

Finally, we prove that, if $f \in L^p(\mathbf{H}^d)$ and

$$A_p[f] = \liminf_{r \downarrow 0} \left(\frac{1}{r^p} \int_{\mathbf{H}^d} \fint_{B_r(x)} |f(x) - f(y)|^p \, dy \, dx \right)^{1/p} < \infty, \tag{2.27}$$

then $f \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$ and hence (2.19) holds. We claim that for such an f, for any $v \in \mathbb{C}^d$ with $|v| = 1$ and for any $\varphi \in C_c^\infty(\mathbf{H}^d)$ we have

$$C_{1,d} \left| \int_{\mathbf{H}^d} f(x) \, \nabla_{\mathbf{H}} \varphi(x) \cdot v \, dx \right| \leq A_p[f] \|\varphi\|_{L^{p'}(\mathbf{H}^d)},$$

where $1/p + 1/p' = 1$. By Theorem 2.1.7, this will imply that $f \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$, completing the proof.

For any $r > 0$, $v \in \mathbb{C}$ with $|v| = 1$ and $\varphi \in C_c^\infty(\mathbf{H}^d)$, let

$$\mathcal{I}_r(\varphi; v) = \left| \frac{1}{r} \int_{\mathbf{H}^d} f(x) \fint_{B_r^+(x;v)} (\varphi(x) - \varphi(y)) \, dy \, dx \right|. \tag{2.28}$$

By Lemma 2.0.4 we have

$$\lim_{r \downarrow 0} \mathcal{I}_r(\varphi; v) = \lim_{r \downarrow 0} \mathcal{I}_r(\varphi; -v) = C_{1,d} \left| \int_{\mathbf{H}^d} f(x) \, \nabla_{\mathbf{H}} \varphi(x) \cdot v \, dx \right|. \tag{2.29}$$

Observe that, identifying $v \in \mathbb{C}^d$ with $(v, 0) \in \mathbf{H}^d$, we have that $\pi(x^{-1} * y) \cdot v \geq 0$ if and only if $(y - x) \cdot v \geq 0$. Since for any $r > 0$, the kernel $\varrho(x,y) = \chi_{B_r}(x^{-1} * y)$ satisfies the assumption of Lemma 1.3.18 for $\mathbf{H}^d = \mathbb{R}^{2d+1}$, we get

$$\mathcal{I}_r(\varphi, v) = \left| \frac{1}{r} \int_{\mathbf{H}^d} \fint_{B_r^+(x;v)} (f(x) - f(y)) \, \varphi(y) \, dx \, dy \right|.$$

Using Hölder inequality we obtain

$$\begin{aligned}\mathcal{I}_r(\varphi, v) + \mathcal{I}_r(\varphi, -v) &\leq \frac{2}{r} \int_{\mathbf{H}^d} ⨍_{B_r(y)} |f(x) - f(y)||\varphi(y)|\, dx\, dy \\ &\leq \|\varphi\|_{L^{p'}(\mathbf{H}^d)} \frac{2}{r} \left(\int_{\mathbf{H}^d} ⨍_{B_r(x)} |f(x) - f(y)|^p\, dy\, dx \right)^{1/p}.\end{aligned} \tag{2.30}$$

By (2.29), letting $r \downarrow 0$ in (2.30) proves the claim. □

2.1.2 Length of the gradient of $BV_{\mathbf{H}}(\mathbf{H}^d)$ functions

Theorem 2.1.11. *Let $\Omega \subset \mathbf{H}^d$ be an open set, $f \in BV_{\mathbf{H}}(\mathbf{H}^d) \cap BV(\mathbf{H}^d)$ and let μ_r be defined as*

$$\mu_r(A) = \frac{1}{r} \int_A \left(⨍_{B_r(x)\cap\Omega} |f(x) - f(y)|\, dy \right) dx,$$

for any $A \subset \mathbf{H}^d$ Borel. Then $\mu_r \rightharpoonup C_{1,d}|\nabla_{\mathbf{H}} f|$ as $r \downarrow 0$ weakly in the sense of Radon measures in Ω. Here $C_{1,d}$ is the geometric constant defined in (2.11)*.*

The following Lemma is the equivalent of Lemma 1.3.13 in $\mathbf{H}^d$.

Lemma 2.1.12. *Under the assumptions of Theorem 2.1.11, let E be a Borel subset of $\mathbf{H}^d$ and let $E_r = \bigcup_{y\in E} B_r(y)$ be the r-neighborhood of E. Then there exists a geometric constant $\kappa > 0$ depending only on the dimension d, such that*

$$\mu_r(E) \leq C_{1,d}|\nabla_{\mathbf{H}} f|(E_r) + \kappa r |\nabla f|(E_r), \tag{2.31}$$

where $C_{1,d}$ is defined in (2.11) *and $|\nabla f|$ is the Euclidean total variation measure of f.*

Proof. We prove (2.31) in the case of $f \in C^1(\mathbf{H}^d)$, the general case will follow by a density argument using Theorem 2.1.8.

By the triangle inequality we have, identifying $\pi(w) \in \mathbb{C}^d$ with $(\pi(w), 0) \in \mathbf{H}^d$, that

$$|f(y * w) - f(y)| \leq |f(y * w) - f(y * \pi(w))| + |f(y * \pi(w)) - f(y)|.$$

Hence we get the estimate

$$\begin{aligned}\mu_r(E) &= \frac{1}{r} \int_E ⨍_{B_r(x)} |f(x) - f(y)|\, dy\, dx \\ &= \frac{1}{r} \int_E ⨍_{B_r} |f(y * w) - f(y)|\, dw\, dx \\ &\leq \frac{1}{r} \int_E ⨍_{B_r} |f(y * w) - f(y * \pi(w))|\, dw\, dx + \frac{1}{r} \int_E ⨍_{B_r} |f(y * \pi(w)) - f(y)|\, dw\, dx.\end{aligned} \tag{2.32}$$

To estimate the second term in the r.h.s. we use the fundamental theorem of calculus and the fact that $\pi(\dot{\gamma}_{\pi(w)}(s)) = \pi(w)$ for any $s \in [0,1]$. Therefore we get

$$\begin{aligned}\frac{1}{r}\int_E \fint_{B_r} |f(y*\pi(w)) - f(y)|\,dw\,dx &= \frac{1}{r}\int_E \fint_{B_r} \left|\int_0^1 \nabla_{\mathbf{H}} f(y*\gamma_{\pi(w)}(s))\cdot \pi(w)\,ds\right| dw\,dx \\ &\le \frac{1}{r}\int_E \fint_{B_r} \int_0^1 |\nabla_{\mathbf{H}} f(y*\gamma_{\pi(w)}(s))\cdot \pi(w)|\,ds\,dw\,dx \\ &\le \frac{1}{r}\fint_{B_r}\int_{E_r} |\nabla_{\mathbf{H}} f(z)\cdot \pi(w)|\,dz\,dw \\ &= C_{1,d}\int_{E_r} |\nabla_{\mathbf{H}} f(z)|\,dz \\ &= C_{1,d}|\nabla_{\mathbf{H}} f|(E_r).\end{aligned} \tag{2.33}$$

For $w = (\pi(w), \tau)$ for some $\tau \in \mathbb{R}$, we let $\hat{w} = (0, t)$. Hence we can write $y * w = y * (\pi(w) + \hat{w})$ and get

$$\begin{aligned}|f(y*w) - f(y*\pi(w))| &= \left|\int_0^\tau \frac{\partial}{\partial s} f\big(y*(\pi(w)+(0,s)\big)\,ds\right| \\ &= \left|\int_0^1 \tau\frac{\partial}{\partial s} f\big(y*(\pi(w)+s\hat{w})\big)\,ds\right| \\ &\le |\tau|\int_0^1 \left|\frac{\partial}{\partial s} f\big(y*(\pi(w)+s\hat{w})\big)\right|\,ds.\end{aligned}$$

Observe that if $w \in B_r$, then $\pi(w) + s\hat{w} \in B_r$ for any $s \in [0,1]$ and that by Proposition 2.0.1 follows that $|\tau| \le \kappa r^2$ for some geometric constant $\kappa > 0$. Then, with the same computations as in (2.33), we can estimate the first term in the r.h.s. of (2.32) as

$$\frac{1}{r}\int_E \fint_{B_r} |f(y*w) - f(y*\pi(w))|\,dw\,dx \le \kappa r\int_{E_r}\left|\frac{\partial f}{\partial t}(z)\right|\,dz \le \kappa r|\nabla f|(E_r). \tag{2.34}$$

The thesis follows using the estimates (2.33) and (2.34) in (2.32). □

Proof of Theorem 2.1.11. Let $(\mu_{r_n})_{n\in\mathbb{N}}$ be any subsequence of $(\mu_r)_{r>0}$, with $r_n \longrightarrow 0$ as $n \to \infty$. As in the proof of Theorem 1.3.12, by (2.31) follows that there exists a subsequence $(\mu_{r_n})_{n\in\mathbb{N}}$, with $\lim_{n\to+\infty} r_n = 0$, of $(\mu_r)_{r>0}$ which converges weakly to a Radon measure μ in Ω. We show that, for all Borel subsets B of Ω, this holds:

$$\mu(B) \le C_{1,d}|\nabla_{\mathbf{H}} f|(B). \tag{2.35}$$

In fact, if K is a compact subset of Ω and $R > 0$ is small enough, there exists n_0

such that $\forall n \geq n_0$ holds $K_{R+r_n} \subset \Omega$. Then, by (2.31),

$$\begin{aligned}
\mu(K) &\leq \mu(K_R) \leq \liminf_{n\to\infty} \mu_{r_n}(K_R) \\
&\leq \lim_{n\to\infty} \Big[C_{1,d} |\nabla_{\mathbf{H}} f| \left(K_{R+r_n}\right) + \kappa r_n |Df| \left(K_{R+r_n}\right) \Big] \\
&= C_{1,d} |\nabla_{\mathbf{H}} f| \left(\bigcap_{n=1}^{\infty} K_{R+r_n} \right) + \lim_{n\to\infty} \kappa r_n |Df| \left(K_{R+r_n}\right) \\
&\leq C_{1,d} |\nabla_{\mathbf{H}} f| \left(\bigcap_{n=1}^{\infty} K_{R+r_n} \right) + \kappa |Df| \left(K_{R+r_{n_0}}\right) \lim_{n\to\infty} r_n \\
&= C_{1,d} |\nabla_{\mathbf{H}} f| \left(\overline{K_R}\right).
\end{aligned}$$

Here we used the continuity from above of measures and the lower semicontinuity on open subsets of the weak convergence of Radon measures (see, for example, [EG92]). Letting $R \downarrow 0$, since $K = \bigcap_{R>0} \overline{K_R}$, we obtain

$$C_{1,d} |\nabla_{\mathbf{H}} f| \big(\overline{K_R}\big) \searrow C_{1,d} |\nabla_{\mathbf{H}} f| (K).$$

Thus our claim (2.35) holds for compact subsets of Ω and therefore for all Borel sets.

To complete the proof, due to the uniqueness of the limit, it suffices to prove that

$$\mu \geq C_{1,d} |\nabla_{\mathbf{H}} f|. \tag{2.36}$$

To prove (2.36), let $v \in \mathbb{C}^d$ be such that $|v| = 1$ and let $\varphi \in C_c^\infty(\mathbf{H}^d)$ be non negative, $\varphi \geq 0$. As in Theorem 2.1.9, we can show that

$$C_{1,d} \left| \int_{\mathbf{H}^d} f(y) \, \nabla_{\mathbf{H}} \varphi(y) \cdot v \, dy \right| \leq \int_{\mathbf{H}^d} \varphi \, d\mu.$$

Integrating by parts in the l.h.s. and approximating the characteristic function of an open set with non negative $C_c^\infty(\mathbf{H}^d)$ functions, yields

$$C_{1,d} \left| \nabla_{\mathbf{H}} f(A) \cdot v \right| \leq \mu(A), \tag{2.37}$$

for any $v \in \mathbb{C}^d$ with $|v| = 1$ and A open.

Let g and σ denote the Radon-Nykodim derivatives with respect to $|\nabla_{\mathbf{H}} f|$ of the measures μ and $\nabla_{\mathbf{H}} f$, respectively. Namely

$$g(x) = \lim_{R\to 0} \frac{\mu(U_R(x))}{|\nabla_{\mathbf{H}} f|(U_R(x))} \quad \text{and} \quad \sigma(x) = \lim_{R\to 0} \frac{\nabla_{\mathbf{H}} f(U_R(x))}{|\nabla_{\mathbf{H}} f|(U_R(x))},$$

for $|\nabla_{\mathbf{H}} f|$-a.e. $x \in \mathbf{H}^d$ and $|\sigma| = 1$ a.e. . Here $U_R(x)$ is the euclidean ball of radius $R > 0$ centered in x. On the other hand $g(x) \geq C_{1,d}$ for $|\nabla_{\mathbf{H}} f|$-a.e. $x \in \mathbf{H}^d$. In fact by (2.37) follows that $\mu(U_R(x)) \geq C_1 |\nabla_{\mathbf{H}} f(U_R(x))|$ for any x and $R > 0$. Therefore we get

$$\mu(A) = \int_A g \, d|\nabla_{\mathbf{H}} f| \geq C_1 |\nabla_{\mathbf{H}} f|(A),$$

that proves (2.36) and hence the theorem. □

Theorem 2.1.13. *A function $f \in L^1(\mathbf{H}^d)$ belongs to $BV_{\mathbf{H}}(\mathbf{H}^d)$ if and only if*

$$\liminf_{r \downarrow 0} \frac{1}{r} \int_{\mathbf{H}^d} \fint_{B_r(x)} |f(x) - f(y)| \, dy \, dx < +\infty. \tag{2.38}$$

Moreover, if $f \in BV_{\mathbf{H}}(\mathbf{H}^d) \cap BV(\mathbf{H}^d)$ then we have

$$\lim_{r \downarrow 0} \frac{1}{r} \int_{\mathbf{H}^d} \fint_{B_r(x)} |f(x) - f(y)| \, dy \, dx = C_{1,d} |\nabla_{\mathbf{H}} f|(\mathbf{H}^d), \tag{2.39}$$

where $C_{1,d}$ is defined in (2.12)*.*

Proof. The proof of the first statement is done as in Theorem 2.1.9, thanks to Theorem 2.1.7.

The second statement follows directly from Theorem 2.1.11, with $\Omega = \mathbf{H}^d$.

□

Chapter 3

Rearrangements in metric spaces

Let (X,d) be a metric space with balls $B_r(x)=\{y\in X:\ d(x,y)<r\}$ for any $x\in X$ and $r>0$. We also let $\partial B_r(x)=\{y\in X:\ d(x,y)=r\}$. When the ball is centered at the origin we let $B_r=B_r(0)$ and $\partial B_r=\partial B_r(0)$. A metric space is *proper* if closed balls are compact. For any set $E\subset X$ we let the *diameter* of E to be

$$\operatorname{diam} E=\sup\big\{d(x,y):\ x,y\in E\big\}. \tag{3.1}$$

For any function $f:X\to\mathbb{R}$ and for any open set $U\subset X$, we define the *Lipschitz constant* of f in U as

$$\operatorname{Lip}(f;U)=\sup\left\{\frac{|f(x)-f(y)|}{d(x,y)}:\ x,y\in U,\ x\neq y\right\}. \tag{3.2}$$

We let $\operatorname{Lip}(f;X)=\operatorname{Lip}(f)$. If $\operatorname{Lip}(f)<+\infty$, we say that f is a *Lipschitz function* and write $f\in\operatorname{Lip}(X)$. We say that f is locally Lipschitz if for any $x\in X$ there exists a neighborhood U of x such that $\operatorname{Lip}(f;U)<\infty$. In this case, we write $f\in\operatorname{Lip}_{\mathrm{loc}}(X)$.

Given a continuous path $\gamma:[0,1]\to X$, we define the length of γ as

$$L(\gamma)=\sup\left\{\sum_{i=0}^{n-1}d\big(\gamma(t_i),\gamma(t_{i+1})\big):\ 0=t_0<t_2<\cdots<t_n=1,\ n\in\mathbb{N}\right\}.$$

If $L(\gamma)<\infty$ we say that γ is rectifiable. We let the *intrinsic metric* d_{I} on X to be

$$d_{\mathrm{I}}(x,y)=\inf\big\{L(\gamma):\ \gamma \text{ is a rectifiable curve s.t. } \gamma:x\mapsto y\big\}, \tag{3.3}$$

where with $\gamma:x\mapsto y$ we mean that $\gamma(0)=x$, $\gamma(1)=y$. If $d(x,y)=d_{\mathrm{I}}(x,y)$ for any $x,y\in X$ we say that X is a *length space*.

Let μ be a Borel measure on (X,d). The triple (X,d,μ) is then called a metric measure space. Using the measure μ we construct the usual $L^p(X,\mu)$ and $L^p_{\mathrm{loc}}(X,\mu)$ spaces. For any function $f\in L^1_{\mathrm{loc}}(X,\mu)$ and for any Borel set $B\subset X$ with positive and finite measure, let

$$\fint_B f(x)\,d\mu(x)=\frac{1}{\mu(B)}\int_B f(x)\,d\mu(x)$$

denote the averaged integral of f over B.

We say that the measure μ is *non-degenerate* if for any $x \in X$ and $r > 0$

$$0 < \mu(B_r(x)) < \infty, \tag{3.4}$$

and *diffuse*[1] if

$$\mu(\partial B_r(x)) = 0. \tag{3.5}$$

Finally we say that the metric measure space (X, d, μ) has the *Lebesgue property* if for any Borel set $A \subset X$ we have that μ-a.e. $x \in A$ is a point of density for A, i.e.

$$\lim_{r \downarrow 0} \frac{\mu(A \cap B_r(x))}{\mu(B_r(x))} = 1. \tag{3.6}$$

We remark that if (X, d, μ) is doubling, in the sense that there exists a constant $D > 0$ such that $\mu(B_{2r}(x)) \leq \mu(B_r(x))$ for any $r > 0$ and $x \in X$, then it has the Lebesgue property. For a proof of this fact we refer to [Hei01, p. 4].

Given a Borel function $\phi : X \to Y$ between the metric measure space (X, d, μ) and the metric space (Y, δ), we define the *push-forward* of μ with respect to ϕ as

$$\phi_\sharp \mu(B) = \mu(\phi^{-1}(B)), \qquad \text{for any Borel set } B \subset Y.$$

If $X = Y$ and $\phi_\sharp \mu = \mu$, we say that the Borel measure μ is *ϕ-invariant.*

3.1 Function spaces in a metric measure space

The results of the previous chapters (Theorems 1.2.4, 1.3.14, 2.1.9 and 2.1.13) suggest the following definition of length of the gradient of a real valued function.

Definition 3.1.1. Let (X, d, μ) be a metric measure space and let $f \in L^1_{\mathrm{loc}}(X, \mu)$. For $1 \leq p < \infty$ we let

$$\|\nabla f\|^-_{L^p(X,\mu)} = \liminf_{r \downarrow 0} \left(\frac{1}{r^p} \int_X \fint_{B_r(x)} |f(x) - f(y)|^p \, d\mu(y) \, d\mu(x) \right)^{1/p}, \tag{3.7}$$

$$\|\nabla f\|^+_{L^p(X,\mu)} = \limsup_{r \downarrow 0} \left(\frac{1}{r^p} \int_X \fint_{B_r(x)} |f(x) - f(y)|^p \, d\mu(y) \, d\mu(x) \right)^{1/p}. \tag{3.8}$$

If $\|\nabla f\|^+_{L^p(X,\mu)} = \|\nabla f\|^-_{L^p(X,\mu)}$, then we let the their common value to be

$$\|\nabla f\|_{L^p(X,\mu)} = \lim_{r \downarrow 0} \left(\frac{1}{r^p} \int_X \fint_{B_r(x)} |f(x) - f(y)|^p \, d\mu(y) \, d\mu(x) \right)^{1/p}. \tag{3.9}$$

[1] The term *diffuse* is used sometimes to refer to non-atomic measures.

Let $1 < p < \infty$. If $f \in L^p(X,\mu)$ and $\|\nabla f\|_{L^p(X,\mu)} < +\infty$ we say that f is a p-Sobolev class function. Similarly, if $f \in L^1(X,\mu)$ and $\|\nabla f\|_{L^1(X,\mu)} < +\infty$, we say that f is a function of bounded variation. We point out that it is not clear wether, if f and g are two p-Sobolev class function, $f+g$ is also a p-Sobolev class function or not. In particular the limit in (3.9) could not even exists. The same is true for functions of bounded variation.

Having defined what a function of bounded variation is, we can define the notion of perimeter in a metric measure space.

Definition 3.1.2. For any Borel set $E \subset X$ let the *lower perimeter* and *upper perimeter* of E be defined as

$$P^-(E; X, d, \mu) = \|\nabla \chi_E\|^-_{L^1(X,\mu)} \quad \text{and} \quad P^+(E; X, d, \mu) = \|\nabla \chi_E\|^+_{L^1(X,\mu)}.$$

Here χ_E is the characteristic function of the set E, namely

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \in X \setminus E. \end{cases}$$

If the lower and upper perimeter coincide, then we let

$$P(E; X, d, \mu) = \|\nabla \chi_E\|_{L^1(X,\mu)}. \tag{3.10}$$

3.2 Compactness

In this section we prove a compactness result for families of functions in $L^p(X,\mu)$. Theorem 3.2.5 below is needed in section 3.6.

Definition 3.2.3. Let (X, d, μ) be a metric measure space and let Φ be a family of functions in $L^p_{\text{loc}}(X,\mu)$.

(i) We say that Φ is *locally uniformly bounded in* $L^p_{loc}(X,\mu)$ if for any compact set $K \subset X$ it holds

$$\sup_{f\in\Phi} \int_K |f|^p \, d\mu < +\infty. \tag{3.11}$$

(ii) We say that Φ is *locally uniformly absolutely continuous in* $L^p_{loc}(X,\mu)$ if for any compact set $K \subset X$ and for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all Borel sets $B \subset K$ with $\mu(B) < \delta$ it holds

$$\sup_{f\in\Phi} \int_B |f|^p \, d\mu < \varepsilon. \tag{3.12}$$

Lemma 3.2.4. *Let (X, d, μ) be a proper metric measure space such that the measure μ is non-degenerate and diffuse, in the sense of* (3.4) *and* (3.5)*. Let $\Phi \subset L^1_{loc}(X,\mu)$ be a* *locally uniformly bounded and locally uniformly absolutely continuous family of functions. Then letting*

$$f_r(x) = \fint_{B_r(x)} f\,d\mu, \tag{3.13}$$

it follows that the family $\Phi_r = \{f_r : f \in \Phi\} \subset C(X)$, $r > 0$, is locally uniformly bounded. Moreover, all $f_r \in \Phi_r$, $r > 0$, are uniformly continuous on compact sets.

Proof. Because the balls $B_r(x)$ are precompact, the functions f_r in (3.13) are well defined. We now prove that $f_r \in C(X)$ for any $r > 0$ and $f \in \Phi$. In fact, since $\mu(\partial B_r(x)) = 0$ for any $r > 0$ and $x \in X$, we have that, for any $x_0 \in X$, $\chi_{B_r(x)} \longrightarrow \chi_{B_r(x_0)}$ μ-a.e. as $x \to x_0$. Thus the dominated convergence theorem yields, for any $g \in L^1_{\text{loc}}(X, \mu)$,

$$\lim_{x \to x_0} \int_{B_r(x)} g\,d\mu = \int_{B_r(x_0)} g\,d\mu.$$

Since this proves, in particular, that $x \mapsto \mu(B_r(x))$ is continuous, we are finished.

We claim that Φ_r is locally uniformly bounded. In fact, if $K \subset X$ is a compact set and if we let $K_r = \{x \in X : \operatorname{dist}(x, K) \le r\}$ to be the compact r-neighborhood of K, we have that $|f_r(x)| \le C_1 C_2$. Here

$$C_1 = \max_{x \in K} \frac{1}{\mu(B_r(x))} \quad \text{and} \quad C_2 = \sup_{f \in \Phi} \int_{K_r} |f|\,d\mu. \tag{3.14}$$

We remark that it is $C_2 < +\infty$, since Φ is locally uniformly bounded.

Finally we show that f_r is uniformly continuous on compact sets for any $r > 0$ and $f \in \Phi$. To this aim, let K be a compact set, as above, and let $x, x_0 \in K$. We have

$$\begin{aligned} |f_r(x) - f_r(x_0)| &\le \max\left\{\frac{1}{\mu(B_r(x))}, \frac{1}{\mu(B_r(x_0))}\right\} \int_{B_r(x)\Delta B_r(x_0)} |f|\,d\mu + \\ &\qquad + \frac{|\mu(B_r(x)) - \mu(B_r(x_0))|}{\mu(B_r(x))\mu(B_r(x_0))} \int_{B_r(x)\cap B_r(x_0)} |f|\,d\mu \\ &\le C_1 \int_{B_r(x)\Delta B_r(x_0)} |f|\,d\mu + C_1^2 C_2 |\mu(B_r(x)) - \mu(B_r(x_0))|. \end{aligned} \tag{3.15}$$

Here the constants C_1 and C_2 are the ones defined in (3.14). The function $m : X \times X \to [0, +\infty)$, $m(x, x_0) = \mu\big(B_r(x)\Delta B_r(x_0)\big)$, is continuous and hence absolutely continuous on $K \times K$. Since $m(x_0, x_0) = 0$, for any $\delta > 0$ there exists an $\eta > 0$ such that, if $d(x, x_0) < \eta$, $m(x, x_0) < \delta$. This, together with the local uniform absolutely continuity of Φ, implies that for a given $\varepsilon > 0$ there exists an $\eta > 0$ such that, if $d(x, x_0) < \eta$, we have

$$\sup_{f \in \Phi} \int_{B_r(x)\Delta B_r(x_0)} |f|\,d\mu < \varepsilon. \tag{3.16}$$

Using (3.16) in (3.15) we complete the proof. □

Now we prove the main result of this section.

Theorem 3.2.5 (Compactness). *Let (X, d, μ) be a proper metric measure space such that the measure μ is non-degenerate and diffuse, in the sense of* (3.4) *and* (3.5). *Let $1 \leq p < +\infty$ and let $\Phi \subset L^p_{loc}(X, \mu)$ be a family of functions such that:*

(i) *Φ is uniformly locally bounded in $L^p_{loc}(X, \mu)$. Moreover, if $p = 1$ we assume Φ to be locally uniformly absolutely continuous in $L^1_{loc}(X, \mu)$.*

(ii) *there exists $\psi \in L^p(X, \mu)$ with*

$$\Big(\|\nabla \psi\|^-_{L^p(X,\mu)} \Big)^p = \liminf_{r \downarrow 0} \frac{1}{r^p} \int_X \fint_{B_r(x)} |\psi(x) - \psi(y)|^p \, d\mu(y) \, d\mu(x) < +\infty, \quad (3.17)$$

such that for all $r \in (0, 1)$ we have

$$\sup_{f \in \Phi} \int_X \fint_{B_r(x)} |f(x) - f(y)|^p \, d\mu(y) \, d\mu(x) \leq \int_X \fint_{B_r(x)} |\psi(x) - \psi(y)|^p \, d\mu(y) \, d\mu(x). \quad (3.18)$$

Then Φ is precompact in $L^p_{loc}(X, \mu)$.

Proof. The family Φ satisfies the assumptions of Lemma 3.2.4. If $p = 1$ this is trivial. Otherwise let $K \subset X$ be a compact set and let q such that $1/p + 1/q = 1$. Then for any Borel set $B \subset K$, the Hölder inequality yields

$$\int_B |f| \, d\mu \leq \mu(B)^{1/q} \left(\int_B |f|^p \, d\mu \right)^{1/p} \leq \|f\|_{L^p(K,\mu)} \mu(B)^{1/q}.$$

This implies that Φ is locally uniformly absolutely continuous.

By Lemma 3.2.4, $\Phi_r = \{f_r : f \in \Phi\} \subset C(K)$ is equibounded and equicontinuous. By Ascoli-Arzelà theorem, Φ_r is totally bounded with respect to the max norm, and hence with respect to the $L^p(K, \mu)$ norm.

Next we claim that

$$\liminf_{r \downarrow 0} \sup_{f \in \Phi} \|f_r - f\|_{L^p(K,\mu)} = 0. \quad (3.19)$$

This follows from (3.18):

$$\begin{aligned} \int_K |f_r - f|^p \, d\mu &= \int_K \left| \fint_{B_r(x)} (f(y) - f(x)) \, d\mu(y) \right|^p \, d\mu(x) \\ &\leq \int_K \fint_{B_r(x)} |f(y) - f(x)|^p \, d\mu(y) \, d\mu(x) \\ &\leq \int_K \fint_{B_r(x)} |\psi(y) - \psi(x)|^p \, d\mu(y) \, d\mu(x), \end{aligned}$$

the inequality holding for $r \in (0, 1)$. By assumption (3.17), this implies (3.19).

Finally, from (3.19) follows that Φ is totally bounded in $L^p(X, \mu)$ and hence precompact. □

3.3 Two-points rearrangement in metric spaces

In this section we study a technique that is very useful in proving the central theorems regarding rearrangements: the two-points rearrangement.

We say that $\mathcal{P} = \{H^-, H, H^+\}$ is a *partition* of X if H^-, H, H^+ are pairwise disjoint subsets of X and $X = H^- \cup H \cup H^+$.

Definition 3.3.6. A *reflection system* $\mathcal{R} = \{\mathcal{P}, \varrho\}$ of the metric space (X, d) is a partition $\mathcal{P} = \{H^-, H, H^+\}$ of X such that H^- and H^+ are open, together with a mapping $\varrho : X \to X$ such that:

(R1) the map ϱ is an involutive isometry of X (i.e. $d(\varrho x, \varrho y) = d(x, y)$ for any $x, y \in X$ and $\varrho^2 = \mathrm{Id}$) such that $\varrho H^+ = H^-$;

(R2) for all $x, y \in H \cup H^+$, we have $d(x, y) \leq d(x, \varrho y)$.

For the sake of brevity, here and henceforth we write $\varrho x = \varrho(x)$ and $\varrho E = \varrho(E)$ for $x \in X$ and $E \subset X$.

Proposition 3.3.7. *Let (X, d) be a length space. Let $\mathcal{P} = \{H^-, H, H^+\}$ be a partition such that H^- and H^+ are open and $\partial H^- = \partial H^+ = H$. Moreover let $\varrho : X \to X$ be a mapping satisfying (R1) and such that $\varrho\big|_H = Id$. Then $\mathcal{R} = \{\mathcal{P}, \varrho\}$ is a reflection system of (X, d).*

Proof. It suffices to prove that condition (R2) holds. Since ϱ is an isometry and $\varrho\big|_H = \mathrm{Id}$, we only need to check that for any $x, y \in H^+$ it holds that $d(x, y) \leq d(x, \varrho y)$.

Let γ be a rectifiable curve joining x with ϱy. By (R1), $\varrho y \in H^-$ and hence γ intersects $H = \partial H^- = \partial H^+$ at some point $z \in H$. Then we can split $\gamma = \gamma_{xz} + \gamma_{zy}$, where the sum is a concatenation of curves and $\gamma_{xz} : x \mapsto z$, $\gamma_{zy} : z \mapsto \varrho y$. Let $\gamma' = \gamma_{xz} + \varrho\gamma_{zy}$. Since $\varrho\big|_H = \mathrm{Id}$, γ' is continuous, and since ϱ is an isometry, $L(\gamma') = L(\gamma)$. By the arbitrariness of γ and since X is a length space (see (3.3)) we get that $d(x, y) \leq d(x, \varrho y)$. □

We describe some examples of reflection systems.

Example 3.3.8. Let $X = Z \oplus V$ be a vector space, where V is a 1-dimensional subspace of X. We may then decompose $x \in X$ as $x = z + v$, for uniquely determined $z \in Z$ and $v \in V$. We fix a total ordering on V.

We define the partition $\mathcal{P} = \{Z^-, Z, Z^+\}$, where $Z^- = \{z + v \in X : z \in Z, v < 0\}$ and $Z^+ = \{z + v \in X : z \in Z, v > 0\}$. Moreover let $\varrho : X \to X$ be defined as $\varrho(x) = \varrho(z + v) = z - v$ for any $x \in X$. Let $\|\cdot\|$ be a norm on X with respect to which ϱ is isometric. Then $\mathcal{R} = \{\mathcal{P}, \varrho\}$ is a reflection system of the metric space X endowed with the distance induced by the norm. Condition (R1) is trivially satisfied. We now prove (R2).

Let $v \in V$ be such that $v > 0$. We start by claiming that for any $x \in X$, the function $\phi_x : t \mapsto \|x + tv\|$ is non-decreasing for $t \geq 0$. In fact, if $0 \leq t < s$ we have

$$x + tv = \sigma(\varrho x - tv) + (1 - \sigma)(x + sv), \quad \text{where } \sigma = \frac{s - t}{s + t} \in (0, 1),$$

and therefore

$$\phi_x(t) = \|x + tv\| = \sigma\|\varrho x - tv\| + (1 - \sigma)\|x + sv\| = \sigma\phi_x(t) + (1 - \sigma)\phi_x(s).$$

This implies $\phi_x(t) \le \phi_x(s)$ and proves the claim.

Writing $y = z + tv$ and using the claim just proved, we get

$$\|x - y\| = \phi_{x-y}(0) \le \phi_{x-y}(2t) = \|x - \varrho y\|,$$

that implies that (R2) is satisfied.

The previous example applies to the Euclidean space. In fact we can always split $\mathbb{R}^d = \mathbb{R}^{d-1} \oplus \mathbb{R}$.

Example 3.3.9. Let $\varrho : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping $\varrho(x, y) = (x, -y)$ and let $\|\cdot\|$ be a norm on $\mathbb{R}^2$ such that $\|\varrho z\| = \|z\|$ for any $z \in \mathbb{R}^2$.

Let $\phi \in \mathrm{Lip}_{loc}(\mathbb{R})$ be a locally Lipschitz function that is not identically zero and consider the vector fields

$$X = \frac{\partial}{\partial x}, \qquad Y = \phi(x)\frac{\partial}{\partial y}.$$

A Lipschitz curve $\gamma : [0, 1] \to \mathbb{R}^2$ is admissible if $\dot\gamma(t) = h_1(t)X\big(\gamma(t)\big) + h_2(t)Y\big(\gamma(t)\big)$ for $h_1, h_2 \in L^1(0, 1)$. We define the length of an admissible curve γ as

$$L(\gamma) = \int_0^1 \|h(t)\|\, dt,$$

where $h = (h_1, h_2)$. We can then define a distance d on letting, for $x, y \in \mathbb{R}^2$,

$$d(x, y) = \inf\big\{L(\gamma) :\ \gamma \in \mathrm{Lip}([0, 1], \mathbb{R}^2) \text{ is admissible and s.t. } \gamma : x \mapsto y\big\}.$$

Then $(\mathbb{R}^2, d)$ is a length space and the mapping ϱ is an isometry. In fact, $\gamma : x \mapsto y$ is an admissible curve if and only if $\varrho \circ \gamma : \varrho x \mapsto \varrho y$ is admissible and moreover $L(\gamma) = L(\varrho \circ \gamma)$.

Let $\mathcal{P} = \{H^-, H, H^+\}$ be the partition of $\mathbb{R}^2$ such that $H^- = \{(x, y) \in \mathbb{R}^2 :\ y < 0\}$, $H = \{(x, y) \in \mathbb{R}^2 :\ y = 0\}$ and $H^+ = \{(x, y) \in \mathbb{R}^2 :\ y > 0\}$. Then $\mathcal{R} = \{\mathcal{P}, \varrho\}$ is a reflection system of $(\mathbb{R}^2, d)$ by Proposition 3.3.7.

If ϕ is an even function, the standard reflection with respect to the y-axis also defines a reflection system of $(\mathbb{R}^2, d)$.

Example 3.3.10. Let (X, d_X) be a metric space with reflection system $\mathcal{R} = \{\mathcal{P}, \varrho\}$ and let (Y, d_Y) be any metric space. The product $Z = X \times Y$ is still a metric space, when endowed with the metric $d_Z = (d_X^2 + d_Y^2)^{1/2}$. The reflection system $\mathcal{R}$ can be extended to a reflection system on Z. Namely, if $\pi : Z \to X$ is the standard projection on X, $\mathcal{R}_Z = \{\pi^{-1}\mathcal{P}, \varrho \times \mathrm{Id}_Y\}$ is a reflection system on Z.

Next we introduce the notion of two-points rearrangement for functions and sets in a metric space.

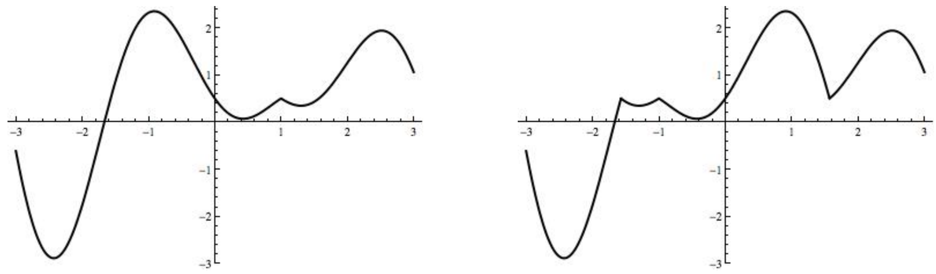

Figure 3.1: A function $f : \mathbb{R} \to \mathbb{R}$ and its two-points rearrangement with respect to the reflection system $\mathcal{R} = \{\mathcal{P}, x \mapsto -x\}$, where $\mathcal{P} = \{\mathbb{R}_-, \{0\}, \mathbb{R}_+\}$.

Definition 3.3.11. Let (X, d) be a metric space with reflection system $\mathcal{R} = \{\mathcal{P}, \varrho\}$, $\mathcal{P} = \{H^-, H, H^+\}$. Let $f : X \to \mathbb{R}$, then the function $f_{\mathcal{R}} : X \to \mathbb{R}$ defined by

$$f_{\mathcal{R}}(x) = \begin{cases} \min\{f(x), f(\varrho x)\} & \text{if } x \in H^- \\ f(x) & \text{if } x \in H \\ \max\{f(x), f(\varrho x)\} & \text{if } x \in H^+ \end{cases}, \tag{3.20}$$

is called the *two-points rearrangement* of f with respect to $\mathcal{R}$.

The definition of two-points rearrangement for sets can be obtained specializing (3.20) to the case of characteristic functions. Namely, for any $E \subset X$ we can define the set $E_{\mathcal{R}}$ via the identity $\chi_{E_{\mathcal{R}}} = (\chi_E)_{\mathcal{R}}$. This is equivalent with the following definition.

Definition 3.3.12. Let (X, d) be a metric space with reflection system $\mathcal{R} = \{\mathcal{P}, \varrho\}$. Let $E \subset X$, then the set $E_{\mathcal{R}}$ defined by

$$E_{\mathcal{R}} = \big(E \cap \varrho E \cap H^-\big) \cup (E \cap H) \cup \big((E \cup \varrho E) \cap H^+\big), \tag{3.21}$$

is called the *two-points rearrangement* of E with respect to $\mathcal{R}$.

The importance of the two-points rearrangement is that it "regularizes" the rearranged function or set. Henceforth we prove various theorems regarding this fact.

Proposition 3.3.13. *Let (X, d) be a metric space with reflection system $\mathcal{R} = \{\mathcal{P}, \varrho\}$. For any $f : X \to \mathbb{R}$, it holds that*

$$\mathrm{Lip}(f_{\mathcal{R}}) \le \mathrm{Lip}(f),$$

where $\mathrm{Lip}(f)$ *is the Lipschitz constant of f, as defined in* (3.2).

Proof. Let $x, y \in X$, $x \neq y$. To prove the assertion it suffices to show that

$$\frac{|f_{\mathcal{R}}(x) - f_{\mathcal{R}}(y)|}{d(x, y)} \le \mathrm{Lip}(f).$$

We have three cases:

1. $f_{\mathcal{R}}(x) = f(x)$ and $f_{\mathcal{R}}(y) = f(y)$;
2. $f_{\mathcal{R}}(x) \neq f(x)$ and $f_{\mathcal{R}}(y) \neq f(y)$;
3. $f_{\mathcal{R}}(x) = f(x)$ and $f_{\mathcal{R}}(y) \neq f(y)$, or viceversa.

In the first case the claim is clear. In the second case, since ϱ is an isometry and by (3.20), we have

$$\frac{|f_{\mathcal{R}}(x) - f_{\mathcal{R}}(y)|}{d(x,y)} = \frac{|f(\varrho x) - f(\varrho y)|}{d(x,y)} = \frac{|f(\varrho x) - f(\varrho y)|}{d(\varrho x, \varrho y)} \leq \mathrm{Lip}(f).$$

We are left with the third case. Here, since $f_{\mathcal{R}}(x) = f(x)$, it must be that $f(x) \geq f(\varrho x)$ and since $f_{\mathcal{R}}(y) \neq f(y)$, it must be that $f(y) < f(\varrho y)$. Then we distinguish three subcases

3a. $x, y \in H^+$, or $x, y \in H^-$;

3b. $x \in H^+$ and $y \in H^-$, or viceversa;

3c. $x \in H$, or $y \in H$.

Assume that we are in case 3a. Thus the claim follows from

$$f_{\mathcal{R}}(x) - f_{\mathcal{R}}(y) = f(x) - f(\varrho y) < f(x) - f(y) \leq |f(x) - f(y)| \leq \mathrm{Lip}(f) d(x,y),$$
$$f_{\mathcal{R}}(y) - f_{\mathcal{R}}(x) = f(\varrho y) - f(x) \leq f(\varrho y) - f(\varrho x) \leq \mathrm{Lip}(f) d(\varrho x, \varrho y) = \mathrm{Lip}(f) d(x,y).$$

Consider now the case 3b. Letting $z = \varrho y \in H^+$ we get, by (R1) and (R2), that

$$\frac{|f_{\mathcal{R}}(x) - f_{\mathcal{R}}(y)|}{d(x,y)} = \frac{|f(x) - f(\varrho y)|}{d(x,y)} = \frac{|f(x) - f(z)|}{d(x, \varrho z)} \leq \frac{|f(x) - f(\varrho y)|}{d(x,z)} \leq \mathrm{Lip}(f).$$

The same computation holds for the case 3c. □

Proposition 3.3.14. *Let (X,d) be a metric space with reflection system $\mathcal{R} = \{\mathcal{P}, \varrho\}$. For any set $E \subset X$, it holds that*

$$\operatorname{diam} E_{\mathcal{R}} \leq \operatorname{diam} E,$$

where $\operatorname{diam} E$ *is defined in* (3.1).

Proof. Let $x, y \in E_{\mathcal{R}}$. If $x, y \in E$ or $x, y \in \varrho E$, then $d(x,y) = d(\varrho x, \varrho y) \leq \operatorname{diam} E$.

Now we claim that for all $x, y \in E_{\mathcal{R}}$ such that $x \in E \setminus \varrho E$ and $y \in \varrho E \setminus E$, it holds that $d(x,y) \leq \operatorname{diam} E$. This will finish the proof. From (3.21) it is clear that $x \in H \cup H^+$ and $y \in H^-$. Therefore $\varrho y \in E \setminus \varrho E$ and, by (R2), we have

$$d(x,y) \leq d(x, \varrho y) \leq \operatorname{diam} E.$$

□

We now investigate the monotonicity property of the two points rearrangement regarding quantities like (3.9). To this aim, let $\phi : [0, +\infty) \mapsto [0, +\infty)$ be a function such that:

(P1) ϕ is strictly increasing;

(P2) ϕ is convex.

The basic inequality we need concerning ϕ is described in the following Lemma.

Lemma 3.3.15. *Let $\phi : [0, +\infty) \mapsto [0, +\infty)$ be a function satisfying (P1) and (P2). Then for all real numbers $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\gamma < \alpha$ and $\delta < \beta$ there holds*

$$\phi(|\alpha - \beta|) + \phi(|\gamma - \delta|) \leq \phi(|\alpha - \delta|) + \phi(|\gamma - \beta|). \tag{3.22}$$

If, in addition, ϕ is strictly convex then the inequality in (3.22) *is strict.*

Proof. Possibly interchanging α with β and γ with δ, we can assume $\beta \leq \alpha$. We have three cases:

1. $\gamma \leq \delta \leq \beta \leq \alpha$;
2. $\delta < \beta \leq \alpha < \gamma$;
3. $\delta \leq \gamma \leq \beta \leq \alpha$.

In the first case, by (P1), we get

$$\phi(\alpha - \beta) + \phi(\gamma - \delta) \leq \phi(\alpha - \delta) + \phi(\gamma - \beta).$$

In the second case, the convexity of ϕ yields

$$\phi(\alpha - \beta) = \phi\big(t(\alpha - \delta) + (1 - t)(\gamma - \beta)\big) \leq t\phi(\alpha - \delta) + (1 - t)\phi(\gamma - \beta), \tag{3.23}$$

where

$$t = \frac{\alpha - \gamma}{\alpha + \beta - (\gamma + \delta)} \quad \text{and} \quad 1 - t = \frac{\beta - \delta}{\alpha + \beta - (\gamma + \delta)}.$$

Since $\alpha > \gamma$ and $\beta > \delta$, then $t \in (0, 1)$. In the same way

$$\phi(\gamma - \delta) = \phi\big((1 - t)(\alpha - \delta) + t(\gamma - \beta)\big) \leq (1 - t)\phi(\alpha - \delta) + t\phi(\gamma - \beta). \tag{3.24}$$

Summing up inequalities (3.23) and (3.24) we get (3.22). If ϕ is strictly convex, then the inequality is strict.

Finally, in the third case we get, by (P1),

$$\phi(\alpha - \beta) \leq \phi(\alpha - \gamma) \quad \text{and} \quad \phi(\gamma - \delta) \leq \phi(\beta - \delta).$$

Then we conclude as in the second case. □

Let μ be a Borel measure on X and let $\mathcal{B}(X)$ denote the set of all Borel functions from X to $\mathbb{R}$. For any $r > 0$ let $Q_r : \mathcal{B}(X) \times \mathcal{B}(X) \to [0, +\infty)$ be the functional

$$Q_r(f, g) = \int_X \fint_{B_r(x)} \phi(|f(x) - g(y)|)\, d\mu(y)\, d\mu(x). \tag{3.25}$$

We omit reference to ϕ in our notation. For $\phi(t) = t^p$ with $1 \le p < \infty$, by (3.7) we have that $\|\nabla f\|^-_{L^p(X,\mu)} = \liminf_{r \downarrow 0} Q_r(f, f)$. In this case we let

$$Q_{r,p}(f, g) = \int_X \fint_{B_r(x)} |f(x) - g(y)|^p\, d\mu(y)\, d\mu(x), \tag{3.26}$$

and $Q_{r,p}(f) = Q_{r,p}(f, f)$.

Theorem 3.3.16. *Let $\mathcal{R} = \{\mathcal{P}, \varrho\}$ be a reflection system of the metric space (X, d). Let μ be a non-degenerate, ϱ-invariant Borel measure such that $\mu(H) = 0$ and let $\phi : [0, +\infty) \to [0, +\infty)$ be a function satisfying (P1) and (P2). Then for any $r > 0$ and all functions $f, g \in \mathcal{B}(X)$ we have*

$$Q_r(f_{\mathcal{R}}, g_{\mathcal{R}}) \le Q_r(f, g). \tag{3.27}$$

Moreover, if ϕ is strictly convex,

$$\mu\{x \in H^+ : f(x) > f(\varrho x)\} > 0 \text{ and } \mu\{y \in H^+ : g(y) < g(\varrho y)\} > 0, \tag{3.28}$$

then the inequality (3.27) *is strict, as soon $Q_r(f, g) < +\infty$.*

Proof. Let $\chi_r : X \times X \to \mathbb{R}$ be the function

$$\chi_r(x, y) = \begin{cases} \frac{1}{\mu(B_r(x))} & \text{if } d(x, y) < r, \\ 0 & \text{otherwise.} \end{cases}$$

As μ is ϱ-invariant, we have $\mu(B_r(\varrho x)) = \mu(\varrho B_r(x)) = \mu(B_r(x))$. Then also χ_r is ϱ-invariant. Namely, since ϱ is an involutive isometry, it holds

$$\chi_r(\varrho x, \varrho y) = \chi_r(x, y) \text{ and } \chi_r(x, \varrho y) = \chi_r(\varrho x, y). \tag{3.29}$$

Then, writing

$$Q_r(f, g) = \int_{X \times X} \phi(|f(x) - g(y)|)\chi_r(x, y)\, d\mu \otimes \mu(x, y),$$

we may replace the integration domani $X \times X$ with

$$(X \setminus H) \times (X \setminus H) = (H^+ \times H^+) \cup (H^+ \times H^-) \cup (H^- \times H^+) \cup (H^- \times H^-).$$

In fact we are assuming H to be a μ-negligible set. By (3.29) and $\varrho_\sharp\mu = \mu$, we obtain

$$\int_{H^+\times H^-} \phi(|f(x)-g(y)|)\chi_r(x,y)\,d\mu\otimes\mu(x,y) = \int_{H^+\times H^+} \phi(|f(x)-g(\varrho y)|)\chi_r(x,\varrho y)\,d\mu\otimes\mu(x,y),$$
$$\int_{H^-\times H^+} \phi(|f(x)-g(y)|)\chi_r(x,y)\,d\mu\otimes\mu(x,y) = \int_{H^+\times H^+} \phi(|f(\varrho x)-g(y)|)\chi_r(x,\varrho y)\,d\mu\otimes\mu(x,y),$$
$$\int_{H^-\times H^-} \phi(|f(x)-g(y)|)\chi_r(x,y)\,d\mu\otimes\mu(x,y) = \int_{H^+\times H^+} \phi(|f(\varrho x)-g(\varrho y)|)\chi_r(x,y)\,d\mu\otimes\mu(x,y).$$

Summing up we obtain

$$Q_r(f,g) = \int_{H^+\times H^+} Q(f,g;x,y)\,d\mu\otimes\mu(x,y),$$

where we let

$$\begin{aligned} Q(f,g;x,y) = \big[\phi(|f(x)-g(y)|) + \phi(|f(\varrho x)-g(\varrho y)|)\big]\chi_r(x,y) + \\ + \big[\phi(|f(x)-g(\varrho y)|) + \phi(|f(\varrho x)-g(y)|)\big]\chi_r(x,\varrho y). \end{aligned}$$

We claim that for all $x, y \in H^+$ we have

$$Q(f_{\mathcal{R}}, g_{\mathcal{R}}; x, y) \le Q(f,g;x,y). \tag{3.30}$$

This implies (3.27). There are only three cases:

1. $d(x,y) \ge r$ and $d(x,\varrho y) \ge r$;
2. $d(x,y) \le d(x,\varrho y) < r$;
3. $d(x,y) < r \le d(x,\varrho y)$.

In fact, by (R2), the case $d(x,y) \ge r$ and $d(x,\varrho y) < r$ cannot occur.

In the first two cases it holds $\chi_r(x,y) = \chi_r(x,\varrho y)$. Hence in case 1 it holds $Q(f,g;x,y) = Q(f_{\mathcal{R}}, g_{\mathcal{R}}; x,y) = 0$. In the second case, we have

$$\begin{aligned} Q(f,g;x,y) &= \frac{1}{\mu(B_r(x))}\big[\phi(|f(x)-g(y)|) + \phi(|f(\varrho x)-g(\varrho y)|) + \\ &\qquad + \phi(|f(x)-g(\varrho y)|) + \phi(|f(\varrho x)-g(y)|)\big] \\ &= Q(f_{\mathcal{R}}, g_{\mathcal{R}}; x, y). \end{aligned}$$

Finally we consider the third case. In such case it holds $\chi_r(x,\varrho y) = 0$ and thus inequality (3.30) is equivalent to

$$\phi(|f_{\mathcal{R}}(x)-g_{\mathcal{R}}(y)|)+\phi(|f_{\mathcal{R}}(\varrho x)-g_{\mathcal{R}}(\varrho y)|) \le \phi(|f(x)-g(y)|)+\phi(|f(\varrho x)-g(\varrho y)|). \tag{3.31}$$

If $f(x) = f(\varrho x)$ or $g(y) = g(\varrho y)$, inequality (3.31) holds as equality. We are then left with the following cases:

3a. $f(x) > f(\varrho x)$ and $g(y) > g(\varrho y)$;

3b. $f(x) < f(\varrho x)$ and $g(y) < g(\varrho y)$;

3c. $f(x) > f(\varrho x)$ and $g(y) < g(\varrho y)$;

3d. $f(x) < f(\varrho x)$ and $g(y) > g(\varrho y)$.

In the first case we have $f_{\mathcal{R}}(x) = f(x)$ and $g_{\mathcal{R}}(y) = g(y)$, and hence (3.31) holds as equality. The same is true in case 3b, since $f_{\mathcal{R}}(x) = f(\varrho x)$ and $g_{\mathcal{R}}(y) = g(\varrho y)$.

Possibly interchanging f and g it is enough to consider only one of case 3c and case 3d. We consider case 3c. Here, inequality (3.31) reduces to

$$\phi(|\alpha - \beta|) + \phi(|\gamma - \delta|) \leq \phi(|\alpha - \delta|) + \phi(|\gamma - \beta|), \tag{3.32}$$

with $\alpha = f(x)$, $\beta = g(\varrho y)$, $\gamma = f(\varrho x)$, $\delta = g(y)$. Since we are in case 3c, we have $\gamma < \alpha$ and $\delta < \beta$. Hence inequality (3.32) holds by Lemma 3.3.15.

To prove the last part of the Theorem, we first observe that, if ϕ is strictly convex, by Lemma 3.3.15 we get that (3.32) is strict. Then if (3.28) holds and if $Q_r(f, g) < +\infty$, on integrating (3.30) we get a strict inequality. □

Remark 3.3.17. In the case $\phi(t) = t^2$ there is a precise version of inequality (3.27). Let

$$\Sigma_f^+ = \{x \in H^+ : f(x) > f(\varrho x)\} \quad \text{and} \quad \Sigma_f^- = \{x \in H^- : f(x) > f(\varrho x)\}, \tag{3.33}$$

denote the sets appearing in cases 3a-3d.

In the proof of Theorem 3.3.16, inequality (3.27) is an equality possibly but for the cases 3c and 3d. In such cases, for $\phi(t) = t^2$, we can replace inequality 3.32 with the identity

$$(\alpha - \beta)^2 + (\gamma - \delta)^2 = (\alpha - \delta)^2 + (\gamma - \beta)^2 + 2(\alpha - \gamma)(\delta - \beta).$$

Now, on integrating the resulting identity, we obtain

$$\begin{aligned} Q_{r,2}(f_{\mathcal{R}}, g_{\mathcal{R}}) &= Q_{r,2}(f, g) + \\ &+ 2 \int_{\substack{\Sigma_f^+ \times \Sigma_g^- \cup \Sigma_f^- \times \Sigma_g^+ \\ d(x,\varrho y) \geq r}} (f(x) - f(\varrho x))(g(y) - g(\varrho y))\chi_r(x, y)\, d\mu \otimes \mu(x, y). \end{aligned} \tag{3.34}$$

Now we state two corollaries of Theorems 3.3.16, regarding the Sobolev norms and the perimeter of a set, defined in (3.9) and (3.10).

Theorem 3.3.18. *Let $\mathcal{R} = \{\mathcal{P}, \varrho\}$ be a reflection system of the metric space (X, d). Let μ be a non-degenerate, ϱ-invariant Borel measure such that $\mu(H) = 0$. Then for any function $f \in \mathcal{B}(X)$ and $1 \leq p < \infty$ there holds*

$$\|f_{\mathcal{R}}\|_{L^p(X,\mu)} = \|f\|_{L^p(X,\mu)} \quad \textit{and} \quad \|\nabla f_{\mathcal{R}}\|^-_{L^p(X,\mu)} \leq \|\nabla f\|^-_{L^p(X,\mu)}. \tag{3.35}$$

Moreover, if we have $\|\nabla f_{\mathcal{R}}\|_{L^2(X,\mu)} = \|\nabla f\|_{L^2(X,\mu)} < +\infty$*, then*

$$\lim_{r\downarrow 0} \frac{1}{r^2} \int_{\Sigma_f^+} \int_{\Sigma_f^- \cap (B_r(x)\setminus B_r(\varrho x))} \frac{(f(x)-f(\varrho x))(g(y)-g(\varrho y))}{\mu(B_r(x))}\, d\mu(y)\, d\mu(x) = 0, \qquad (3.36)$$

where Σ_f^+ *and* Σ_f^- *are defined in* (3.33)*. The same conclusion holds in* (3.36) *interchanging* Σ_f^+ *and* Σ_f^-*.*

Proof. The identity in (3.35) is trivial. By Theorem 3.3.16 we have

$$\frac{1}{r^p} Q_{r,p}(f_{\mathcal{R}}) \leq \frac{1}{r^p} Q_{r,p}(f) \qquad (3.37)$$

for any $r > 0$. Taking the $\liminf$ in (3.37) as $r \downarrow 0$, we get the inequality in(3.35).

Assume that both $\|\nabla f_{\mathcal{R}}\|_{L^2(X,\mu)}$ and $\|\nabla f\|_{L^2(X,\mu)}$ exist, are equal and finite. Then, by (3.34) for $f = g$, we get

$$\lim_{r\downarrow 0} \frac{1}{r^2} \int_{\substack{\Sigma_f^+ \times \Sigma_f^- \\ d(x,\varrho y)\geq r}} (f(x)-f(\varrho x))(f(y)-f(\varrho y))\chi_r(x,y)\, d\mu\otimes\mu(x,y) = 0,$$

where χ_r is the function defined in (3.29). By the Fubini theorem, this is equivalent to (3.36) or to the same limit with interchanged Σ_f^+ and Σ_f^-. □

For the perimeter we have the following theorem.

Theorem 3.3.19. *Let* $\mathcal{R} = \{\mathcal{P}, \varrho\}$ *be a reflection system of the metric space* (X, d) *and let* μ *be a non-degenerate,* ϱ*-invariant Borel measure such that* $\mu(H) = 0$*. Then for any Borel set* $E \subset X$ *there holds*

$$\mu(E_{\mathcal{R}}) = \mu(E) \quad \textit{and} \quad P^-(E_{\mathcal{R}}) \leq P^-(E). \qquad (3.38)$$

Moreover if $P(E_{\mathcal{R}}) = P(E) < +\infty$*, then*

$$\begin{aligned} &\lim_{r\downarrow 0} \frac{1}{r} \int_{H^+\cap(E\setminus\varrho E)} \frac{\mu\big((\varrho E\setminus E)\cap H^+\cap(B_r(x)\setminus B_r(\varrho x))\big)}{\mu(B_r(x))}\, d\mu(x) = 0, \\ &\lim_{r\downarrow 0} \frac{1}{r} \int_{H^+\cap(\varrho E\setminus E)} \frac{\mu\big((E\setminus \varrho E)\cap H^+\cap(B_r(x)\setminus B_r(\varrho x))\big)}{\mu(B_r(x))}\, d\mu(x) = 0. \end{aligned} \qquad (3.39)$$

Proof. As above, the identity in (3.38) is trivial, while the inequality follows from Theorem 3.3.16. Identities (3.39) follows from (3.34) with $f = g = \chi_E$, observing that $|\chi_E(x) - \chi_E(y)| = |\chi_E(x) - \chi_E(y)|^2$. □

We end this section presenting a simplified version of Theroem 3.3.16. Here we do not require the mapping ϱ to be an isometry.

Theorem 3.3.20. *Let $\mathcal{P} = \{H^-, H, H^+\}$ be a partition of the metric space (X, d), let $\varrho : X \to X$ be an involutive Borel map such that $\varrho H^+ = H^-$ and let μ be a ϱ-invariant Borel measure on X such that $\mu(H) = 0$. Finally let $\phi : [0, +\infty) \to [0, +\infty)$ be a function satisfying (P1) and (P2). Then for all $f, g \in \mathcal{B}(X)$ we have*

$$\int_X \phi(|f_{\mathcal{R}}(x) - g_{\mathcal{R}}(x)|)\, d\mu(x) \leq \int_X \phi(|f(x) - g(x)|)\, d\mu(x). \tag{3.40}$$

Moreover, if ϕ is strictly convex and

$$\mu\{x \in H^+ :\ f(x) > f(\varrho x) \text{ and } g(x) < g(\varrho x)\} > 0, \tag{3.41}$$

then the inequality (3.40) *is strict, as soon as the right hand side of* (3.40) *is finite.*

Proof. Using $\mu(H) = 0$ and the ϱ-invariance of μ, we obtain

$$\begin{aligned}\int_X \phi(|f(x) - g(x)|)\, d\mu(x) &= \int_{H^-} \phi(|f(x) - g(x)|)\, d\mu(x) + \int_{H^+} \phi(|f(x) - g(x)|)\, d\mu(x)\\ &= \int_{H^+} \Big[\phi(|f(x) - g(x)|) + \phi(|f(\varrho x) - g(\varrho x)|)\Big]\, d\mu(x).\end{aligned}$$

Since the same computation holds for the r.h.s of (3.40), to complete the proof it suffices to establish the pointwise inequality

$$\phi(|f_{\mathcal{R}}(x) - g_{\mathcal{R}}(x)|) + \phi(|f_{\mathcal{R}}(\varrho x) - g_{\mathcal{R}}(\varrho x)|) \leq \phi(|f(x) - g(x)|) + \phi(|f(\varrho x) - g(\varrho x)|).$$

This is inequality (3.31) in the proof of Theorem 3.3.16. The argument is then concluded as in the final part of that proof. In fact, if $f(x) > f(\varrho x)$ and $g(x) < g(\varrho x)$, or viceversa, the inequality is strict, provided that ϕ is strictly convex. □

3.4 Rearrangement systems

Let $\mathcal{S}(X, \mu)$ denote the set of all non-negative Borel functions $f : X \to \mathbb{R}$, such that $\mu\{f > t\} < +\infty$, for any $t > 0$. Here and henceforth, we let $\{f > t\} = \{x \in X : f(x) > t\}$ denote the t-superlevel of f. For any $f \in \mathcal{S}(X, \mu)$ we have the representation formula

$$f(x) = \int_0^{+\infty} \chi_{\{f>t\}}(x)\, dt, \quad x \in X. \tag{3.42}$$

To any $f \in \mathcal{S}(X, \mu)$ we can associate $\psi_f : (0, +\infty) \to (0, +\infty)$, defined as $\psi_f(t) = \mu\{f > t\}$, $t > 0$, called *distribution function* of f. Such function is non-increasing and lower semicontinuous. In fact, for any $s > 0$ we have

$$\lim_{t \downarrow s} \psi_f(t) = \lim_{t \downarrow s} \mu\{f > t\} = \mu\left(\bigcup_{t>s} \{f > s\}\right) = \mu\{f > s\} = \psi_f(s). \tag{3.43}$$

A function $g \in \mathcal{S}(X,\mu)$ is said to be a rearrangement of $f \in \mathcal{S}(X,\mu)$ if it has the same distribution function of f (i.e. $\psi_g \equiv \psi_f$). In such case we write $g \sim f$. It is clear that $\sim$ defines an equivalence relation on $\mathcal{S}(X,\mu)$.

If $f \in L^p(X,\mu)$, $1 \le p < +\infty$, then it is in $\mathcal{S}(X,\mu)$. In such case, by (3.42), it holds

$$\int_X f(x)^p \, d\mu(x) = \int_0^{+\infty} \mu\{f > t^{1/p}\} \, dt. \tag{3.44}$$

This implies that if $g \in \mathcal{S}(X,\mu)$ is a rearrangement of f (i.e. $g \sim f$), then also $g \in L^p(X,\mu)$ and $\|g\|_{L^p(X,\mu)} = \|f\|_{L^p(X,\mu)}$.

3.4.1 The Euclidean Steiner rearrangement

We start by defining the Steiner rearrangement of sets and functions in the d-dimensional Euclidean case.

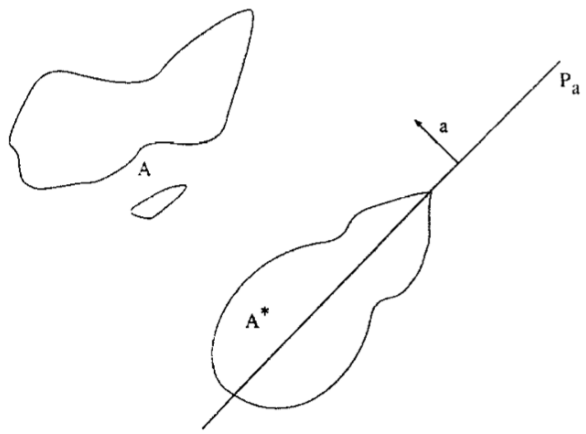

Figure 3.2: An example of Steiner simmetrization in $\mathbb{R}^2$. Image taken from [EG92].

Fix $a \in \mathbb{R}^d$, $|a| = 1$. Let $\Gamma = (\tau_t)_{t\in\mathbb{R}}$ be the family of the translations with direction a, namely $\tau_t(x) = x + ta$, $x \in \mathbb{R}^d$. The orbit of a point $x \in \mathbb{R}^d$ is the line through x of direction a:

$$L_{x,a} = \{x + ta : \, t \in \mathbb{R}\}.$$

The orbit relation, $x \sim y$ if and only if $y \in L_{x,a}$, is an equivalence relation. We can identify $\mathbb{R}^d/\Gamma$ with the plane through the origin perpendicular to a:

$$P_a = \{x \in \mathbb{R}^d : \, x \cdot a \ge 0\}.$$

Given a Borel set $A \subset \mathbb{R}^d$, we define its x-section in the direction a as $A_x = A \cap L_{x,a}$. By the Fubini theorem we have

$$\int_A dz = \int_{P_a} \mathcal{H}^1(A_x) \, d\mathcal{L}^{d-1}(x). \tag{3.45}$$

Here, $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure. We let $A_x^* = B_s(x) \cap L_{x,a}$ where $s \in [0, +\infty)$ is such that $\mathcal{H}^1(A_x) = \mathcal{H}^1(B_s(x) \cap L_{x,a})$. Here $B_s(x) = \{y \in \mathbb{R}^d : |y - x| < s\}$ is the ball of radius $s > 0$ centered at $x \in \mathbb{R}^d$. Such an s exists and is unique for any $x \in P_a$. In fact, for any $x \in \mathbb{R}^d$, $s \mapsto \mathcal{H}^1(B_s(x) \cap L_{x,a})$ is a strictly increasing function that maps $[0, +\infty)$ in itself.

Definition 3.4.21.

(i) Let $A \subset \mathbb{R}^d$ be a Borel set, with $|A| < +\infty$. The *Steiner rearrangement* of A with respect to the direction a is defined as

$$A^* = \bigcup_{x \in P_a} A_x^*.$$

(ii) Let $f \in \mathcal{S}(\mathbb{R}^d, \mathcal{L}^d)$. The *Steiner rearrangement* of f with respect to the direction a is defined as

$$f^*(z) = \int_0^{+\infty} \chi_{\{f>t\}^*}(z)\, dt, \qquad z \in \mathbb{R}^d.$$

The main results on the Steiner rearrangement are the following Theorems.

Theorem 3.4.22. *(i) Let $f \in W^{1,p}(\mathbb{R}^d, \mathcal{L}^d) \cap \mathcal{S}(\mathbb{R}^d, \mathcal{L}^d)$ be compactly supported. Then the Steiner rearrangement f^* of f satisfies*

$$\|f\|_{L^p(\mathbb{R}^d, \mathcal{L}^d)} = \|f^*\|_{L^p(\mathbb{R}^d, \mathcal{L}^d)} \quad \text{and} \quad \|Df^*\|_{L^p(\mathbb{R}^d, \mathcal{L}^d)} \le \|Df\|_{L^p(\mathbb{R}^d, \mathcal{L}^d)}.$$

(ii) Let $A \subset \mathbb{R}^d$ be a bounded Borel set of finite perimeter. Then the Steiner rearrangement A^* of A satisfies

$$|A| = |A^*| \quad \text{and} \quad P(A^*) \le P(A).$$

We do not prove this theorem. Indeed it is a particular case of Theorems 3.6.37 and 3.6.38. We only remark that the translations $\{\tau_t\}_{t>0}$ play a great role in the proofs.

3.4.2 The general case

In order to generalize the above concepts to a metric measure space (X, d, μ), we need some additional structure.

Let Γ be a group of isometries of X. Let $\Gamma_x = \{\gamma x : \gamma \in \Gamma\}$ be the orbit of a point $x \in X$. The orbit relation, $x \sim y$ if and only if $y \in \Gamma_x$, is clearly an equivalence relation. Let the quotient space X/Γ be identified with some subset of X. Finally, for any Borel set E, we let $E_x = E \cap \Gamma_x$ be the x-section of E.

Definition 3.4.23. Let μ be a Borel measure on the metric space (X, d) and let Γ be a group of isometries of (X, d). We say that μ is *disintegrable along* Γ if there are Borel measures μ_x on Γ_x, for all $x \in X/\Gamma$, and a Borel measure $\bar{\mu}$ on X/Γ such that for any Borel set $E \subset X$ we have:

(D1) the function $x \mapsto \mu_x(E_x)$ is Borel measurable from X/Γ to $[0, +\infty)$;

(D2) we have $\mu(E) = \int_{X/\Gamma} \mu_x(E_x)\, d\bar{\mu}(x)$.

The existence of a disintegration satisfying (D1)-(D2) holds under general assumptions. We will address this question later, in Section 3.5.

For any $x \in X/\Gamma$ let the number $s_0(x) > 0$ be the minimum number, possibly $+\infty$, such that the sets $B_r(x) \cap \Gamma_x$ are stable for $r > s_0(x)$.

Definition 3.4.24. Let (X, d, μ) be a metric measure space and let Γ be a group of isometries of (X, d). Let $(\mu_x)_{x\in X/\Gamma}$ be Borel measures on the orbits Γ_x and let $\bar{\mu}$ a Borel measure on the quotient X/Γ. We say that the triple $(\Gamma, (\mu_x)_{x\in X/\Gamma}, \bar{\mu})$ is a *rearrangement system of* (X, d, μ) if

(RS1) μ is disintegrable along Γ in the measures $(\mu_x)_{x\in X/\Gamma}$ and $\bar{\mu}$, as in Definition 3.4.23;

(RS2) the function $s \mapsto \mu_x(B_s(x) \cap \Gamma_x)$ from $[0, s_0(x))$ to $[0, +\infty)$ is strictly increasing and continuous.

Condition (RS2) is needed to define E^*_x for any $x \in X/\Gamma$. In fact, in analogy with the Euclidean case, possibly letting $E^*_x = \varnothing$ on a $\bar{\mu}$ negligible set, we define

$$E^*_x = B_s(x) \cap \Gamma_x, \text{ for } s \text{ such that } \mu_x(E_x) = \mu_x(B_s(x) \cap \Gamma_x). \tag{3.46}$$

If (RS2) holds such an s exists and is unique for $\bar{\mu}$-a.e. $x \in X/\Gamma$, while in general it might not even exists.

Definition 3.4.25. Let $(\Gamma, (\mu_x)_{x\in X/\Gamma}, \bar{\mu})$ be a rearrangement system of the metric measure space (X, d, μ).

(i) For any Borel set $E \subset X$ such that $\mu(E) < +\infty$ let the *rearrangement of* E in $(\Gamma, (\mu_x)_{x\in X/\Gamma}, \bar{\mu})$ be

$$E^* = \bigcup_{x\in X/\Gamma} E^*_x, \tag{3.47}$$

where E^*_x is defined in (3.46).

(ii) For any $f \in \mathcal{S}(X, \mu)$ let the *rearrangement of* f in $(\Gamma, (\mu_x)_{x\in X/\Gamma}, \bar{\mu})$ be

$$f^*(x) = \int_0^{+\infty} \chi_{\{f>t\}^*}(x)\, dt, \qquad x \in X. \tag{3.48}$$

Moreover, we say that the rearrangement system $(\Gamma, (\mu_x)_{x\in X/\Gamma}, \bar{\mu})$ is *regular* if for any Borel set $E \subset X$, the rearrangement E^* is a Borel set.

The problem of determining wether a rearrangement system is regular or not is in general very subtle. However, in most of the relevant examples, the system is indeed regular.

Notice that the definition of rearrangement depends on the choice of the representative of X/Γ in X. In 3.6 we will fix such a representative by a reflection system.

In the following Lemma we prove some properties of the rearrangements E^* and f^*.

Lemma 3.4.26. *Let* $(\Gamma, (\mu_x)_{x\in X/\Gamma}, \bar{\mu})$ *be a rearrangement system of the metric measure space* (X, d, μ). *For any* $f \in \mathcal{S}(X, \mu)$ *the rearrangement* f^* *of* f *enjoys the following properties:*

(i) $\{f^* > t\} = \{f > t\}^*$, $t > 0$, *and in particular* $f^* \in \mathcal{S}(X, \mu)$;

(ii) $\mu_x\{f^* > t\}_x = \mu_x\{f > t\}_x$, $t > 0$, *for* $\bar{\mu}$*-a.e.* $x \in X/\Gamma$ *and, in particular,* $f \sim f^*$;

(iii) $f^*(y) = f^*(z)$ *if* $y, z \in \Gamma_x$ *for some* $x \in X/\Gamma$ *and* $d(x, y) = d(x, z)$;

(iv) $f^*(y) = f^*(z)$ *if* $y, z \in \Gamma_x$ *for some* $x \in X/\Gamma$ *and* $d(x, y) \geq d(x, z)$.

Proof. We start by proving statement (i). We show that $\{f^* > t\} \subset \{f > t\}^*$ for any $t > 0$. Notice that the family $(\{f > t\}^*)_{t>0}$ is non-increasing in t. For any $x \in \{f^* > t\}$ we have

$$t < f^*(x) = \int_0^{+\infty} \chi_{\{f>s\}^*}(x)\, ds,$$

and thus $x \in \{f > s\}^*$ for any $0 \leq s \leq t$ and the claim follows.

To show the converse inclusion $\{f > t\}^* \subset \{f^* > t\}$, we start by noticing that

$$\{f > t\}^* = \bigcup_{s>t} \{f > s\}^*. \tag{3.49}$$

In fact, by the lower semicontinuity of the distribution function (see (3.43)), we have that for any $x \in X/\Gamma$ it holds

$$\begin{aligned} \lim_{s\downarrow t} \mu_x(\{f > s\}^* \cap \Gamma_x) &= \lim_{s\downarrow t} \mu_x(\{f > s\} \cap \Gamma_x) \\ &= \mu_x(\{f > t\} \cap \Gamma_x) \\ &= \mu_x(\{f > t\}^* \cap \Gamma_x). \end{aligned} \tag{3.50}$$

Moreover, by assumption, the function $r \mapsto \mu_x(B_r(x) \cap \Gamma_x)$ is strictly increasing for $r > 0$. Thus, if $z \in \{f > t\}^* \cap \Gamma_x = B_r(x) \cap \Gamma_x$, then for some $\bar{r} < r$ we have $z \in B_{\bar{r}}(x)$ and, by (3.50), there exists $s > 0$ such that $z \in \{f > s\}^*$. This completes the proof of (3.49).

Finally, by (3.49), $z \in \{f > t\}^*$ implies $z \in \{f > s\}^*$ for some $s > t$, that, by the definition of f^*, implies $f^*(z) \geq s > t$.

The statement (ii) follows from (i). Using the definition of rearrangement f^*, statements (iii) and (iv) are clear.

□

3.5 Disintegration of a measure

In this section we address the problem of the existence of a disintegration of a Borel measure μ along Γ, an isometry group of the metric space (X, d). We recall that μ is disintegrable along Γ if there are Borel measures μ_x on Γ_x, for all $x \in X/\Gamma$, and a Borel measure $\bar{\mu}$ on X/Γ such that (D1) and (D2) in Definition 3.4.23 are satisfied for any Borel set $E \subset X$. We observe that for (D1) to hold, it suffices that $y \mapsto \mu_y(A)$ is a Borel map for any open set $A \subset X$. We now state a general measure theoretic fact: the Monotone Class Theorem. For a proof we refer to [DM78, I-21]. We recall that, given a family $\mathcal{C}$ of real-valued functions defined on a set X, we denote with $\sigma(\mathcal{C})$ the smallest σ-field of subset of X with respect to which all of the functions in $\mathcal{C}$ are measurable.

Theorem 3.5.27 (Monotone Class Theorem)**.** *Let X be a set. Let $\mathcal{F}$ be a vector space of real valued and bounded functions on X, which contains the constants, is closed with respect to uniform convergence and such that for any increasing and uniformly bounded sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ of non-negative functions it holds that $f = \lim_{n\to+\infty} f_n \in \mathcal{F}$ (i.e. $\mathcal{F}$ is closed with respect to monotone convergence). Let $\mathcal{C} \subset \mathcal{F}$ be a vector space closed with respect to multiplication. Then $\mathcal{F}$ contains all the $\sigma(\mathcal{C})$-measurable functions.*

When $X/\Gamma = \{x\}$ consists of only one element, i.e. if Γ acts transitively on X, choosing $\mu_x = \mu$ and $\bar{\mu}$ to be the Dirac mass on X/Γ yields a trivial disintegration. In a general setting, the disintegration is provided by the following disintegration theorem for probability measures. The proof we present is essentially the one in [DM78, III.70-73], integrated using the one of [AFP00, Theorem 2.28]. We recall that a Borel measure μ on a topological space is *inner regular* if $\mu(B) = \sup\{\mu(K) : K \subset B \text{ compact}\}$ for any Borel set $B \subset X$.

Theorem 3.5.28. *Let $(X, d_X), (Y, d_Y)$ be separable metric spaces, let μ be an inner regular Borel probability measure on X and let $\pi : X \to Y$ be a Borel map. Then, letting $\bar{\mu} = \pi_\sharp\mu$, there exist Borel probability measures μ_y, $y \in Y$, supported in $\pi^{-1}(y)$ such that, for any Borel set $E \subset X$, the function $y \mapsto \mu_y(E)$ is a Borel map and*

$$\mu(E) = \int_Y \mu_y(E)\, d\bar{\mu}(y). \tag{3.51}$$

Proof. Assume for simplicity that X is compact. In this case we can drop the assumption for μ to be inner regular. For a proof of the general case we refer to [DM78, III.70-73]. We start by showing that we can associate to any $f \in C(X)$ a finite signed measure $\bar{\mu}[f] \ll \bar{\mu}$ on Y: simply let $\bar{\mu}[f] = \pi_\sharp(f\mu)$. In fact, for any Borel set $B \subset Y$,

$$\left|\bar{\mu}[f](B)\right| \le \int_{\pi^{-1}(B)} |f| d\mu \le \|f\|_\infty \mu(\pi^{-1}(B)) = \|f\|_\infty \bar{\mu}(B),$$

and so $\bar{\mu}[f] \ll \bar{\mu}$.

By the Radon-Nikodym differentiation theorem (see [Bar66, Theorem 8.9]), for any $f \in C(X)$ there exists a function $d_f \in L^\infty(Y, \bar{\mu})$ such that $\|d_f\|_\infty \le \|f\|_\infty$ and $\bar{\mu}[f] = d_f\bar{\mu}$. This construction is additive, i.e. for any $f, g \in C(X)$,

$$\bar{\mu}[f + g] = \bar{\mu}[f] + \bar{\mu}[g] = d_f\bar{\mu} + d_g\bar{\mu} = (d_f + d_g)\bar{\mu}.$$

Since X is separable, there exists a countable dense set $D \subset X$. Let $\mathcal{D} = \{g \in C(X) : g(x) \in \mathbb{Q} \text{ for any } x \in D\}$. It is clear that $\mathcal{D} \subset C(X)$ is a countable vector space over

$\mathbb{Q}$ which is closed with respect to the maximum and the minumum operations, dense in $C(X)$ and such that $1 \in \mathcal{D}$. Then we can find $N \subset Y$ such that $\bar{\mu}(N) = 0$, $d_1(y) = 1$ and $T_y : f \mapsto d_f(y)$ is a $\mathbb{Q}$-linear functional over $\mathcal{D}$ for any $y \in Y \setminus N$. By the inequality $|T_y(f)| = |d_f(y)| \leq \|f\|_\infty$ and applying Hahn-Banach Theorem, we can extended T_y to a continuous linear functional of norm 1 defined on the whole $C(X)$. Then, by the Riesz representation theorem, there exists a unique measure μ_y over X such that

$$T_y(f) = \int_X f\, d\mu_y, \quad f \in C(X).$$

We can extend the map $y \mapsto \mu_y$ to all of Y by setting μ_y to be any probability measure over X if $y \in N$. By construction and since $y \mapsto d_f(y)$ is Borel measurable by the Radon-Nikodym theorem, the function $y \mapsto \int_X f\, d\mu_y$ is Borel measurable for any $f \in \mathcal{D}$. By uniform convergence, the same property is still true if $f \in C(X)$. Since $C(X)$ is closed with respect to multiplication, by the Monotone Class Theorem 3.5.27 the vector space $\mathcal{F} = \{f : X \to \mathbb{R} :\ f \text{ is bounded and } y \mapsto \int_X f\, d\mu_y \text{ is Borel measurable}\}$ contains all the Borel functions. This proves that $y \mapsto \int_X f\, d\mu_y$ is Borel measurable if f is a bounded Borel measurable function, in particular if f is the characteristic function of an open set $A \subset X$.

Now we claim that (3.51) holds. In fact, for any Borel set $B \subset Y$ and $f \in \mathcal{D}$, it holds

$$\int_{\pi^{-1}(B)} f\, d\mu = (f\mu)(\pi^{-1}(B)) = \bar{\mu}[f](B) = \int_B d_f(y)\, d\bar{\mu}(y) = \int_B \left(\int_X f\, d\mu_y \right) d\bar{\mu}(y), \tag{3.52}$$

where the last equality is justified since for $\bar{\mu}$-a.e. $y \in Y$, $d_f(y) = \int_X f\, d\mu_y$. By approximation and Theorem 3.5.27 again, identity (3.52) is true for f a bounded Borel function and hence if $f = \chi_A$ with $A \subset X$ open. If $B = Y$, this proves (3.51).

Finally we prove that μ_y is supported in $\pi^{-1}(y)$ for $\bar{\mu}$-a.e. $y \in Y$. Let $G = X \times Y$, endowed with the product metric, and let $\psi : X \to G$ be the mapping $\psi(x) = (x, \pi(x))$. We claim that for any Borel set $A \subset G$ it holds

$$\psi_\sharp \mu(A) = \int_Y (\mu_y \otimes \delta_y)(A)\, d\bar{\mu}(y), \tag{3.53}$$

where δ_y is the Dirac measure concentrated in y, $\delta_y(B) = \chi_B(y)$. In fact, if $E \times B \subset X \times Y$ is a Borel rectangle of G, it holds that

$$\psi_\sharp \mu(E \times B) = \mu\big(\psi^{-1}(E \times B)\big) = \mu\big(E \cap \pi^{-1}(B)\big).$$

Then, by (3.52), we get

$$\begin{aligned} \int_Y (\mu_y \otimes \delta_y)(E \times B)\, d\bar{\mu}(y) &= \int_B \mu_y(E)\, d\bar{\mu}(y) = \int_B \left(\int_X \chi_E\, d\mu_y \right) d\bar{\mu}(y) \\ &= \int_{\pi^{-1}(B)} \chi_E\, d\mu = \mu(E \cap \pi^{-1}(B)) = \psi_\sharp \mu(E \times B). \end{aligned}$$

Since the rectangles are a basis for the Borel subsets of G, this proves the claim.

Let now $E \subset X$ be a Borel set, we claim that $\psi_\sharp\mu(E \times \pi(E)) = \mu(E)$. In fact, since $E \subset \pi^{-1}(\pi(E))$, we have that $\psi^{-1}(E \times \pi(E)) = E \cap \pi^{-1}(\pi(E)) = E$. Therefore $\psi_\sharp\mu(E \times \pi(E)) = \mu(\psi^{-1}(E \times \pi(E))) = \mu(E)$. By (3.53) and (3.51), the previous claim implies that

$$\int_Y (\mu_y \otimes \delta_y)(E \times \pi(E))\, d\bar{\mu}(y) = \int_Y \mu_y(E)\, d\bar{\mu}(y).$$

This yields

$$\int_Y \mu_y(E)\chi_{\pi(E)}(y)\, d\bar{\mu}(y) = \int_Y \mu_y(E)\, d\bar{\mu}(y).$$

Hence, for $\bar{\mu}$-a.e. $y \in Y$ and for any Borel set $E \subset X$, we have that $\mu_y(E) \neq 0$ only if $y \in \pi(E)$, or equivalently if $E \cap \pi^{-1}(y) \neq \varnothing$. This implies that μ_y is supported in $\pi^{-1}(y)$ for $\bar{\mu}$-a.e. $y \in Y$, completing the proof of the Theorem. □

In the following section we assume the measure μ to be invariant with respect to some 1-parameter group of isometries of the space X. In this case we can relax some of the assumptions of Theorem 3.5.28. We recall that a topological space X is said to be σ-compact if there exists a countable covering of X consisting of compact sets.

Proposition 3.5.29. *Let (X, d) be a σ-compact metric space. Let $\mathcal{P} = \{H^-, H, H^+\}$ be a partition of X, and let $T = \{\tau_t\}_{t\in\mathbb{R}}$ be a 1-parameter group of isometries such that:*

(i) *the projection $\pi : X \to X/T$ is continuous;*

(ii) *for any $t \in \mathbb{R}$ the map $(x, t) \mapsto \tau_t(x)$ is continuous from $H \times \mathbb{R}$ to X;*

(iii) *$H^- = \bigcup_{t<0} \tau_t(H)$ and $H^+ = \bigcup_{t>0} \tau_t(H)$, with disjoint union.*

Moreover, let μ be a T-invariant, locally finite and inner regular Borel measure on X. Then the measure μ is disintegrable along T in measures μ_x on T_x, for all $x \in X/T$, and $\bar{\mu}$ on X/T, where μ_x, $x \in X/T$, is locally finite and $\bar{\mu}$-a.e. non-atomic. Here $T_x = \{\tau_t(x) : t \in \mathbb{R}\}$ is the orbit of $x \in X$ under the action of the group T.

Proof. Under the previous assumptions, $H = X/T$ and H is σ-compact. Without loss of generality we can assume H to be compact. In fact, if $\{K_n\}_{n\in\mathbb{N}}$ is a covering of H consisting of compact sets, we have that

$$X = \bigcup_{n\in\mathbb{N}} \left(\bigcup_{t\in\mathbb{R}} \tau_t(K_n) \right),$$

and hence it suffices to disintegrate the measure μ on each set $\bigcup_{t\in\mathbb{R}} \tau_t(K_n)$.

We claim that there exists $R > 0$ such that if $t_1, t_2 \in \mathbb{R}$ and $0 < t_2 - t_1 < R$, then

$$\mu\left(\bigcup_{t\in[t_1,t_2)} \tau_t(H) \right) < \infty. \tag{3.54}$$

In fact, by the local finiteness of μ, for any $x \in H$ there exists $r(x) > 0$ such that $\mu(B_{r(x)}(x)) < \infty$. Let now $\{x_1, \ldots, x_m\}$ be a finite set of points in H such that $H \subset C = B_{r(x_1)}(x_1) \cup \ldots \cup B_{r(x_m)}(x_m)$. This set exists by the compactness of H. The map $x \mapsto \sup\{t > 0 : \tau_s(x) \in C \text{ for all } |s| < t\}$ is continuous from H to $(0, +\infty)$, since $(x, t) \mapsto \tau_t(x)$ is continuous from $H \times \mathbb{R}$ to X and C is open. By the Weierstrass theorem, it attains a minimum $R > 0$ on H. Therefore, for any $s < R$, the monotonicity of the measure implies

$$\mu\left(\bigcup_{t\in[0,s)} \tau_t(H)\right) \leq \mu(C) \leq \sum_{j=1}^{m} \mu\big(B_{r(x_j)}(x_j)\big) < +\infty.$$

The claim (3.54) follows by the T-invariance of μ.

For any $k \in \mathbb{Z}$, let

$$X_k = \bigcup_{[t_k,t_{k+1})} \tau_t(H).$$

Here $t_k = \frac{R}{2}k$. The Borel sets $\{X_k\}_{k\in\mathbb{Z}}$ are bounded, since $(x,t) \mapsto \tau_t(x)$ is continuous from $H \times \mathbb{R}$ to X, and they form a partition of X. The measure $\mu_k = \mu \llcorner X_k$ is then finite by (3.54) and moreover the measure $\bar{\mu} = \pi_\sharp \mu_k$ is independent of k, because μ is T-invariant. By Theorem 3.5.28 there are probability measures μ_x^k, $x \in X/T$, supported in $T_x \cap X_k$ such that

$$\mu(E) = \int_{X/T} \mu_y^k(E \cap T_x)\, d\bar{\mu}(x),$$

for any Borel set $E \subset X_k$. Here T_x is the orbit of a point $x \in X/T$ under the action of T, i.e. $T_x = \{\tau_t(x) : t \in \mathbb{R}\}$. Letting $\mu_x = \sum_{k\in\mathbb{Z}} \mu_x^k$ we obtain a disintegration of μ along T. The measure μ_x are then locally finite, by definition.

Finally we prove that μ_x is non-atomic $\bar{\mu}$-a.e. . Let $E \subset H$ be a Borel set and, for $-\infty < r < s < +\infty$, let

$$E_{r,s} = \bigcup_{t\in(r,s)} \tau_t(E). \tag{3.55}$$

Since μ is T-invariant we have $\mu(E_{r,s}) = \mu(E_{r+t,s+t})$ for any $t \in \mathbb{R}$. The disintegration formula implies that

$$\int_E \mu_x(E_{r,s})\, d\bar{\mu}(x) = \int_E \mu_x(E_{r+t,s+t})\, d\bar{\mu}(x).$$

By the arbitrariness of E it follows that, for fixed r, s, t, there holds

$$\mu_x(E_{r,s}) = \mu_x(E_{r+t,s+t}) \tag{3.56}$$

for $\bar{\mu}$-a.e. $x \in H$. Finally, this implies that there exists $N \subset H$ with $\bar{\mu}(N) = 0$ such that (3.56) holds for any $x \in H \setminus N$ and for all $r, s, t \in \mathbb{Q}$ with $r < s$. We claim that this implies that μ_x is non-atomic for all $x \in H \setminus N$, i.e. for any $z \in T_x$ there holds $\mu_x\{z\} = 0$. In fact, if $\mu_x\{z\} = \delta > 0$ for some $z \in T_x$ then by (3.56) this holds for all $z \in T_x$ and the measure μ_x is not locally finite. □

3.6 Steiner and Schwarz rearrangements

Let $\mathcal{R} = \{\mathcal{P}, \varrho\}$ be a reflection system of X with $\mathcal{P} = \{H^-, H, H^+\}$. Let T be a 1-parameter group of isometries endowed with the natural topology. Finally, let $\pi : X \to X/T$ be the natural projection.

Definition 3.6.30. We say that $(\mathcal{R}, T)$ is a *Steiner system* of the metric space (X, d) if we have:

(St1) $X/T \subset H$ and $\pi : X \to X/T$ is continuous;

(St2) $\tau^{-1}x = \varrho\tau x$ for any $x \in X/T$ and $\tau \in T$;

(St3) $(x, \tau) \mapsto \tau x$ is continuous and proper from $X/T \times T$ in X.

By $X/T \subset H$ we mean that any equivalence class of X/T is determined by one single element of H.

In order to rearrange functions and sets, we need a rearrangement system associated with the family T. In general the existence of such a system is a separate assumption. However, we have the following proposition.

Proposition 3.6.31. *Let (X, d) be a σ-compact metric space. Let $(\mathcal{R}, T)$ be a Steiner system of (X, d), with $\mathcal{R} = \{\{H^-, H, H^+\}, \varrho\}$, and $T = \{\tau_t\}_{t\in\mathbb{R}}$ such that*

$$H^+ = \bigcup_{t>0} \tau_t(H), \text{ with disjoint union.}$$

Finally let μ be a T-invariant, locally finite, inner regular and non-degenerate Borel measure on X. Then we have that:

(i) $H = X/T$;

(ii) *the measure μ is disintegrable along T in the Borel measures $(\mu_x)_{x\in X/T}$ and $\bar{\mu}$, as per Definition 3.4.23;*

(iii) *μ_x is locally finite for any $x \in X/T$ and non-atomic for $\bar{\mu}$-a.e. $x \in X/T$;*

Moreover, if for $\bar{\mu}$-a.e. $x \in X/T$ the orbit T_x intersects the spheres $\partial B_s(x)$, $s > 0$, in isolated points, $(T, (\mu_x)_{x\in X/T}, \bar{\mu})$ is a rearrangement system of (X, d, μ), in the sense of Definition 3.4.25.

Proof. The fact that $H = X/T$ is clear, while statement (ii) and (iii) follow from Proposition 3.5.29. To complete the proof it suffices to show that, if for $\bar{\mu}$-a.e. $x \in X/T$ the orbit T_x intersects the spheres $\partial B_s(x)$, $s > 0$, in isolated points, then, for $\bar{\mu}$-a.e. $x \in H$, the function $s \mapsto \mu_x(B_s(x) \cap T_x)$ is strictly increasing and continuous for $s \geq 0$. This will prove that $(T, (\mu_x)_{x\in X/T}, \bar{\mu})$ is a rearrangement system of (X, d, μ).

We claim that for $\bar{\mu}$-a.e. $x \in X/T$ the function $s \mapsto \mu_x\left(\bigcup_{t\in(0,s)} \tau_t(H)\right)$ is either identically zero or continuous and strictly increasing. We let $E \subset H$ to be a Borel set and, for $r, s \in \mathbb{R}$, $r < s$, we define $E_{r,s}$ as in (3.55). Then, by identity (3.56), it follows that if $\mu_x(E_{r,s}) = 0$ for some $x \in E$ and $r < s$, then $\mu_x \equiv 0$. This and the fact that for $\bar{\mu}$-a.e. $x \in X/T$ the measures μ_x are non-atomic, proves the claim.

By the non-degeneracy of the measure μ, the previous claim implies that the function $s \mapsto \mu_x(B_s(x) \cap T_x)$ is strictly increasing. Moreover we have that $\mu_x(\partial B_s(x) \cap T_x) = 0$, for any $s > 0$ and for $\bar{\mu}$-a.e. $x \in X/T$. This follows by the fact that for $\bar{\mu}$-a.e. $x \in X/T$ the orbit T_x intersects the spheres $\partial B_s(x)$, $s > 0$, in isolated points and the measures μ_x are non-atomic. This proves that the function $s \mapsto \mu_x(B_s(x) \cap T_x)$ is continuous for $s \geq 0$, thus completing the proof. □

The Steiner system is enough for many applications, as the one seen in Section 3.4.1. However, by enriching the family of isometries acting on X we can obtain a more refined and general result. Namely, let G be a compact group of isometries acting on X and let $\Gamma = \Gamma(T, G)$ be the group generated by T and G. With abuse of notation, let $\pi : X \to X/\Gamma$ be the natural projection.

Definition 3.6.32. We say that $(\mathcal{R}, T, G)$ is a *Schwarz system* of the metric space (X, d) if we have:

(Sc1) $X/T \subset H$ (and thus $X/\Gamma \subset H$) and $\pi : X \to X/\Gamma$ is continuous;

(Sc2) $\tau^{-1}x = \varrho\tau x$ for any $x \in X/T$ and $\tau \in T$;

(Sc3) $\Gamma_x = \{\gamma\tau x :\ \gamma \in G, \tau \in T\}$ and $\gamma x = x$ for any $\gamma \in G$ and $x \in X/\Gamma$;

(Sc4) $(x, \gamma, \tau) \mapsto \gamma\tau x$ is continuous and proper from $X/\Gamma \times \Gamma \times T$ in X.

It is clear that when G consists only of the identity, condition (Sc3) automatically holds true, thus reducing the Schwarz system to a Steiner system. Therefore, from now on, we will always refer to the former.

For a Schwarz system we do not have a result analogous to Proposition 3.6.31. However the following Proposition will be enough for our purposes. In fact, in Theorem 3.6.37 the functions are supposed to have compact support. Then we could localize the rearrangement in some compact set and restrict the measure to this set.

Proposition 3.6.33. *Let (X, d) be a compact metric space. Let $(\mathcal{R}, T, G)$ be a Schwarz system of (X, d). Then any finite, inner regular Borel measure μ is disintegrable along $\Gamma = \Gamma(T, G)$ in the Borel measures $(\mu_x)_{x \in X/\Gamma}$ and $\bar{\mu}$, as in Definition 3.4.23. Moreover if for $\bar{\mu}$-a.e. $x \in X/\Gamma$ the function $s \mapsto \mu_x(B_s(x) \cap \Gamma_x)$ is strictly increasing and continuous for $s \geq 0$, then $(T, (\mu_x)_{x \in X/\Gamma}, \bar{\mu})$ is a rearrangement system of X.*

Proof. It suffices to prove that the measure μ is disintegrable along Γ. This follows from Theorem 3.5.28 with $Y = X/\Gamma = \pi(X)$. In fact Y is compact, and hence separable, due to the continuity of π. □

Condition (Sc3) in the definition of Schwarz system, is used only to prove the following Lemma. Indeed, one could replace (Sc3) with the thesis of this Lemma, which is more general.

Lemma 3.6.34. *Let $(\mathcal{R}, T, G)$ be a Schwarz system of the metric space (X, d). Then for any $x \in X/\Gamma$ and for any $z_+, z_- \in \Gamma_x$ there exists a reflection system $\bar{\mathcal{R}} = \{\bar{\mathcal{P}}, \bar{\varrho}\}$ such that $\bar{\varrho} z_+ = z_-$,*

Proof. By (Sc3) there exist $\gamma_+, \gamma_- \in G$ and $\tau_+, \tau_- \in T$ such that $z_+ = \gamma_+ \tau_+ x$ and $z_- = \gamma_- \tau_- x$. Since $\gamma_+ \tau_+ \gamma_- \tau_- x \in \Gamma_x$, there exist $\gamma \in G$ and $\tau \in T$ such that

$$\gamma \tau x = \gamma_+ \tau_+ \gamma_- \tau_- x. \tag{3.57}$$

Let $\sqrt{\tau} \in T$ be such that $\tau = \sqrt{\tau}\sqrt{\tau}$. Such a $\sqrt{\tau}$ exists because T is a 1-parameter group. Let us define $\iota = \gamma_- \tau_- \gamma \sqrt{\tau} \in \Gamma$, and let

$$\bar{H}^- = \iota(H^-), \quad \bar{H} = \iota(H), \quad \bar{H}^+ = \iota(H^+), \quad \bar{\varrho} = \iota \varrho \iota^{-1}.$$

We claim that $\bar{\varrho} z_+ = z_-$. In fact, by (3.57), (Sc2) and the second part of (Sc3), we have

$$\bar{\varrho} z_+ = \gamma_- \tau_- \gamma \sqrt{\tau} \varrho \sqrt{\tau} x = \gamma_- \tau_- \gamma \sqrt{\tau} \sqrt{\tau}^{-1} x = \gamma_- \tau_- \gamma x = \gamma_- \tau_- x = z_-.$$

Finally we claim that, letting $\bar{\mathcal{P}} = \{\bar{H}^-, \bar{H}, \bar{H}^+\}$, $\bar{\mathcal{R}} = \{\bar{\mathcal{P}}, \bar{\varrho}\}$ is a reflection system of (X, d). By the definition of $\bar{\varrho}$ it is clear that $\bar{\varrho}^2 = \mathrm{Id}$ and that $\bar{\varrho}\bar{H}^+ = \bar{H}^-$. Moreover, for $x, y \in \bar{H}^+$, we have

$$d(x, \bar{\varrho} y) = d(\iota^{-1} x, \varrho \iota^{-1} y) \geq d(\iota^{-1} x, \iota^{-1} y) = d(x, y).$$

Here we used the fact that Γ is a group of isometries. Thus the axioms (R1) and (R2) of Definition 3.3.6, are satisfied and the claim is proved. □

Now we give a criterion for condition (Sc3).

Proposition 3.6.35. *Let T, G be two groups of isometries of the metric space (X, d) and let $\Gamma = \Gamma(T, G)$ be the group generated by them. If we have that:*

$$\text{(i)} \quad \gamma x = x \textit{ for all } x \in X/\Gamma \textit{ and } \gamma \in \Gamma, \tag{3.58}$$

$$\text{(ii)} \quad \textit{it holds } TGT \subset GTG; \tag{3.59}$$

then for any $x \in X/\Gamma$ and $y \in \Gamma_x$ there exist $\gamma \in G$ and $\tau \in T$ such that $y = \gamma \tau x$.

Proof. Let $x \in X/\Gamma$ and $y \in \Gamma_x$. Then there exists $\xi \in \Gamma$ such that $y = \xi x$. By (3.59) we have that $\xi = \gamma \tau \gamma'$ for some $\gamma, \gamma' \in G$ and $\tau \in T$. To complete the proof it suffices to observe that by (3.58) we have $\gamma' x = x$, and thus $y = \gamma \tau x$. □

Using Proposition 3.6.35 we present an example of Schwarz system: a generalization of the Euclidean Steiner rearrangement presented in Section 3.4.1.

Example 3.6.36. Let us factorize $\mathbb{R}^d = \mathbb{R}^m \times \mathbb{R}^{d-m}$ for some $1 \leq m \leq d$. If $m = d$ we agree to set $\mathbb{R}^{d-m} = \{0\}$. Let $G = O(m) \subset O(d)$ be the group of orthogonal transformation of $\mathbb{R}^d$ fixing the $\mathbb{R}^{d-m}$ factor. Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$ such that

$\|\gamma(x)\| = \|x\|$ for any $x \in \mathbb{R}^d$ and $\gamma \in G$. We endow $\mathbb{R}^d$ with the metric $d_{\|\cdot\|}$ induced by this norm.

Let $v \in \mathbb{R}^m$, $v \neq 0$. With abuse of notation we identify v and $(v, 0) \in \mathbb{R}^m \times \mathbb{R}^{d-m}$. Let $H = \pi_v$ be the hyperplane orthogonal to v. We have a natural partition $\mathcal{P} = \{H^-, H, H^+\}$ of $\mathbb{R}^d$ and a natural reflection ϱ with respect to H. As noted in Example 3.3.8, $\mathcal{R} = \{\mathcal{P}, \varrho\}$ is a reflection system.

Let $T = (\tau_t)_{t\in\mathbb{R}}$ be the 1-parameter group of the isometries $\tau_t : \mathbb{R}^d \mapsto \mathbb{R}^d$, $\tau_t(x) = x + tv$. Finally let $\Gamma = \Gamma(T, G)$ be the group generated by T and G. We have $X/T = H$ and $X/\Gamma = \mathbb{R}^m$.

We show that condition (3.59) holds: for any $\gamma \in G$ and $s, t \in \mathbb{R}$, there exist $\xi, \vartheta \in G$ and $r \in \mathbb{R}$ such that $\tau_s\gamma\tau_t = \xi\tau_r\vartheta$. In fact, we have $\tau_s\gamma\tau_t x = \gamma x + t\gamma v + sv$ and $\xi\tau_r\vartheta x = \xi\vartheta x + r\xi v$, for any $x \in \mathbb{R}^d$. Thus we have to solve the system

$$\begin{cases} \xi\vartheta = \gamma, \\ r\xi v = t\gamma v + sv. \end{cases}$$

From the second equation we determine r up to the sign, $|r| = \|t\gamma v + sv\|/\|v\|$. If $r = 0$ we are finished. If $r \neq 0$, we choose $\xi \in O(m)$ such that

$$\xi v = \frac{t}{r}\gamma v + \frac{s}{r}v.$$

Such a ξ does exists, because $v = (v, 0) \in \mathbb{R}^m \times \mathbb{R}^{d-m}$ and the same holds for γv. Finally we determine ϑ by the first equation, $\vartheta = \xi^{-1}\gamma$. This proves that (3.59) holds, thus proving that condition (Sc3) holds, since (3.58) is trivially satisfied.

Therefore $(\mathcal{R}, T, G)$ is a Schwarz system of $(\mathbb{R}^d, d_{\|\cdot\|})$, since it is clear that conditions (Sc1), (Sc2) and (Sc4) holds.

Now we present the main result on the Schwarz rearrangement of functions. We say that a Borel measure μ is *invariant with respect to the Schwarz system* $(\mathcal{R}, T, G)$, $\mathcal{R} = \{\{H^-, H, H^+\}, \varrho\}$, if $\mu(H) = 0$, μ is ϱ-invariant and μ is $\Gamma(T, G)$-invariant, i.e. $\gamma_\sharp\mu = \mu$ for any $\gamma \in \Gamma(T, G)$.

Theorem 3.6.37. *Let (X, d) be a proper metric space endowed with a Schwarz system $(\mathcal{R}, T, G)$. Let μ be a non-degenerate and diffuse Borel measure, in the sense of* (3.4) *and* (3.5)*, that is invariant with respect to the Schwarz system $(\mathcal{R}, T, G)$ and let $(\Gamma, (\mu_x)_{x\in X/\Gamma}, \bar{\mu})$ be a regular rearrangement system of (X, μ), where $\Gamma = \Gamma(T, G)$. Finally, let the metric measure space (X, d, μ) have the Lebesgue property* (3.6)*. Then the rearrangement f^* of any compactly supported and non-negative $f \in L^p(X, \mu)$, $1 < p < \infty$, satisfies*

$$\|f^*\|_{L^p(X,\mu)} = \|f\|_{L^p(X,\mu)} \quad \textit{and} \quad \|\nabla f^*\|^-_{L^p(X,\mu)} \leq \|\nabla f\|^-_{L^p(X,\mu)}. \tag{3.60}$$

Proof. The identity $\|f^*\|_{L^p(X,\mu)} = \|f\|_{L^p(X,\mu)}$ follows from identity (3.44) and statement (ii) in Lemma 3.4.26. We assume $\|\nabla f\|^-_{L^p(X,\mu)} < +\infty$. In fact, if $\|\nabla f\|^-_{L^p(X,\mu)} = +\infty$ the inequality in (3.60) is trivial and we are finished.

Now we claim that there exists a compact set $K \subset X$, such that $(\operatorname{supp} f) \cup (\operatorname{supp} f^*) \subset K$. In fact, by assumption (Sc4), the action $\alpha : X/\Gamma \times \Gamma \times T \to X$, $\alpha(x, \gamma, \tau) = \gamma\tau x$ is proper, and thus $\alpha^{-1}(\operatorname{supp} f) \subset X/\Gamma \times \Gamma \times T$ is compact in the product topology. It follows that there exists a compact set $T_0 \subset T$ such that, letting

$$K = \{\gamma\tau x : \ \gamma \in G, \tau \in T_0, x \in \pi(\operatorname{supp} f)\},$$

we have $\operatorname{supp} f \subset K$. The set K is compact because $K = \alpha\big(\pi(\operatorname{supp} f) \times G \times T_0\big)$, and hence it is the continuous image of a compact set. Here, we used the fact that G is compact. Possibly enlarging T_0, we may assume it to be symmetric (i.e. $\tau \in T_0$ if and only if $\tau^{-1} \in T_0$), connected and such that $\mathrm{Id} \in T_0$. Then we also have $\operatorname{supp} f^* \subset K$, proving the claim. By (Sc2) we may also assume that $K = \varrho K$.

Let us recall the notation introduced in (3.26), Section 3.3:

$$Q_{r,p}(f) = \int_X \fint_{B_r(x)} |f(x) - f(y)|^p \, d\mu(y) \, d\mu(x).$$

Let $\mathcal{A}_f$ be the family of all non-negative functions $g \in L^p(X, \mu)$ such that:

(A1) $\mu_x\{g > t\}_x = \mu_x\{f > t\}_x$ for $\bar{\mu}$-a.e. $x \in X/\Gamma$ and for all $t > 0$;

(A2) $g(x) = 0$ for all $x \in X \setminus K$;

(A3) $Q_{r,p}(g) \le Q_{r,p}(f)$ for all $r \in (0, 1)$.

The set $\mathcal{A}_f$ is non-empty, since $f \in \mathcal{A}_f$. Now we show that $\mathcal{A}_f$ is compact in $L^p(X, \mu)$. To this aim we apply Theorem 3.2.5 to $\mathcal{A}_f$. Here is where the properness of X is required.

By identity (3.44) and (A1), exploiting assumption (D2) on the disintegration of μ along Γ, we have, for any $g \in \mathcal{A}_f$,

$$\begin{aligned} \int_X g^p \, d\mu &= \int_0^\infty \mu\{g > t^{1/p}\} \, dt \\ &= \int_0^\infty \int_{X/\Gamma} \mu_x\{g > t^{1/p}\}_x \, d\mu(x) \, dt \\ &= \int_0^\infty \int_{X/\Gamma} \mu_x\{f > t^{1/p}\}_x \, d\mu(x) \, dt \\ &= \int_0^\infty \mu\{f > t^{1/p}\} \, dt \\ &= \int_X f^p \, d\mu. \end{aligned}$$

Thus, $\mathcal{A}_f$ is uniformly bounded in $L^p(X, \mu)$. The uniform bound (3.19) holds by (A3). By Theorem 3.2.5, $\mathcal{A}_f$ is then precompact in $L^p(X, \mu)$. Finally we prove that $\mathcal{A}_f$ is also closed in $L^p(X, \mu)$ and thus compact. Let $(g_j)_{j \in \mathbb{N}} \subset \mathcal{A}_f$ be a sequence such that $g_j \longrightarrow g$ in $L^p(X, \mu)$ and μ-almost everywhere. Then $g \in \mathcal{A}_f$.

In fact, by the Fatou Lemma, g satisfies (A2) and (A3). We check (A1). We first claim that for a function $g \in L^p(X,\mu)$, the set $\mathcal{I} = \{t > 0 : \mu\{g = t\} > 0\}$ is at most countable. In fact we have

$$\mathcal{I} = \bigcup_{k=0}^{\infty} \mathcal{I}_k, \qquad \text{where } \mathcal{I}_k = \left\{ t > 0 : \frac{1}{2^k} \leq \mu\{g = t\} < \frac{1}{2^{k-1}} \right\}.$$

For any $k \in \mathbb{N}$, it holds

$$\mu\left\{ g > \frac{1}{2^{k-1}} \right\} \geq \sum_{t \in \mathcal{I}_k} \mu\{g = t\} \geq \#(\mathcal{I}_k)\frac{1}{2^k}. \tag{3.61}$$

This implies that $\mathcal{I}_k$ is of finite cardinality for any $k \in \mathbb{N}$, indeed $L^p(X,\mu) \subset \mathcal{S}(X,\mu)$ and hence $\mu\{g > t\} < \infty$ for all $t > 0$. Since $\mathcal{I}$ is a countable union of finite sets, the claim is proved.

Using the previous claim, for $\bar{\mu}$-a.e. $x \in X/\Gamma$ and for $\mathcal{L}^1$-a.e. $t > 0$ we have

$$\begin{aligned} &\lim_{j\to\infty} \mu_x\big(\{g > t\}_x \cap \{g_j \leq t\}_x\big) = 0, \\ &\lim_{j\to\infty} \mu_x\big(\{g \leq t\}_x \cap \{g_j > t\}_x\big) = \lim_{j\to\infty} \mu_x\big(\{g < t\}_x \cap \{g_j > t\}_x\big) = 0. \end{aligned} \tag{3.62}$$

This implies $\mu_x\big(\{g_j > t\}_x \Delta \{g > t\}_x\big) \longrightarrow 0$ as $j \to \infty$, and (A1) follows for $\mathcal{L}^1$-a.e. $t > 0$. By right continuity (A1) follows for all $t > 0$.

The functional $J : \mathcal{A}_f \to [0, +\infty)$, defined as

$$J(g) = \int_X |g - f^*|^p \, d\mu,$$

is continuous in $L^p(X,\mu)$. In fact $J(g) = \|g - f^*\|_{L^p(X,\mu)}$. Since $\mathcal{A}_f$ is compact in $L^p(X,\mu)$, the Weierstrass theorem guarantees the existence of $\bar{f} \in \mathcal{A}_f$ such that

$$J(\bar{f}) = \min_{f \in \mathcal{A}_f} J(f). \tag{3.63}$$

If $J(\bar{f}) = 0$ we are finished. In fact, this imply that $\bar{f} = f^*$ μ-a.e. and hence, by (A3), that

$$\int_X \fint_{B_r(x)} |f^*(x) - f^*(y)|^p \, d\mu(y)\, d\mu(x) \leq \int_X \fint_{B_r(x)} |f(x) - f(y)|^p \, d\mu(y)\, d\mu(x),$$

for any $r \in (0,1)$. Dividing this inequality by r^p and taking the lim inf as $r \downarrow 0$ we get $\|\nabla f^*\|_{L^p(X,\mu)} \leq \|\nabla f\|_{L^p(X,\mu)}$.

Now we show that the case $J(\bar{f}) > 0$ cannot occur by contradicting the minimality of $\bar{f}$. If $J(\bar{f}) > 0$, by the representation formula (3.42) and the disintegration of μ along

Γ, we have

$$
\begin{aligned}
0 &< \left[\int_X |\bar f(x) - f^*(x)|^p \, d\mu(x)\right]^{1/p} \\
&= \left[\int_X \left|\int_0^{+\infty} \big(\chi_{\{\bar f > t\}}(x) - \chi_{\{f^* > t\}}(x)\big)\, dt\right|^p d\mu(x)\right]^{1/p} \\
&\leq \int_0^{+\infty} \left[\int_X \left|\chi_{\{\bar f > t\}}(x) - \chi_{\{f^* > t\}}(x)\right|^p d\mu(x)\right]^{1/p} dt \\
&= \int_0^{+\infty} \mu\big(\{\bar f > t\}\Delta\{f^* > t\}\big)^{1/p}\, dt.
\end{aligned}
$$

Therefore there exists $t > 0$ such that, letting $A = \{\bar f > t\}$ and $B = \{f^* > t\}$, it holds $\mu(A \,\Delta\, B) > 0$. This implies that $\mu(A \setminus B) = \mu(B \setminus A) > 0$. In fact by (A1) follows that both f and f^* are rearrangements of f, and thus $\mu(A) = \mu(B)$. By the Lebesgue property (3.6), μ-a.e. $z \in A \setminus B$ is a point of density of $A \setminus B$, i.e.

$$\lim_{r \downarrow 0} \frac{\mu\big((A \setminus B) \cap B_r(z)\big)}{\mu(B_r(x))} = 1.$$

For any Borel set $E \subset X$, let

$$\Lambda_E = \{x \in X/\Gamma : \text{ there exists } z \in \Gamma_x \text{ point of density of } E\}.$$

We claim that $\bar\mu(\Lambda_{A\setminus B} \cap \Lambda_{B\setminus A}) > 0$. Since both $\bar f$ and f^* satisfies (A1), we have that $\mu_x(A_x \setminus B_x) = \mu_x(B_x \setminus A_x)$ for $\bar\mu$-a.e. $x \in X/\Gamma$. Therefore

$$\int_{\Lambda_{A\setminus B}} \mu_x(B_x \setminus A_x)\, d\bar\mu(x) = \int_{\Lambda_{A\setminus B}} \mu_x(A_x \setminus B_x)\, d\bar\mu(x) = \mu(A \setminus B) > 0,$$

and thus there exists a set $\Lambda \subset \Lambda_{A\setminus B}$ such that $\bar\mu(\Lambda) > 0$ and $\mu_x(B_x \setminus A_x) > 0$ for any $x \in \Lambda$. This implies that $\Lambda \subset \Lambda_{B\setminus A}$ and proves the claim.

By the previous claim, there exists $x \in X/\Gamma$ and $z_-, z_+ \in \Gamma_x$ such that z_- is a point of density for $A \setminus B$ and z_+ is a point of density for $B \setminus A$. Let $\bar R = \{\bar{\mathcal P}, \bar\varrho\}$, $\bar{\mathcal P} = \{\bar H^-, \bar H, \bar H^+\}$, be the reflection system given by Lemma 3.6.34 and let $\eta > 0$ be such that

$$\mu\big(B_\eta(z_-) \cap (A \setminus B)\big) \geq \frac{1}{2}\mu(B_\eta(z_-)) \quad \text{and} \quad \mu\big(B_\eta(z_+) \cap (B \setminus A)\big) \geq \frac{1}{2}\mu(B_\eta(z_+)). \tag{3.64}$$

Possibly choosing a smaller η we may also assume that $B_\eta(z_-) \subset \bar H^-$ and $B_\eta(z_+) \subset \bar H^+$.

From (3.64) we deduce that $\mu\big(\bar H^+ \cap (B \setminus A \cap \bar\varrho(A \setminus B))\big) > 0$. In view of

$$(B \setminus A) \,\cap\, \bar\varrho(A \setminus B) = (B \setminus \bar\varrho B) \,\cap\, (\bar\varrho A \setminus B),$$

we eventually obtain

$$\mu\{x \in \bar H^+ :\ \bar f(x) > \bar f(\bar\varrho x) \text{ and } f^*(x) < f^*(\bar\varrho x)\} > 0. \tag{3.65}$$

This is assumption (3.41) in Theorem 3.3.20. As $\phi(t) = t^p$, $1 < p < \infty$, is strictly convex, by the statement concerning the strict inequality in Theorem 3.3.20 we have

$$\int_X |\bar{f}_{\bar{\mathcal{R}}} - f^*|^p \, d\mu = \int_X |\bar{f}_{\bar{\mathcal{R}}} - f^*_{\bar{\mathcal{R}}}|^p \, d\mu < \int_X |\bar{f} - f|^p \, d\mu. \tag{3.66}$$

Here we used the fact that $f^*_{\bar{\mathcal{R}}} = f^*$ by statement (iii) and (iv) in Lemma 3.4.26.

We claim that $\bar{f}_{\bar{\mathcal{R}}} \in \mathcal{A}_f$. Since inequality (3.66) can be rewritten as $J(\bar{f}_{\bar{\mathcal{R}}}) < J(\bar{f})$, this contradicts the minimality of $\bar{f}$, thus completing the proof of the Theorem. To prove the claim, observe that the function $\bar{f}_{\bar{\mathcal{R}}}$ is supported in K, hence satisfying (A2). This follows from the fact that if $\bar{f}(z) > 0$ for some $z \in \bar{H}^-$ then $\varrho z \in K$. The function $\bar{f}_{\bar{\mathcal{R}}}$ satisfies (A1), since μ_x is ϱ and T-invariant. This follows from the fact that for any Borel set $E \subset X$ and $\tau \in T$, it holds

$$\int_{X/\Gamma} \mu_x(E_x) \, d\bar{\mu}(x) = \mu(E) = \mu(\tau E) = \int_{X/\Gamma} \mu_x(\tau E_x) \, d\bar{\mu}(x).$$

Finally (A3) follows from Theorem 3.3.16.

□

We have an analogous Theorem for the rearrangements of sets.

Theorem 3.6.38. *Let (X, d) be a proper metric space endowed with a Schwarz system $(\mathcal{R}, T, G)$. Let μ be a non-degenerate and diffuse Borel measure, in the sense of* (3.4) *and* (3.5)*, that is invariant with respect to the Schwarz system $(\mathcal{R}, T, G)$ and let $(\Gamma, (\mu_x)_{x \in X/\Gamma}, \bar{\mu})$ be a regular rearrangement system of (X, μ), where $\Gamma = \Gamma(T, G)$. Finally, let the metric measure space (X, d, μ) have the Lebesgue property* (3.6)*. Then the Schwarz rearrangement E^* of any bounded Borel set $E \subset X$, satisfies*

$$\mu(E^*) = \mu(E) \quad \textit{and} \quad P^-(E^*) \le P^-(E). \tag{3.67}$$

Proof. The proof is analogous to the one of Theorem 3.6.37 and we only sketch it. First we fix a suitable compact set K, as in the above proof. Then we introduce the set $\mathcal{A}_E$ of all Borel subsets F of X such that (A1)–(A3) hold with $g = \chi_F$, $f = \chi_E$ and $p = 2$ (or equivalently $p = 1$). The functional $J(F) = \mu(F \Delta E^*)$ attains the minimum on $\mathcal{A}_E$ at some $\bar{F}$. The compactness Theorem 3.2.5 does apply to this situation. As in the above proof, we show that it must be $\bar{F} = E^*$ and the proof is finished. □

Chapter 4

Rearrangements in the Heisenberg Group

Let $\mathbf{H}^d$ be the Heisenberg Group defined in Chapter 2. In this chapter we develop a rearrangement theory specifically for $\mathbf{H}^d$. In fact the natural reflection systems in $\mathbf{H}^d$ do not satisfy condition (R2) in Definition 3.3.6 with respect to the Carnot-Carathéodory metric. However, these systems are indeed reflection systems with respect to the Euclidean metric. Then it is enough to require the functions or sets we rearrange to have some symmetry.

4.1 Two-points rearrangements with symmetry

In this section we introduce what we may call the horizontal and the vertical reflection system with symmetry on $\mathbf{H}^d$. Let $\varrho : \mathbf{H}^d \to \mathbf{H}^d$ denote the mapping

$$\varrho(z,t) = (\bar{z}, -t), \quad (z,t) \in \mathbf{H}^d. \tag{4.1}$$

Here, let $\bar{z}$ be the complex conjugate of z, namely, if $z = x + iy$, then $\bar{z} = x - iy$. The mapping ϱ is an involutive isometry of $(\mathbf{H}^d, d)$. This follows from the fact that a curve γ is horizontal if and only if $\varrho \circ \gamma$ is horizontal and, moreover, $L(\gamma) = L(\varrho \circ \gamma)$. In fact, for any $(x,y,t) \in \mathbf{H}^d$ the differential of ϱ at (x,y,t) is

$$d\varrho_{(x,y,t)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4.2}$$

We remark that $d\varrho_{(x,y,t)}$ is independent of the point (x,y,t), thus henceforth we omit the dependence of the point. By (4.2) have that, for any $j = 1, \dots, d$, $d\varrho\, X_j(x,y,t) = X_j(x,-y,-t) = X_j\big(\varrho(x,y,t)\big)$ and $d\varrho\, Y_j(x,y,t) = -Y_j(x,y,t) = -Y_j\big(\varrho(x,y,t)\big)$. Here we used the fact that $Y_j(x,y,t) = Y_j(\varrho(x,y,t))$ for any $(x,y,t) \in \mathbf{H}^d$ and for any j. Hence, if α_j and β_j, $j = 1, \dots, d$, are the components of $\dot{\gamma}$ with respect to the horizontal vector fields, the chain rule yields

$$\frac{d}{ds}(\varrho \circ \gamma)(s) = d\varrho\,\dot\gamma(s) = \sum_{j=1}^{d} \big(\alpha_j(s) d\varrho\, X_j(\gamma(s)) + \beta_j(s) d\varrho\, Y_j(\gamma(s))\big)$$
$$= \sum_{j=1}^{d} \big(\alpha_j(s) X_j(\varrho \circ \gamma(s)) - \beta_j(s) Y_j(\varrho \circ \gamma(s))\big).$$

This proves that γ is horizontal if and only if $\varrho \circ \gamma$ and that $L(\gamma) = L(\varrho \circ \gamma)$.

Definition 4.1.1. The *horizontal reflection system with symmetry* σ *of* $\mathbf{H}^d$ is the 3-tuple $\mathcal{R} = \{\mathcal{P}, \varrho, \sigma\}$, where $\mathcal{P} = \{H^-, H, H^+\}$ is the partition of $\mathbf{H}^d$ composed by the sets $H^- = \{(z,t) \in \mathbf{H}^d : t < 0\}$, $H = \{(z,t) \in \mathbf{H}^d : t = 0\}$ and $H^+ = \{(z,t) \in \mathbf{H}^d : t > 0\}$; the mapping $\varrho : \mathbf{H}^d \to \mathbf{H}^d$ is defined in (4.1) and the symmetry $\sigma : \mathbf{H}^d \to \mathbf{H}^d$ is defined by

$$\sigma(z,t) = (\bar z, t), \quad (z,t) \in \mathbf{H}^d. \tag{4.3}$$

It is clear that the reflection ϱ maps H^+ in H^-. However ϱ does not satisfy (R2) with respect to neither the Carnot-Carathéodory metric nor the Euclidian metric. Choosing $x = (z,0)$ and $y = (\bar z, 0)$ gives a counterexample.

On the other hand, the mapping $\varrho \circ \sigma : (z,t) \mapsto (z,-t)$ satisfies both (R1) and (R2) with respect to the partition $\mathcal{P}$ and the Euclidean metric. Thus $(\mathcal{P}, \varrho \circ \sigma)$ is a reflection system of $(\mathbf{H}^d, |\cdot|)$. However, $\varrho\circ\sigma$ is not an isometry of $(\mathbf{H}^d, d)$ and (R2) is not satisfied with respect to the Carnot-Carathéodory metric. In fact, for $(z,t), (\zeta,\tau) \in \mathbf{H}^d$, we have that

$$d((z,t),(\zeta,\tau)) = d(0,(z,t)^{-1} * (\zeta,\tau)) = d\big(0,(\zeta - z, -t + \tau - 2\mathrm{Im}(z\cdot\bar\zeta))\big), \tag{4.4}$$

while

$$d(\varrho\circ\sigma(z,t), \varrho\circ\sigma(\zeta,\tau)) = d((z,-t),(\zeta,-\tau)) = d\big(0,\big(\zeta - z, t - \tau - 2\mathrm{Im}(z\cdot\bar\zeta)\big)\big).$$

This proves that for $t \neq \tau$ the reflection $\varrho\circ\sigma$ is not an isometry with respect to d.

To prove that (R2) is not satisfied with respect to d, let $(z,t), (\zeta,\tau) \in H \cup H^+$ (i.e. $t, \tau \geq 0$) be such that $0 < -(t + 2\mathrm{Im}(z\cdot\bar\zeta)) \leq \tau$. In this case, we claim that it holds $d((z,t),(\zeta,\tau)) > d((z,t), \varrho\circ\sigma(\zeta,\tau))$. This is equivalent to $d(0,(z,t)^{-1} * (\zeta,\tau)) > d(0,(z,t)^{-1} * (\zeta,-\tau))$. Let $h : \mathbf{H}^d \to \mathbb{R}$ be defined by $h(z,t) = |t|$ for any $(z,t) \in \mathbf{H}^d$. By (4.4) and the choice of (z,t) and (ζ,τ), we get

$$h\big((z,t)^{-1} * (\zeta,-\tau)\big) = |-t-\tau-2\mathrm{Im}(z\cdot\bar\zeta)| < |-t+\tau-2\mathrm{Im}(z\cdot\bar\zeta)| = h\big((z,t)^{-1} * (\zeta,\tau)\big).$$

Since $\pi\big((z,t)^{-1} * (\zeta,-\tau)\big) = \pi\big((z,t)^{-1} * (\zeta,\tau)\big)$, this proves the claim. Therefore $\varrho\circ\sigma$ does not satisfy (R2) with respect to the Carnot-Carathéodory metric.

Definition 4.1.2. The *vertical reflection system with symmetry* σ *of* $\mathbf{H}^d$, is the 3-tuple $\mathcal{R} = \{\mathcal{P}, \varrho, \sigma\}$, where $\mathcal{P} = \{H^-, H, H^+\}$ is the partition of $\mathbf{H}^d$ composed by the sets

$H^- = \{(z,t) \in \mathbf{H}^d : \operatorname{Im}(z_1) < 0\}$, $H = \{(z,t) \in \mathbf{H}^d : \operatorname{Im}(z_1) = 0\}$ and $H^+ = \{(z,t) \in \mathbf{H}^d : \operatorname{Im}(z_1) > 0\}$; the mapping $\varrho : \mathbf{H}^d \to \mathbf{H}^d$ is defined in (4.1) and the symmetry $\sigma : \mathbf{H}^d \to \mathbf{H}^d$ is defined by

$$\sigma(z,t) = (z_1, \bar{z}_2, \dots, \bar{z}_n, -t) \quad (z,t) \in \mathbf{H}^d. \tag{4.5}$$

The same consideration as above apply to this situation. We show that neither in this case $\varrho \circ \sigma : (z,t) \mapsto (\tilde{z}, t) = (\bar{z}_1, z_2, \dots, z_n, t)$ is an isometry of $(\mathbf{H}^d, d)$ nor it satisfies (R2). In fact, for any $(z,t), (\zeta, \tau) \in \mathbf{H}^d$ we have that

$$\big(\varrho \circ \sigma(z,t)\big)^{-1} * \big(\varrho \circ \sigma(\zeta,\tau)\big) = \left(\tilde{\zeta} - \tilde{z}, -t + \tau + 2\operatorname{Im}(z_1 \cdot \bar{\zeta}_1) - 2 \sum_{j=2}^{n} \operatorname{Im}(z_j \cdot \bar{\zeta}_j) \right).$$

The above identity implies that the reflection $\varrho \circ \sigma$ is not isometric if $z_1, \zeta_1 \neq 0$. In a similar manner $(z,t)^{-1} * \big(\varrho \circ \sigma(\zeta,\tau)\big) = \Big(\tilde{\zeta} - z, T\big((z,t),(\zeta,\tau)\big)\Big)$, where we let

$$T\big((z,t),(\zeta,\tau)\big) = -t + \tau - 2\operatorname{Im}(z_1 \cdot \zeta_1) - 2 \sum_{j=2}^{n} \operatorname{Im}(z_j \cdot \bar{\zeta}_j).$$

Hence if $(z,t) \in H \cup H^+$ and $(\zeta,\tau) \in H^+$ (i.e. $\operatorname{Im}(\zeta_1) > 0$) are such that $T\big((z,t),(\zeta,\tau)\big) > 0$ and $\operatorname{Re}(z_1) < 0$ or $T\big((z,t),(\zeta,\tau)\big) < 0$ and $\operatorname{Re}(z_1) > 0$, then $d((z,t),(\zeta,\tau)) > d((z,t), \varrho \circ \sigma(\zeta,\tau))$ and hence condition (R2) is not satisfied.

Having defined a rearrangement system with symmetry of $\mathbf{H}^d$, we let the two-points rearrangement of functions or sets to be as in Definitions 3.3.11 and 3.3.12.

Definition 4.1.3. Let $\mathcal{R}$ be either a horizontal or vertical rearrangement system with symmetry σ of $\mathbf{H}^d$.

(i) Let $f : \mathbf{H}^d \to \mathbb{R}$, then the function $f_{\mathcal{R}} : \mathbf{H}^d \to \mathbb{R}$ defined by

$$f_{\mathcal{R}}(x) = \begin{cases} \min\{f(x), f(\varrho x)\} & \text{if } x \in H^- \\ f(x) & \text{if } x \in H \\ \max\{f(x), f(\varrho x)\} & \text{if } x \in H^+ \end{cases}, \tag{4.6}$$

is called the *two-points rearrangement* of f with respect to $\mathcal{R}$.

(ii) Let $\mathcal{R}$ be either a horizontal or vertical rearrangement system with symmetry σ of $(\mathbf{H}^d, d)$. Let $E \subset \mathbf{H}^d$, then the set $E_{\mathcal{R}}$ defined by

$$E_{\mathcal{R}} = \big(E \cap \varrho E \cap H^-\big) \cup (E \cap H) \cup \big((E \cup \varrho E) \cap H^+\big), \tag{4.7}$$

is called the *two-points rearrangement* of E with respect to $\mathcal{R}$.

Here we give the analogous of Theorem 3.3.18 in this setting. We recall that $f : \mathbf{H}^d \to \mathbb{R}$ is σ-symmetric if $f = f \circ \sigma$.

Theorem 4.1.4. *Let $\mathcal{R} = \{\mathcal{P}, \varrho, \sigma\}$ be either a horizontal or vertical rearrangement system with symmetry σ of $(\mathbf{H}^d, d)$ and let $1 < p < \infty$. For any σ-symmetric function $f \in C^1_c(\mathbf{H}^d)$ we have that $f_{\mathcal{R}} \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^d) \cap BV_{\mathbf{H}}(\mathbf{H}^d)$ and moreover it holds*

$$\|\nabla_{\mathbf{H}} f_{\mathcal{R}}(z,t)\|_{L^p(\mathbf{H}^d)} \leq \|\nabla_{\mathbf{H}} f(z,t)\|_{L^p(\mathbf{H}^d)}, \tag{4.8}$$

$$|\nabla_{\mathbf{H}} f_{\mathcal{R}}|(\mathbf{H}^d) \leq |\nabla_{\mathbf{H}} f|(\mathbf{H}^d). \tag{4.9}$$

Here $|\nabla_{\mathbf{H}} f|$ denotes the horizontal total variation of f.

Proof. For any $r \in (0,1)$ let

$$Q_{r,p}(f) = \int_{\mathbf{H}^d \times \mathbf{H}^d} |f(x) - f(y)|^p \chi_r(x,y)\, dx\, dy,$$

where

$$\chi_r(x,y) = \begin{cases} \frac{1}{|B_r(x)|} & \text{if } d(x,y) < r, \\ 0 & \text{otherwise.} \end{cases}$$

Here $B_r(x)$ denote the Carnot-Carathéodory ball centered at x of radius $r > 0$. Notice that, by the left-invariance of the metric d and using the dilations δ_λ, $|B_r(x)| = r^{2d+2}|B_1(0)|$. Moreover, as the Lebesgue measure $\mathcal{L}^{2d+1}$ is invariant with respect to the reflection ϱ, we have that

$$\chi_r(\varrho x, \varrho y) = \chi_r(x,y) \text{ and } \chi_r(x, \varrho y) = \chi_r(\varrho x, y), \quad x, y \in \mathbf{H}^d. \tag{4.10}$$

Let L denote the Lipschitz constant of f with respect to the Euclidean metric. Namely

$$L = \mathrm{Lip}(f) = \sup_{\substack{x,y \in \mathbf{H}^d \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|}.$$

Let $K \subset \mathbf{H}^d$ be a compact cube centered at 0, with axes parallel to the coordinate axes and such that

$$\mathrm{dist}_{\mathbf{H}}(\mathbf{H}^d \setminus K, \operatorname{supp} f) \geq 1. \tag{4.11}$$

Here we let $\mathrm{dist}_{\mathbf{H}}$ be the Carnot-Carathéodory distance. By a well-known estimate, there exists a constant $C_K > 0$ such that

$$|x - y| \leq C_K d(x,y) \quad \text{for any } x, y \in \mathbf{H}^d. \tag{4.12}$$

Finally let H be the reflection hyperplane of $\mathcal{R} = \{\mathcal{P}, \varrho, \sigma\}$ and let $(H \cap K)_r$ denote the $C_K\, r$-neighborhood of $H \cap K$ in the Euclidean metric, namely

$$(H \cap K)_r = \{x \in \mathbf{H}^d : \mathrm{dist}(x, H \cap K) < C_k\, r\}.$$

By Theorems 2.1.9 and 2.1.13 if

$$\liminf_{r \downarrow 0} \frac{1}{r^p} Q_{r,p}(f_{\mathcal{R}}) < \infty \quad \text{and} \quad \liminf_{r \downarrow 0} \frac{1}{r} Q_{r,1}(f_{\mathcal{R}}) < \infty,$$

then $f_{\mathcal{R}} \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^d) \cap BV_{\mathbf{H}}(\mathbf{H}^d)$. Moreover, the first lim inf is in fact a limit and

$$\lim_{r\downarrow 0} \frac{1}{r^p} Q_{r,p}(f_{\mathcal{R}}) = C_{p,d} \|\nabla_{\mathbf{H}} f_{\mathcal{R}}(z,t)\|_{L^p(\mathbf{H}^d)}.$$

If $f_{\mathcal{R}} \in BV(\mathbf{H}^d)$ then also the second lim inf is a limit and it holds

$$\lim_{r\downarrow 0} \frac{1}{r} Q_{r,1}(f_{\mathcal{R}}) = C_{1,d} |\nabla_{\mathbf{H}} f_{\mathcal{R}}|(\mathbf{H}^d).$$

Here $C_{p,d}$ is the geometric constant, depending only on the dimension d and the exponent p, defined in (2.11). Then in order to complete the proof, it suffices to prove that $f_{\mathcal{R}} \in BV(\mathbf{H}^d)$ and that, for any $r \in (0,1)$ and $1 \le p < \infty$, it holds

$$Q_{r,p}(f_{\mathcal{R}}) \le Q_{r,p}(f) + 2L^p C_K^p r^p \, |(H \cap K)_r|. \tag{4.13}$$

In fact $f \in C^1_c(\mathbf{H}^d) \subset W^{1,p}_{\mathbf{H}}(\mathbf{H}^d) \cap BV_{\mathbf{H}}(\mathbf{H}^d) \cap BV(\mathbf{H}^d)$ and $\lim_{r\to 0} |(H \cap K)_r| = 0$.

As in the proof of Theorem 3.3.16, by (4.10) we get that

$$\begin{aligned} Q_{r,p}(f) = &\int_{H^+}\int_{H^+} \big\{|f(x)-f(y)|^p + |f(\varrho x) - f(\varrho y)|^p\big\}\chi_r(x,y)\,dx\,dy \\ &+ \int_{H^+}\int_{H^+} \big\{|f(x)-f(\varrho y)|^p + |f(\varrho x) - f(y)|^p\big\}\chi_r(x,\varrho y)\,dx\,dy. \end{aligned}$$

By the symmetries $f(\varrho\sigma y) = f(\varrho y)$ and $f(\sigma y) = f(y)$ we obtain

$$Q_{r,p}(f) = \int_{H^+}\int_{H^+} Q(f;x,y)\,dx\,dy,$$

where we let

$$\begin{aligned} Q(f;x,y) = &\big\{|f(x)-f(y)|^p + |f(\varrho x) - f(\varrho y)|^p\big\}\chi_r(x,y) \\ &+ \big\{|f(x)-f(\varrho y)|^p + |f(\varrho x) - f(y)|^p\big\}\chi_r(x,\varrho\sigma y). \end{aligned}$$

Let $x, y \in H^+$. We have the following four cases:

1. $d(x,y) \ge r$ and $d(x,\varrho\sigma y) \ge r$;
2. $d(x,y) \le d(x,\varrho\sigma y) < r$;
3. $d(x,y) < r \le d(x,\varrho\sigma y)$;
4. $d(x,\varrho\sigma y) < r \le d(x,y)$.

In the proof of Theorem 3.3.16, we had no case 4. In the cases 1, 2 and 3 we have

$$Q(f_{\mathcal{R}};x,y) \le Q(f;x,y). \tag{4.14}$$

The proof is the same as in Theorem 3.3.16. We study case 4. Let

$$E_r = \{(x,y) \in H^+ \times H^+ : \ d(x,\varrho\sigma y) < r \le d(x,y)\}.$$

If $(x,y) \in E_r$, we have

$$Q(f;x,y) = \{|f(x) - f(\varrho y)|^p + |f(\varrho x) - f(y)|^p\}\chi_r(x, \varrho\sigma y).$$

The function $f_{\mathcal{R}}$ is σ-symmetric. Moreover, $f_{\mathcal{R}}$ is the Euclidean two-points rearrangement of f with respect to the reflection system $\{\mathcal{P}, \varrho \circ \sigma\}$. By Proposition 3.3.13, we have $\mathrm{Lip}(f_{\mathcal{R}}) \le \mathrm{Lip}(f) = L$, and hence $f_{\mathcal{R}} \in BV(\mathbf{H}^d)$, since $\operatorname{supp} f_{\mathcal{R}} \subset K$. By (4.12), we have

$$|f_{\mathcal{R}}(x) - f_{\mathcal{R}}(\varrho y)| = |f_{\mathcal{R}}(x) - f_{\mathcal{R}}(\varrho\sigma y)| \le L|x - \varrho\sigma y| \le LC_K d(x, \varrho\sigma y) \le LC_K r.$$

Analogously, we get also that $|f_{\mathcal{R}}(\varrho x) - f_{\mathcal{R}}(y)| \le LC_K r$.

By (4.11), we may assume $x, y \in K$. In fact, if $x \in \mathbf{H}^d \setminus K$, or $y \in \mathbf{H}^d \setminus K$, and $r < 1$, by the fact that $(x,y) \in E_r$ we have

$$f(x) = f(y) = f(\varrho x) = f(\varrho y) = 0,$$

and thus $Q(f_{\mathcal{R}}; x, y) = Q(f;x,y) = 0$ and we are finished. Let then $x, y \in H^+ \cap K$. In this case we have

$$\mathrm{dist}(\varrho x, H \cap K) = \mathrm{dist}(x, H \cap K) < |x - \varrho\sigma y| \le C_K d(x, \varrho\sigma y) < C_K r.$$

Then we have

$$\begin{aligned}
\int_{E_r} Q(f_{\mathcal{R}}; x, y)\,dx\,dy &\le \int_{H^+}\int_{H^+\cap(H\cap K)_r} \{|f(x) - f(\varrho y)|^p + |f(\varrho x) - f(y)|^p\}\chi_r(x, \varrho\sigma y)\,dx\,dy \\
&\le 2L^p C_K^p r^p \int_{H^+}\int_{H^+\cap(H\cap K)_r} \chi_r(x, \varrho\sigma y)\,dx\,dy \\
&\le 2L^p C_K^p r^p |(H\cap K)_r|.
\end{aligned}$$

This is (4.13) and completes the proof. □

We extend Theorem 4.1.4 to the case of Sobolev functions in $W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$ and to sets with finite horizontal perimeter. We proceed by approximation, using Theorem 2.1.8. We remark that the approximating functions f_n in that Theorem are obtained as convolutions of the form

$$f_\varepsilon(x) = \int_{\mathbf{H}^d} f(y) J_\varepsilon(|x - y|)\,dy, \quad \varepsilon > 0,\ x \in \mathbf{H}^d.$$

Here $J_\varepsilon = \varepsilon^{2d+1} J(|x|/\varepsilon)$ is a standard convolution kernel. Hence, if f is σ-symmetric, then also f_n is σ-symmetric. Multiplying each f_n by a suitable cut-off function, we may then assume that the functions f_n are compactly supported and σ-symmetric, if f is σ-symmetric.

Corollary 4.1.5. *Let $\mathcal{R}$ be either an horizontal or vertical rearrangement system with symmetry σ of $\mathbf{H}^d$.*

(i) *Let $1 < p < \infty$ and let $f \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$ be a σ-symmetric function. Then $f_{\mathcal{R}} \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$ and moreover*

$$\|\nabla_{\mathbf{H}} f_{\mathcal{R}}\|_{L^p(\mathbf{H}^d)} \leq \|\nabla_{\mathbf{H}} f\|_{L^p(\mathbf{H}^d)}. \tag{4.15}$$

(ii) *Let $f \in BV_{\mathbf{H}}(\mathbf{H}^d)$ be a σ-symmetric function. Then $f_{\mathcal{R}} \in BV_{\mathbf{H}}(\mathbf{H}^d)$ and moreover*

$$|\nabla_{\mathbf{H}} f_{\mathcal{R}}|(\mathbf{H}^d) \leq |\nabla_{\mathbf{H}} f|(\mathbf{H}^d). \tag{4.16}$$

Here $f_{\mathcal{R}}$ is the two-points rearrangement of f defined in (4.6).

Proof. We prove only statement (i). The same argument applies to statement (ii). We proceed by approximation using Theorem 2.1.8 and the above considerations. Let $(f_n)_{n\in\mathbb{N}} \subset C^1_c(\mathbf{H}^d)$ be a sequence of σ-symmetric functions such that

$$\lim_{n\to\infty} \|f_n - f\|_{L^p(\mathbf{H}^d)} = \lim_{n\to\infty} \|\nabla_{\mathbf{H}} f_n - \nabla_{\mathbf{H}} f\|_{L^p(\mathbf{H}^d)} = 0.$$

Possibly taking a subsequence, we can assume that $f_n \to f$ a.e. in $\mathbf{H}^d$.

By Theorem 4.1.4, we have that $(f_n)_{\mathcal{R}} \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$ and

$$\|\nabla_{\mathbf{H}} (f_n)_{\mathcal{R}}\|_{L^p(\mathbf{H}^d)} \leq \|\nabla_{\mathbf{H}} f_n\|_{L^p(\mathbf{H}^d)}.$$

It follows that, up to subsequences, the sequence $\big((f_n)_{\mathcal{R}}\big)_{n\in\mathbb{N}}$ converges weakly in $W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$ to a function g such that

$$\|\nabla_{\mathbf{H}} g\|_{L^p(\mathbf{H}^d)} \leq \liminf_{n\to\infty} \|\nabla_{\mathbf{H}} (f_n)_{\mathcal{R}}\|_{L^p(\mathbf{H}^d)}.$$

We may also assume that $(f_n)_{\mathcal{R}} \to g$ a.e.in $\mathbf{H}^d$. We claim that $g = f_{\mathcal{R}}$.

Let $N \subset \mathbf{H}^d$ be a $\mathcal{L}^{2d+1}$-negligible set such that for all $x \in \mathbf{H}^d \setminus N$ we have that $f_n(x) \to f(x)$ and $(f_n)_{\mathcal{R}}(x) \to g(x)$ as $n \to \infty$. By the continuity of ϱ this implies that $f_n(\varrho x) \to f(\varrho x)$ as $n \to \infty$, as well. If $f(x) > f(\varrho x)$, then there exists a $\bar{n}$ such that for all $n \geq \bar{n}$ it holds that $f_n(x) > f_n(\varrho x)$. Obviously the same holds if $f(x) < f(\varrho x)$. This implies that $(f_n)_{\mathcal{R}}(x) \to f_{\mathcal{R}}(x)$ for all $x \in \mathbf{H}^d \setminus N$ and hence that $g = f_{\mathcal{R}}$. This completes the proof.

□

4.2 Steiner rearrangement

In this section we consider the Steiner rearrangement system associated to the horizontal reflection system with symmetry σ of $\mathbf{H}^d$ (see Definition 4.1.1).

Let $\tau_s : \mathbf{H}^d \to \mathbf{H}^d$, $s \in \mathbb{R}$, be the vertical translation defined as $\tau_s(z,t) = (z, t+s)$ for any $(z,t) \in \mathbf{H}^d$. These translations form a 1-parameter group of isometries, $T = \{\tau_s\}_{s\in\mathbb{R}}$. We may identify the reflection hyperplane $H = \{(z,0) \in \mathbf{H}^d\}$ with $\mathbf{H}^d/T$. The action of T is continuous and proper and the orbits are the vertical lines $T_z = \{(z,t) \in \mathbf{H}^d : t \in \mathbb{R}\}$. Finally, the natural projection is $\pi : \mathbf{H}^d \to H$, $\pi(z,t) = (z,0)$.

By the Fubini theorem, the Lebesgue measure $\mathcal{L}^{2d+1}$ disintegrates naturally along T. Indeed, for any measurable set $E \subset \mathbf{H}^d$ we have

$$|E| = \int_H \mathcal{H}^1(E \cap T_z)\, d\mathcal{H}^{2d}(z),$$

where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure. Since it is clear that the function $s \mapsto \mathcal{H}^1(B_s(x) \cap T_z)$ is strictly increasing from $[0, +\infty)$ to $[0, +\infty)$ for any $z \in \mathbb{C}^d$, the triple $(T, \mathcal{H}^1, \mathcal{H}^{2d})$ is a rearrangement system of $(\mathbf{H}^d, d, \mathcal{L}^{2d+1})$. It is clear that such rearrangement is regular.

For any measurable set $E \subset \mathbf{H}^d$ and for any non-negative, measureable function $f : \mathbf{H}^d \to \mathbb{R}$, we call the rearrangements E^* and f^*, given in Definition 3.4.25, the *Steiner rearrangements of E and f.*

Now we prove a statement analogous to Theorem 3.6.37.

Theorem 4.2.6. *Let $f \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$, $1 < p < \infty$, be a non-negative, σ-symmetric function and let f^* be the Steiner rearrangement of f. Then $f^* \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$ and*

$$\|\nabla_{\mathbf{H}} f^*\|_{L^p(\mathbf{H}^d)} \leq \|\nabla_{\mathbf{H}} f\|_{L^p(\mathbf{H}^d)}. \tag{4.17}$$

Proof. We start by proving the Theorem for $f \in C^1_c(\mathbf{H}^d)$. Here we follow closely the proof of Theorem 3.6.37.

Let $K \subset \mathbf{H}^d$ be a compact cube centered at 0, with axes parallel to the coordinate axes and such that $\operatorname{supp} f \subset K$. Let $\mathcal{A}_f$ be the family of all non-negative σ-symmetric functions $g \in L^p(\mathbf{H}^d)$ such that

(A1) $\mathcal{H}^1\{g > s\}_z = \mathcal{H}^1\{f > s\}_z$ for $\mathcal{L}^{2d}$-a.e. $z \in \mathbb{C}^d$ and for all $s > 0$;

(A2) $g(z, t) = 0$ for all $(z, t) \in \mathbf{H}^d \setminus K$;

(A3) $\|\nabla_{\mathbf{H}} g\|_{L^p(\mathbf{H}^d)} \leq \|\nabla_{\mathbf{H}} f\|_{L^p(\mathbf{H}^d)}$.

By (A3) the family $\mathcal{A}_f$ is uniformly bounded in $W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$ and by (A2) it is boundedly supported. By the compactness theorem in [GN96], $\mathcal{A}_f$ is then compact in $L^p(\mathbb{R}^d)$.

By the Weierstrass theorem, the functional $J : \mathcal{A}_f \to [0, +\infty)$,

$$J(g) = \int_{\mathbf{H}^d} |g - f^*|^p\, dx, \tag{4.18}$$

achieves the maximum at some point $\bar{f} \in \mathcal{A}_f$. If $J(\bar{f}) = 0$, we are finished. In fact this would imply that $\bar{f} = f^*$ a.e., and hence that $f^* \in \mathcal{A}_f$. Condition (A3) is then (4.17). On the other hand, $J(\bar{f}) > 0$ cannot occur. The proof is exactly the same as in Theorem 3.6.37. In fact we can find a vertical translation of H such that the two-points rearrangement $\bar{f}_{\bar{\mathcal{R}}}$ of $\bar{f}$ with respect to the translated vertical rearrangement system, satisfies (A1) and (A2) and moreover is such that $J(\bar{f}_{\bar{\mathcal{R}}}) < J(\bar{f})$. Since by Corollary 4.1.5 $\bar{f}_{\bar{\mathcal{R}}}$ satisfies also (A3), we have that $\bar{f}_{\bar{\mathcal{R}}} \in \mathcal{A}_f$, thus contradicting the minimality of $\bar{f}$.

To prove the general case, we proceed by approximation using Theorem 2.1.8. Let $(f_n)_{n\in\mathbb{N}} \subset C^1_c(\mathbf{H}^d)$ be a sequence such that

$$\lim_{n\to\infty} \|f_n - f\|_{L^p(\mathbf{H}^d)} = \lim_{n\to\infty} \|\nabla_{\mathbf{H}} f_n - \nabla_{\mathbf{H}} f\|_{L^p(\mathbf{H}^d)} = 0.$$

In particular, we have

$$\lim_{n\to\infty} \int_0^{+\infty} \int_{\mathbb{C}^d} \mathcal{L}^1\{t \in \mathbb{R} : \, |f_n(z,t) - f(z,t)| > s^{1/p}\}\, dz\, ds = 0$$

and, up to subsequences, we have that for $\mathcal{L}^{2d}$-a.e. $z \in \mathbb{C}^d$ and for all $s > 0$

$$\lim_{n\to\infty} \mathcal{L}^1\{t \in \mathbb{R} : \, |f_n(z,t) - f(z,t)| > s^{1/p}\} = 0. \tag{4.19}$$

We can also assume that $f_n(z,t) \to f(z,t)$ for a.e. $(z,t) \in \mathbf{H}^d$.

Now we claim that this implies that for all $s > 0$

$$\lim_{n\to\infty} \mathcal{L}^1\big(\{f_n(z,\cdot) > s\}\Delta\{f(z,\cdot) > s\}\big) = 0. \tag{4.20}$$

In fact, for any $\varepsilon > 0$

$$\begin{aligned} &\{f_n(z,\cdot) > s+\varepsilon\}\Delta\{f(z,\cdot) > s+\varepsilon\} \\ &\qquad = \{f_n(z,\cdot) > s+\varepsilon \geq f(z,\cdot)\} \cup \{f(z,\cdot) > s+\varepsilon \geq f_n(z,\cdot)\} \\ &\qquad \subset \big\{|f_n(z,\cdot) - f(z,\cdot)| \geq 2\varepsilon\big\}. \end{aligned}$$

Hence, by (4.3), $\mathcal{L}^1\big(\{f_n(z,\cdot) > s+\varepsilon\}\Delta\{f(z,\cdot) > s+\varepsilon\}\big) \to 0$ as $n \to \infty$. Letting $\varepsilon \downarrow 0$ we get (4.20).

By the first part of the proof, we have

$$\|\nabla_{\mathbf{H}} f_n^*\|_{L^p(\mathbf{H}^d)} \leq \|\nabla_{\mathbf{H}} f_n\|_{L^p(\mathbf{H}^d)}.$$

It follows that, up to subsequences, the sequence $(f_n^*)_{n\in\mathbb{N}}$ converges weakly in $W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$ to a function g such that

$$\|\nabla_{\mathbf{H}} g\|_{L^p(\mathbf{H}^d)} \leq \liminf_{n\to\infty} \|\nabla_{\mathbf{H}} f_n^*\|_{L^p(\mathbf{H}^d)}.$$

We may also assume that $f_n^* \to g$ a.e.in $\mathbf{H}^d$. We claim that $g = f^*$.

The functions $t \mapsto f_n^*(z,t)$ and $t \mapsto g(z,t)$ are even and non-increasing for $t \geq 0$. The sets $I_n(z,s) = \{f_n^*(z,\cdot) > s\}$ and $I(z,s) = \{g(z,\cdot) > s\}$ are essentialy symmetric intervals. Then $I_n(z,s) \to I(z,s)$, in the natural sense, for $\mathcal{L}^{2d}$-a.e. $z \in \mathbb{C}^d$ and $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$. It follows that

$$\lim_{n\to\infty} \mathcal{L}^1\big(\{f_n^*(z,\cdot) > s\}\Delta\{g(z,\cdot) > s\}\big) = 0. \tag{4.21}$$

From (4.20) and (4.21), we deduce that

$$\mathcal{L}^1\{g(z,\cdot) > s\} = \lim_{n\to\infty} \mathcal{L}^1\{f_n^*(z,\cdot) > s\} = \lim_{n\to\infty} \mathcal{L}^1\{f_n(z,\cdot) > s\} = \mathcal{L}^1\{f(z,\cdot) > s\}.$$

This implies that $g = f^*$ a.e. on $\mathbf{H}^d$, thus completing the proof. □

We recall that the $\mathbf{H}$-perimeter of a set $E \subset \mathbf{H}^d$ is $P_{\mathbf{H}}(E) = |\nabla_{\mathbf{H}}\chi_E|(\mathbf{H}^d)$. A set $E \subset \mathbf{H}^d$ is σ-symmetric if $E = \sigma E$ or equivalently if χ_E is a σ-symmetric function.

Theorem 4.2.7. *Let $E \subset \mathbf{H}^d$ be a σ-symmetric set of finite measure and $\mathbf{H}$-perimeter and let E^* be the Steiner rearrangement of E. Then E^* is of finite $\mathbf{H}$-perimeter and*

$$P_{\mathbf{H}}(E^*) \leq P_{\mathbf{H}}(E). \tag{4.22}$$

Proof. The proof is a repetition of the one of Theorem 4.2.6. In a first step, we prove the theorem for $f \in C_c^1(\mathbf{H}^d) \cap BV_{\mathbf{H}}(\mathbf{H}^d)$ with $\|f\|_\infty \leq 1$. In the definition of $\mathcal{A}_f$, we consider functions $g \in L^1(\mathbf{H}^d)$ with $\|g\|_\infty \leq 1$, replacing in (A3) the Sobolev norm with the horizontal total variation. In the functional J in (4.18) we choose $p = 2$. To exclude the case $J(f) > 0$ we use now Corollary 4.1.5, statement (ii). In the second step, we prove the theorem for sets with finite perimeter and measure, on using the approximation Theorem 2.1.8. □

We end this section proving that Theorem 4.2.7 does not hold dropping the assumption on σ-symmetry of the sets. We do so constructing a set $E \subset \mathbf{H}^d$ such that its Steiner rearrangement satisfies

$$P_{\mathbf{H}}(E^*) > P_{\mathbf{H}}(E).$$

In particular, the set E is the left translation of a cylinder.

Example 4.2.8. Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and define the horizontal area of the graph of a Lipschitz function $f : D \to \mathbb{R}$ as

$$A_{\mathbf{H}}(f) = \int_D \sqrt{\left(\frac{\partial f}{\partial x} - 2y\right)^2 + \left(\frac{\partial f}{\partial y} + 2x\right)^2}\, dx\, dy. \tag{4.23}$$

This area is the horizontal perimeter of the epigraph of f inside the cylinder $D \times \mathbb{R}$.

Fix a real number $c > 0$ and let $f_{a,b}$, $a, b \in \mathbb{R}$, be the affine function $f_{a,b}(x, y) = ax + by + c$. The horizontal area of the graph of this function depends only on the parameter $s = \sqrt{a^2 + b^2}$. Namely for $s \geq 0$, by (4.23),

$$A(s) = A_{\mathbf{H}}(f_{a,b}) = \int_0^1 \left(\int_0^{2\pi} \sqrt{s^2 + 4rs\sin\vartheta + 4r^2}\, d\vartheta\right) r\, dr.$$

The derivative in s of the function A is

$$A'(s) = \int_0^1 \left(\int_0^{2\pi} \frac{s + 2r\sin\vartheta}{2\sqrt{s^2 + 4r\sin\vartheta + 4r^2}}\, d\vartheta\right) r\, dr.$$

In particular $A'(0) = 0$. The second derivative is

$$A''(s) = \int_0^1 \left(\int_0^{2\pi} \frac{4r^3\cos^2\vartheta}{(s^2 + 4r\sin\vartheta + 4r^2)^{3/2}}\, d\vartheta\right) r\, dr > 0.$$

Then A' is strictly increasing and thus also A is increasing for $s \geq 0$.

Now, let $C_{a,b} \subset \mathbf{H}^1$ be the cylinder

$$C_{a,b} = \{(x+iy,t) \in \mathbf{H}^1 : x+iy \in D,\ |t-ax-by| < c\}.$$

We claim that for all $a, b \in \mathbb{R}$ we have

$$P_{\mathbf{H}}(C_{0,0}) \leq P_{\mathbf{H}}(C_{a,b}), \tag{4.24}$$

with equality if and only if $a = b = 0$.

In fact, by a standard formula for the horizontal perimeter, there holds

$$P_{\mathbf{H}}(C_{a,b}) = 2A_{\mathbf{H}}(f_{a,b}) + \mathcal{H}^2(\partial D \times \mathbb{R} \cap \partial C_{a,b}),$$

where $\mathcal{H}^2$ is the 2-dimensional Hausdorff measure in $\mathbf{H}^1 = \mathbb{R}^3$. By Fubini theorem, $\mathcal{H}^2(\partial D \times \mathbb{R} \cap \partial C_{a,b}) = 4\pi c$ is independent of a, b. The claim follows from the previous considerations on the function A.

Now, let $p = (z_0, 0) \in \mathbf{H}^1$ be a point such that $z_0 \neq 0$ and let

$$E = p * C_{0,0} = \{(z,t) \in \mathbf{H}^1 :\ |z - z_0| < 1,\ |t - 2\mathrm{Im}(z_0 \bar{z})| < c\}.$$

The Steiner rearrangement of the E is the cylinder

$$E^* = \{(z,t) \in \mathbf{H}^1 :\ |z - z_0| < 1, |t| < c\} = p * C_{a,b},$$

for suitable $a, b \in \mathbb{R}$ that satisfy $a^2 + b^2 \neq 0$, since $z_0 \neq 0$. By the left invariance of the $\mathbf{H}$-perimeter and by the discussion of the equality case in (4.24), we have

$$P_{\mathbf{H}}(E^*) = P_{\mathbf{H}}(C_{a,b}) > P_{\mathbf{H}}(C_{0,0}) = P_{\mathbf{H}}(E).$$

4.3 Cap rearrangement

In this section we define the cap rearrangement associated to the vertical reflection system with symmetry σ of $\mathbf{H}^d$ (see Definition 4.1.2). We refer to [Bae94] for the Euclidean cap rearrangement.

For $z \in \mathbb{C}^d$, we let $z = (z_1, z') \in \mathbb{C} \times \mathbb{C}^{d-1}$. Let $r_\alpha : \mathbf{H}^d \to \mathbf{H}^d$, $\alpha \in \mathbb{S}^1$, be the rotation defined as $r_\alpha(z_1, z', t) = (e^{i\alpha} z_1, z', t)$ for any $(z_1, z', t) \in \mathbf{H}^d$. These rotations form a group of isometries, $R = \{r_\alpha\}_{\alpha \in \mathbb{S}^1}$. We may identify $\mathbf{H}^d/R = \mathbb{R}_+ \times \mathbb{C}^{d-1} \times \mathbb{R} \subset H$, where we let $\mathbb{R}_+ = \{x \in \mathbb{R} :\ x \geq 0\}$. The action of R is continuous and proper and the orbits are the spheres $R_p = \{(e^{i\alpha} x, z', t) \in \mathbf{H}^d :\ \alpha \in \mathbb{S}^1\}$, for $p = (x, z', t) \in \mathbf{H}^d/R$. Finally, the natural projection is $\pi : \mathbf{H}^d \to \mathbf{H}^d/R$, $\pi(z_1, z', t) = (z_1, z', t)$.

The natural disintegration of the Lebesgue measure $\mathcal{L}^{2d+1}$ along R is given by the polar coordinates on $\mathbb{C}$. Indeed, for any measurable set $E \subset \mathbf{H}^d$, by the Fubini theorem and the Coarea formula, we have

$$|E| = \int_{\mathbf{H}^d/R} \mathcal{H}^1(E \cap R_p)\, d\mathcal{H}^{2d}(p),$$

where $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure on $\mathbf{H}^d = \mathbb{R}^{2d+1}$. It is clear that the function $s \mapsto \mathcal{H}^1(B_s(p) \cap R_p)$ is strictly increasing from $[0, s_0(p))$ to $[0, \mathcal{H}^1(R_p))$ for any $p \in \mathbf{H}^d/R$. Here $s_0(p) = \max\{d(x,p) : x \in R_p\}$. Then the triple $(R, \mathcal{H}^1, \mathcal{H}^{2d})$ is a rearrangement system of $(\mathbf{H}^d, d, \mathcal{L}^{2d+1})$. It is clear that such rearrangement is regular.

For any measurable set $E \subset \mathbf{H}^d$ and for any non-negative, measurable function $f : \mathbf{H}^d \to \mathbb{R}$, we call the rearrangements E^* and f^*, given in Definition 3.4.25, the *cap rearrangements of E and f.*

We end this chapter proving two results analogous to Theorems 4.2.6 and 4.2.7.

Theorem 4.3.9. *Let $f \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$, $1 < p < \infty$, be a non-negative, σ-symmetric function and let f^* be the cap rearrangement of f. Then $f^* \in W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$ and*

$$\|\nabla_{\mathbf{H}} f^*\|_{L^p(\mathbf{H}^d)} \leq \|\nabla_{\mathbf{H}} f\|_{L^p(\mathbf{H}^d)}. \tag{4.25}$$

Proof. We can prove the assertion for $f \in C^1_c(\mathbf{H}^d)$ as in Theorem 4.2.6.

To prove the general case, we proceed by approximation using Theorem 2.1.8. Let $(f_n)_{n\in\mathbb{N}} \subset C^1_c(\mathbf{H}^d)$ be a sequence such that

$$\lim_{n\to\infty} \|f_n - f\|_{L^p(\mathbf{H}^d)} = \lim_{n\to\infty} \|\nabla_{\mathbf{H}} f_n - \nabla_{\mathbf{H}} f\|_{L^p(\mathbf{H}^d)} = 0.$$

In particular, we have

$$\lim_{n\to\infty} \int_0^{+\infty} \int_{\mathbf{H}^d/R} \mathcal{H}^1\{\alpha \in \mathbb{S}^1 : |f_n(r_\alpha(\xi)) - f(r_\alpha(\xi))| > s^{1/p}\}\, d\xi\, ds = 0$$

and, up to subsequences, we have that for $\mathcal{H}^{2d}$-a.e. $\xi \in \mathbf{H}^d/R$ and for all $s > 0$

$$\lim_{n\to\infty} \mathcal{H}^1\{\alpha \in \mathbb{S}^1 : |f_n(r_\alpha(\xi)) - f(r_\alpha(\xi))| > s^{1/p}\} = 0.$$

We can also assume that $f_n \to f$ a.e. in $\mathbf{H}^d$. As in the proof of Theorem 4.2.6, we can then show that

$$\lim_{n\to\infty} \mathcal{H}^1\big(\{\alpha : f_n(r_\alpha(\xi)) > s\} \Delta \{\alpha : f(r_\alpha(\xi)) > s\}\big) = 0. \tag{4.26}$$

By the first part of the proof, we have

$$\|\nabla_{\mathbf{H}} f_n^*\|_{L^p(\mathbf{H}^d)} \leq \|\nabla_{\mathbf{H}} f_n\|_{L^p(\mathbf{H}^d)}.$$

It follows that, up to subsequences, the sequence $(f_n^*)_{n\in\mathbb{N}}$ converges weakly in $W^{1,p}_{\mathbf{H}}(\mathbf{H}^d)$ to a function g such that

$$\|\nabla_{\mathbf{H}} g\|_{L^p(\mathbf{H}^d)} \leq \liminf_{n\to\infty} \|\nabla_{\mathbf{H}} f_n^*\|_{L^p(\mathbf{H}^d)}.$$

We may also assume that $f_n^* \to g$ a.e.in $\mathbf{H}^d$. We claim that $g = f^*$.

For fixed $\xi \in \mathbf{H}^d/R$ and parametrizing $\mathbb{S}^1 = [-\pi, \pi)$, $f_n^*(r_\alpha(\xi))$ and $g(r_\alpha(\xi))$ depend only on $|\alpha|$. Then, with abuse of notation, we let $f_n^*(\xi, \alpha) = f_n^*(r_\alpha(\xi))$ and $g(\xi, \alpha) =$

$g(r_\alpha(\xi))$. The functions $\alpha \mapsto f_n^*(\xi,\alpha)$ and $\alpha \mapsto g(\xi,\alpha)$, $\alpha \in [-\pi,\pi)$, are then even and non-increasing for $\alpha \geq 0$. Thus the sets $I_n(\xi,s) = \{\alpha \in [-\pi,\pi) : f_n^*(\xi,\alpha) > s\}$ and $I(\xi,s) = \{\alpha \in [\pi,\pi) : g(\xi,\alpha) > s\}$ are essentially symmetric intervals. Then $I_n(\xi,s) \to I(\xi,s)$, in the natural sense, for $\mathcal{H}^{2d}$-a.e. $\xi \in \mathbf{H}^d/R$ and $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$. It follows that

$$\lim_{n\to\infty} \mathcal{L}^1\big(\{f_n^*(\xi,\cdot) > s\}\Delta\{g(\xi,\cdot) > s\}\big) = 0. \tag{4.27}$$

Let $\varphi : \mathbf{H}^d \to \mathbb{R}$ be a function such that $\varphi(r_\alpha(\xi))$ depends only on $|\alpha|$ for $\xi \in \mathbf{H}^d/R$, as above. We let $\varphi(\xi,\alpha) = \varphi(r_\alpha(\xi))$. Then we claim that it holds

$$\mathcal{H}^1\{\alpha \in \mathbb{S}^1 : \varphi(r_\alpha(\xi)) > s\} = 2|\xi|\mathcal{L}^1\{\alpha \in [-\pi,\pi) : \varphi(\xi,\alpha) > s\}, \quad \text{for a.e. } \xi \in \mathbf{H}^d/R \tag{4.28}$$

In fact, by a change of variables, we get that

$$\begin{aligned}
\mathcal{H}^1\{\alpha : \varphi(r_\alpha(\xi)) > s\} &= \int_{\mathbb{S}^1} \chi_{\{\alpha:\, \varphi(r_\alpha(\xi))>s\}}(\alpha)\, d\mathcal{H}^1(\alpha) \\
&= \int_{-\pi}^{\pi} \chi_{\{\varphi(\xi,\cdot)>s\}}(\alpha)|\xi|\, d\alpha \\
&= 2|\xi| \int_0^{\pi} \chi_{\{\varphi(\xi,\cdot)>s\}}(\alpha)\, d\alpha \\
&= 2|\xi|\mathcal{L}^1\{\varphi(\xi,\cdot) > s\}.
\end{aligned}$$

Here we used the fact that $\varphi(\xi,\alpha) = \varphi(\xi,-\alpha)$. By the same computations, we can also prove that (4.28) holds replacing $>$ with $\leq$.

Both f^* and g satisfies the assumptions on φ in the previous claim. Therefore (4.27) and (4.28) imply that

$$\lim_{n\to\infty} \mathcal{H}^1\big(\{\alpha : f_n^*(r_\alpha(\xi)) > s\}\Delta\{\alpha : g(r_\alpha(\xi)) > s\}\big) = 0 \tag{4.29}$$

From (4.26) and (4.29), we deduce that

$$\begin{aligned}
\mathcal{H}^1\{\alpha : g(r_\alpha(\xi)) > s\} &= \lim_{n\to\infty} \mathcal{H}^1\{\alpha : f_n^*(r_\alpha(\xi)) > s\} \\
&= \lim_{n\to\infty} \mathcal{H}^1\{\alpha : f_n(r_\alpha(\xi)) > s\} \\
&= \mathcal{H}^1\{\alpha : f(r_\alpha(\xi)) > s\}.
\end{aligned}$$

This implies that $g = f^*$ a.e. on $\mathbf{H}^d$, thus completing the proof. □

Theorem 4.3.10. *Let $E \subset \mathbf{H}^d$ be a σ-symmetric set of finite measure and $\mathbf{H}$-perimeter and let E^* be the cap rearrangement of E. Then E^* is of finite $\mathbf{H}$-perimeter and*

$$P_{\mathbf{H}}(E^*) \leq P_{\mathbf{H}}(E). \tag{4.30}$$

Proof. The proof is exactly the same as Theorem 4.2.7, upon using the approximation argument in the proof of Theorem 4.3.9. □

Appendix A

Notation

A.1 Measures and sets

$\overline{E}$	Closure of E.
∂E	Topological boundary of E.
χ_E	Characteristic function of the set E, i.e. $\chi_E(x) = 1$ if $x \in E$, otherwise $\chi_E(x) = 0$.
$\operatorname{diam} E$	$\sup_{x,y \in E} \|x - y\|$, diameter of the set E.
$\#$	Counting measure.
$\mathcal{L}^n$	d-dimensional Lebesgue measure.
$\mathcal{H}^s$	s- dimensional Hausdorff measure in $\mathbb{R}^d$.
ω_d	Lebesgue volume of the unit ball of $\mathbb{R}^d$.
$\|E\|$	Lebesgue measure of the set E.
$⨍_E f \, d\mu$	$\frac{1}{\|E\|} \int f \, d\mu$, averaged integral of f.

A.2 Functions

Id	Identity function.
$f\|_E$	Restriction of f to the set E.
∇f	Gradient of $f : \mathbb{R}^d \to \mathbb{R}$.
$\frac{\partial f_i}{\partial x_j}$	Partial derivative of $f : \mathbb{R}^d \to \mathbb{R}$ with respect to the variable x_j.
$\mathrm{Lip}(f)$	Lipschitz constant of $f : \mathbb{R}^d \to \mathbb{R}$, see (3.2).
$C^k(\mathbb{R}^d)$	$\{ f : \mathbb{R}^d \to \mathbb{R} : \ f$ is differentiable with continuity k times $\}$.
$\\| \cdot \\|_{L^p(X,\mu)}$	The L^p-norm in X with respect to the measure μ.

A.3 Finite dimensional Banach space

$\mathbb{R}^d$	d-dimensional Euclidean space.
e_i	i-th vector of the standard Euclidean basis, $e_i = (\delta_{ij})_{j=1}^d$.
$x = (x_1, \dots, x_d)$	Point of $\mathbb{R}^d$.

$x \cdot y$	$x_1y_1 + x_2y_2 + \ldots + x_ny_n$, Euclidean scalar product.
$\|x\|$	$\sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$, Euclidean norm.
$\\|\cdot\\|$	A norm in $\mathbb{R}^d$.
$\|\nabla f\|$	Euclidean total variation measure of $f : \mathbb{R}^d \to \mathbb{R}$.
$\\|\cdot\\|_{*,p}$	p-mean norm associated to $\\|\cdot\\|$, see (1.2).
$\\|\cdot\\|_{*,1}$	1-mean norm associated to $\\|\cdot\\|$.
$E(v)$	Vector associated to v defined in (1.4).
$K_{p,d}$	Geometric constant defined in (1.3).
B_r	$\{y \in \mathbb{R}^d \mid \\|y\\| < r\}$.
$B_r(x)$	$\{y \in \mathbb{R}^d : \\|x - y\\| < r\}$.
$B_r^{\pm}(x; v)$	$\{y \in B_r(x) : \pm(y - x) \cdot v \geq 0\}$.

A.4 Heisenberg group

$\mathbf{H}^d$	d-dimensional Heisenberg group.
$x = (z, t)$	Point of $\mathbf{H}^d$.
$x * y$	Heisenberg group non-commutative product.
x^{-1}	Inverse of a point, i.e. $x * x^{-1} = 0$.
$\bar{z}$	Complex conjugate of $z \in \mathbb{C}$.
$\mathrm{Re}(z)$, $\mathrm{Im}(z)$	Real and imaginary part of $z \in \mathbb{C}$.
$\triangle$	Horizontal distribution on $\mathbf{H}^d$.
T, X_j, Y_j	Vector fields spanning the Lie algebra of $\mathbf{H}^d$.
$L(\gamma)$	Length of a horizontal curve.
$d(\cdot, \cdot)$	Carnot-Carathéodory metric in $\mathbf{H}^d$.
δ_λ	The non-isotropic dilation $\delta_\lambda(z, t) = (\lambda z, \lambda^2 z)$.
$d_{\mathbf{H}}(\cdot, \cdot)$	A metric equivalent to the Carnot-Carathéodory metric, see Proposition 2.0.1.
π	Standard projection $\pi(z, t) = z$.
B_r	$\{y \in \mathbf{H}^d : d(0, y) < r\}$.
$B_r(x)$	$\{y \in \mathbf{H}^d : d(x, y) < r\}$.
$B_r^{\pm}(x; v)$	$\{y \in B_r(x) : \pm\pi(x^{-1} * y) \cdot v \geq 0\}$.
$C_{p,d}$	Geometric constant defined in (2.11).
$\nabla_{\mathbf{H}} f$	Horizontal gradient of a function $f : \mathbf{H}^d \to \mathbb{R}$, see (2.6).
$\|\nabla_{\mathbf{H}} f\|$	Horizontal total variation measure of $f : \mathbf{H}^d \to \mathbb{R}$.
ϱ	The reflection function defined in (4.1).

A.5 Metric measure space

$\phi_\# \mu$	Push-forward measure of μ with respect to ϕ.
$\\|\nabla f\\|^{\pm}_{L^p(X,\mu)}$	Quantity defined in Definition 3.1.1.
$P^-(E; X, d, \mu)$	Inferior perimeter of $E \subset X$, defined in Definition 3.1.2.

$\mathcal{R} = \{\mathcal{P}, \varrho\}$	A reflection system, see Definition 3.3.6.
$f_{\mathcal{R}}$, $E_{\mathcal{R}}$	Two-points rearrangements of f or E, see Definitions 3.3.11 and 3.3.12.
$Q_r(f, g)$	The quantity defined in (3.26).
$\mathcal{S}(X, \mu)$	$\{f : X \to [0, +\infty) :\ \mu\{f > t\} < +\infty$ for any $t > 0\}$.
$(\Gamma, (\mu_x)_{x \in X/\Gamma}, \bar{\mu})$	A rearrangement system, defined in Definition 3.4.24.
E_x	$E \cap \Gamma_x$, the x-section of $E \subset X$.
f^*, E^*	Rearrangements of f or E, see Definition 3.4.25.
$(\mathcal{R}, T)$, $(\mathcal{R}, T, G)$	A Steiner or a Schwarz system, see Definitions 3.6.30 and 3.6.32.

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