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Rectifiable sets and their characterization through tangent measures

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Chapter 1

Introduction

The notion of an *m-rectifiable set* plays a central role in Geometric Measure Theory. Indeed, $m-$rectifiable sets are a measure-theoretic generalization of C^1 $m-$dimensional submanifolds of $\mathbb{R}^n$.
This concept was first introduced by Besicovitch for 1-dimensional sets in the plane, then his work was extended by Federer to $m-$subsets of $\mathbb{R}^n$, with m an integer, and finally generalized by Marstrand to fractal sets in the plane whose Hausdorff dimension is any positive real number.

In order to appreciate the importance of these sets we mention the decomposition theorem for Borel sets with finite Hausdorff measure ([DL08], Theorem 5.7). We refer to [DL08], Definition 5.6, for the definition of a purely unrectifiable set:

Theorem 1: *Let B be a Borel set such that $\mathcal{H}^m(B) < \infty$. Then there exist two Borel sets $B^u, B^r \subset B$ such that*

- *B^r is rectifiable,*
- *B^u is purely unrectifiable,*
- *$B^u \cup B^r = B$*

and such a decomposition is unique up to sets with zero $\mathcal{H}^m$ measure.

Therefore it's useful to have some criteria to establish if a set is rectifiable or not. Rectifiable sets can be characterized in several different ways:

- through the existence of m-dimensional density (see [AFP00], Besicovitch-Marstrand-Mattila Theorem 2.63, where it is also proved that this density is equal to 1);
- in a more geometric way, using cones (see [DL08], Theorem 4.6);
- using the Besicovitch-Federer projection theorem ([MA95], Theorem 18.1), which instead characterize purely unrectifiable sets to be those sets with finite $\mathcal{H}^m$ measure and null projection on almost every m-dimensional linear plane;
- through tangent measures: this notion was introduced by Preiss in his work on rectifiable sets and the aim of this Thesis is to prove the following theorem, which explaines this characterization.

Theorem 2: *Let $B \subseteq \mathbb{R}^n$ be a Borel set such that $\mu = \mathcal{H}^m \llcorner B$ is a Radon measure, then the following statements are equivalent:*

(*i*) *B is $m-$rectifiable;*

(ii) For $\mu-$a.e. $x \in \mathbb{R}^n$ there exists $W_x \in \mathbb{G}(m, \mathbb{R}^n)$ such that

$$Tan^{(m)}(\mu, x) = \left\{\mathcal{H}^m \llcorner W_x\right\},$$

where $Tan^{(m)}(\mu, x)$ is the set of $m-$tangent measures of μ at x.

For the formal definition of $m-$tangent measure we refer to Definition 37.
The interesting aspect of this characterization is that it explains the description of rectifiable sets as "$m-$dimensional surfaces in Geometric Measure Theory". Indeed, the approximate tangent space to an m-rectifiable measure can be seen as a generalization of the tangent plane to a C^1 submanifold of the Euclidian space in Differential Geometry. Namely, let S be an $m-$dimensional C^1 submanifold of $\mathbb{R}^n$, $\mu = \mathcal{H}^m \llcorner S$ and $x \in S$. If we set

$$S_r = \left(\frac{S - x}{r}\right) = \left\{y \in \mathbb{R}^n : x + ry \in S\right\},$$

then we have

$$r^{-m}(T_{x,r})_* \, \mu = \mathcal{H}^m \llcorner S_r$$

where $T_{x,r}(y) = \dfrac{y - x}{r}$ and $(T_{x,r})_*\mu$ is the push-forward measure.
To get a tangent measure we let r tend to 0. But, since S is C^1, as $r \searrow 0$ the sets S_r converge to the tangent plane T_x to S at x.

In Chapter 2 we see some preliminary definitions for the work of the following chapters: the basic notions of measure theory and of Hasudorff measures, the elementary properties of Lipschitz functions, the definition of weak* convergence for Radon measures. In particular, in Section 2.5 we prove the Area Formula, a result that will be essential for the proof of Theorem 2.

In Chapter 3 we define the notion of m-rectifiable set, adopting the terminology first introduced by Federer in [FE69].
Then we define tangent measures using the weak* convergence for Radon measures. Therefore, in Section 3.2 we show that these two concepts are linked. Indeed, Theorem 2 states a necessary and sufficient condition for a subset B of $\mathbb{R}^n$ to be $m-$rectifiable involving tangents measures: given a Radon measure μ, a set is m-rectifiable if and only if the tangent measure to μ at almost all its points is the m-dimensional Hausdorff measure restricted to an appropriate m-dimensional subspace of $\mathbb{R}^n$. In particular, before starting with the proof we see some properties of the function defined in the support of the measure μ that associetes to each point x a m-plane W_x in $\mathbb{R}^n$ with the property that the tangent measure to μ in x is the $m-$Hausdorff measure restrected to that affine subspace of $\mathbb{R}^n$.

Finally, in Chapter 4, we apply Theorem 2 to sets of finite perimeter: we prove that the reduced boundary of a set E with finite perimeter is $(n-1)-$ rectifiable by proving that the Radon measure $\mu = ||D1_E||$ is $(n-1)-$ rectifiable.

Chapter 2

Preliminaries and notation

In this Chapter we introduce the basic definitions and results of measure theory; we mainly follow [EG92], Section 1.1, part of 1.5, something from 1.7 and 1.8, 1.9, a great part of Chapter 2, Sections 3.1, 3.2, 3.3.

However, Theorem 16 is Theorem 3.1.16 in [FE69] and Theorem 22 is the second part of Theorem 3.2 in [SI84].

2.1 Notation and Basic Notions of Measure Theory

Given $x \in \mathbb{R}^n$ and $r > 0$ we denote by $U(x,r)$ and $B(x,r]$ the open and closed ball of center x and radius r, while we use $\partial B(x,r)$ to refer to its boundary.
In general, for any set $A \subset \mathbb{R}^n$ we denote by $\bar{A}$ its topological closure and by ∂A its topological boundary.

Next, we recall the basic definitions of measure theory. Let X be a set (in our later argument we will have $X = \mathbb{R}^n$):

Definition 1. A family of sets $\mathcal{A} \subset \mathcal{P}(X)$ is called **σ-algebra** if:

(i) $\emptyset, X \in \mathcal{A}$;

(ii) if $A \in \mathcal{A}$, then $X \setminus A \in \mathcal{A}$

(iii) if $\{A_k\}_{k\in\mathbb{N}}$, then $\displaystyle\bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$.

In particular, if (X,τ) is a topological space, we define $\mathcal{B}(X)$, the **Borel σ-algebra of** (X,τ), as the smallest σ-algebra on X which contains all open sets (i.e., $\tau \subset \mathcal{B}(X)$). For instance, when we consider the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$, we refer to $\mathbb{R}^n$ endowed with its standard topology generated by the family of open balls $\{B(x,r) : x \in \mathbb{R}^n, r \in \mathbb{R}_{>0}\}$.

Definition 2. A mapping $\mu : \mathcal{A} \longrightarrow [0,\infty]$ is a **measure on the σ-algebra** $\mathcal{A}$ if:

(i) $\mu(\emptyset) = 0$;

(ii) if $\{A_k\}_{k\in\mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{A}$, then $\displaystyle\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$.

Then $(X, \mathcal{A}, \mu)$ is a measure space.

Definition 3. A set $A \subset X$ is $\mu-$measurable if for each set $B \subset X$,

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A).$$

It's useful to extend the notion of measurability from sets to functions:

Definition 4. Let $(X, \mathcal{A}, \mu)$ be a measure space, (Y, τ) be a topological space and $f : X \longrightarrow Y$ be a function. We say that f is μ-measurable if $f^{-1}(U)$ is $\mu-$measurable for all $U \in \tau$.

Definition 5. Given two measure spaces $(X, \mathcal{M})$, $(Y, \mathcal{N})$, where $\mathcal{N}$ is the Borel $\sigma-$algebra of Y, a measurable function $f : X \to Y$ and a measure μ defined on $\mathcal{M}$, the **pushforward of** μ is the measure $f_*(\mu)$ on Y given by:

$$\big(f_*(\mu)\big)(B) = \mu\big(f^{-1}(B)\big), \quad \text{for} B \in \mathcal{N}.$$

Definition 6. Let $(X, \mathcal{A}, \mu)$ be a measure space, we say that μ is:

- **finite** if $\mu(X) < \infty$;
- σ**- finite** if there exists a family of μ-measurable sets $\{A_k\}_{k \in \mathbb{N}}$ such that $\mu(A_k) < \infty$ and $X = \bigcup_{k=1}^{\infty} A_k$.

Moreover, if there is a topology on X, μ is said to be

- a **Borel measure** if $\mathcal{B}(X) \subset \mathcal{A}$;
- a **Borel regular measure** if it is a Borel measure and for any $A \subset X$ there exists $B \in \mathcal{B}(X)$ so that $A \subset B$ and $\mu(A) = \mu(B)$;
- a **Radon measure** if it is Borel regular and $\mu(K) < \infty$ for each K compact subset of X.

If $E \subset X$ and μ is a measure on X, we denote by $\mu \llcorner E$ the measure defined by

$$\big(\mu \llcorner E\big)(A) := \mu(A \cap E).$$

From now on, we will consider the case $X = \mathbb{R}^n$.

2.2 Hausdorff Measures on $\mathbb{R}^n$

Idea: We want to define in $\mathbb{R}^n$ some "lower dimensional" measures in order to be able to associate a measure and a dimension (not necessarily in $\mathbb{N}$) also to subsets of $\mathbb{R}^n$ which are not n-dimensional. For this reason we introduce on $\mathbb{R}^n$ the so called Hausdorff measures: these measures are defined in terms of the diameters of proper coverings and we say that a set $A \subset \mathbb{R}^n$ is a m-dimensional subset of $\mathbb{R}^n$ if $0 < \mathcal{H}^m(A) < \infty$ for some $0 \leq m \leq n$.

Definition 7. Let $A \subset \mathbb{R}^n$,

(i) if $0 \le m < \infty$, $0 < \delta \le \infty$ we define

$$\mathcal{H}^m_\delta(A) \equiv \inf\left\{ \sum_{j=1}^{\infty} \alpha(m)\Big(\frac{\mathrm{diam}(C_j)}{2}\Big)^m : A \subset \bigcup_{j=1}^{\infty} C_j, \mathrm{diam} C_j \le \delta \right\},$$

where

$$\alpha(m) \equiv \frac{\pi^{m/2}}{\Gamma\big(\frac{m}{2}+1\big)},$$

and the gamma function is defined by $\Gamma(m) \equiv \int_0^\infty e^{-x} x^{m-1} dx$, for $0 < m < \infty$.

(ii)

$$\mathcal{H}^m(A) \equiv \lim_{\delta \to 0} \mathcal{H}^m_\delta(A) = \sup_{\delta > 0} \mathcal{H}^m_\delta(A)$$

and we call it m**-dimensional Hausdorff measure** on $\mathbb{R}^n$. b

Therefore the **Hausdorff dimension** of a set $A \subset \mathbb{R}^n$ is defined to be

$$\dim_{\mathcal{H}}(A) \equiv \inf\{0 \le s < \infty : \mathcal{H}^s(A) = 0\} = \sup\{0 \le s < \infty : \mathcal{H}^s(A) = \infty\}.$$

We observe that $\dim_{\mathcal{H}}(A) \le n$ since $\mathcal{H}^s \equiv 0$ on $\mathbb{R}^n$ if $s > n$.

b But is there any relation between the definition of $\mathcal{L}^n$ as the n-fold product of $\mathcal{L}^1$ and $\mathcal{H}^n$ computed in terms of coverings of arbitrary small diameter? The following important theorem guarantees that equality holds:

Theorem 8. *$\mathcal{H}^n = \mathcal{L}^n$ on $\mathbb{R}^n$.*

As for the proof we refer to [EG92], section 2.2, we only remark that this proof is based on the Isodiametric Inequality:

Theorem 9 (Isodiametric Inequality)**.** *For all sets $A \subset \mathbb{R}^n$,*

$$\mathcal{L}^n(A) \le \alpha(n)\Big(\frac{diam(A)}{2}\Big)^n.$$

Therefore it's useful to recall that $\mathcal{H}^m$, $0 \le m < \infty$, is a Borel regular measure on $\mathbb{R}^n$, but is not a Radon measure if $0 \le m < n$, since $\mathbb{R}^n$ is not σ- finite with respect to $\mathcal{H}^m$ ([EG92], section 2.1, Theorem 1).
However, if B is a Borel measurable set such that $\mathcal{H}^m(B \cap K) < \infty$, for all $K \subset \mathbb{R}^m$ compact, $\mathcal{H}^m \llcorner B$ is a Radon mesure on $\mathbb{R}^m$.

Finally, we introduce some useful results about densities of Hausdorff measures. For the Lebesgue measure, there is a theorem which states that, given $E \subset \mathbb{R}^n$ $\mathcal{L}^n$-measurable, the set of points in E whose neighbourhood is partially in E and partially outside of E has Lebesgue measure equal to 0 (for the proof of this theorem see [EG92], Section 1.7):

Theorem 10 (Lebesgue-Besicovitch Differentiation Theorem)**.** *Let $E \subset \mathbb{R}^n$ be $\mathcal{L}^n-$ measurable, then*

$$\lim_{r \to 0} \frac{\mathcal{L}^n\big(B(x,r) \cap E\big)}{\alpha(n) r^n} = \begin{cases} 1 & \text{for } \mathcal{L}^n \text{a.e. } x \in E, \\ 0 & \text{for } \mathcal{L}^n \text{a.e. } x \in \mathbb{R}^n \setminus E. \end{cases}$$

Therefore the concept of density was generalized by Besicovitch to a general Radon measure in $\mathbb{R}^n$: in particular, in the above hypothesis on the set B (i.e., B is a Borel measurable set with the property that $\mathcal{H}^m(B \cap E) < \infty$ for all $K \subset \mathbb{R}^m$ compact), the following result holds:

Proposition 11. *(i) For $\mathcal{H}^m$- a.e. $x \in B$*

$$\frac{1}{2^m} \leq \Theta^m_*(\mathcal{H}^m \llcorner B, x) \leq \Theta^{m,*}(\mathcal{H}^m \llcorner B, x) \leq 1,$$

where $\Theta^m_*(\mathcal{H}^m \llcorner B, x)$ and $\Theta^{m,*}(\mathcal{H}^m \llcorner B, x)$ are the densities defined by

$$\Theta^m_*(\mathcal{H}^m \llcorner B, x) = \liminf_{r \to 0} \frac{\mathcal{H}^m\big(B(x,r) \cap B\big)}{\alpha(m)\, r^m}$$

and

$$\Theta^{m,*}(\mathcal{H}^m \llcorner B, x) = \limsup_{r \to 0} \frac{\mathcal{H}^m\big(B(x,r) \cap B\big)}{\alpha(m)\, r^m}.$$

(ii) If $B \subset \mathbb{R}^n$, B is $\mathcal{H}^m-$measurable and $\mathcal{H}^m(B) < \infty$, for $\mathcal{H}^m-$a.e. $x \notin B$

$$\Theta^m(\mathcal{H}^m \llcorner B, x) = \lim_{r \to 0} \frac{\mathcal{H}^m\big(B(x,r) \cap B\big)}{\alpha(m)\, r^m} = 0.$$

As far as for its proof we refer to [EG92], Section 2.3; we just observe that it is based on one of the two fundamental covering theorems (the other one is Besicovitch's covering theorem, [EG92], Section 1.5):

Theorem 12 (Vitali's Covering Theorem)**.** *Let $\mathcal{F}$ be any collection of nondegenerate closed balls in $\mathbb{R}^n$ with*

$$\sup\{\operatorname{diam} B \,|\, B \in \mathcal{F}\} < \infty.$$

Then there exists a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in \mathcal{G}} \hat{B},$$

where $\hat{B}$ denotes the concentric closed ball with radius 5 times the radius of B.

Proof: We refer to [EG92], Section 1.5.2, Theorem 2. $\square$

2.3 Lipschitz functions

We recall that a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called **Lipschitz** if there exists a constant C such that

$$|f(x) - f(y)| \leq C|x - y| \text{ for all } x, y \in \mathbb{R}^n.$$

Therefore, we can define the **Lipschitz constant of** f

$$\operatorname{Lip}(f) \equiv \sup\left\{\frac{|f(x) - f(y)|}{|x - y|} \,:\, x, y \in \mathbb{R}^n, x \neq y\right\}.$$

Definition 13 (Differentiability)**.** Let $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a function and x be a point in $\mathbb{R}^m$. We say that f **is differentiable at** x if there exists a linear map

$$L : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

such that

$$\lim_{y \to x} \frac{|f(y) - f(x) - L(x - y)|}{|x - y|} = 0,$$

which is equivalent to require

$$f(y) = f(x) + L(y - x) + o(|y - x|) \quad \text{as} \quad y \to x.$$

If such a map exists, it's unique and we indicate it with $Df(x)$, the **derivative or differential** of f at x.

There is an important link between Lipschitz and differentiable functions, as Rademacher's theorem states:

Theorem 14 (Rademacher's Theorem)**.** *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a (locally) Lipschitz function. Then f is (Fréchet) differentiable at $\mathcal{L}^n-$almost every $x \in \mathbb{R}^n$, i.e., by Definition 13, for $\mathcal{L}^n-$a.e. $x \in \mathbb{R}^n$ there exists a linear map $Df : \mathbb{R}^n \to \mathbb{R}^m$ such that:*

$$\lim_{y \in \mathbb{R}^n, y \to x} \frac{|f(y) - f(x) - Df(x) \cdot (y - x)|}{|y - x|} = 0.$$

Proof: We refer to [EG92], Section 3.1.2, Theorem 2. $\square$

Now we introduce two extension theorems: the first one ensures the possibility to extend a Lipschitz map f which is defined in a subset $A \subset \mathbb{R}^n$ to a Lipschitz map $\bar{f}$ defined in the whole space, without enlarging the Lipschitz constant, while the second one states states that this function f is as closer as we want to a function defined on the whole space $\mathbb{R}^n$ which is of class C^1.

Theorem 15 (Extension Theorem for Lipschitz Functions)**.** *Let A be a subset of $\mathbb{R}^n$ and $f : A \longrightarrow \mathbb{R}^m$ be Lipschitz. Then there exists a Lipschitz function $\bar{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that:*

(1) $\bar{f} = f$ on A,

(2) Lip$(\bar{f})$ = Lip(f).

Proof: *Step 1:* We first assume $m = 1$, $f : A \to \mathbb{R}$. Define:

$$\bar{f}(x) \equiv \inf_{a \in A} \{f(a) + \mathrm{Lip}(f)|x - a|\}.$$

Choose $b \in A$: by the definition of $\bar{f}$ we have $\bar{f}(b) \leq f(b)$, but also, by the definition of Lipschitz function:

$$f(b) \leq f(a) + \mathrm{Lip}(f)|b - a|, \text{ for all } a \in A$$

and so $f(b) \leq \bar{f}(b)$. Now, if $x, y \in \mathbb{R}^n$, we have, using the triangular inequality

$$\bar{f}(x) \leq \inf_{a \in A} \{f(a) + \mathrm{Lip}(f)(|y - a| + |x - y|)\} = \bar{f}(y) + \mathrm{Lip}(f)|x - y|$$

and, changing the role of x and y, we have also $\bar{f}(y) \leq \bar{f}(x) + \mathrm{Lip}(f)|x - y|$.

Step 2: If $f : A \to \mathbb{R}^m$, $f = (f^1, \ldots, f^m)$, we define $\bar{f} \equiv (\bar{f}^1, \ldots, \bar{f}^m)$. Now we have found a function $\bar{f}$ such that

$$|\bar{f}(x) - \bar{f}(y)|^2 = \sum_{i=1}^{m} |\bar{f}^i(x) - \bar{f}^i(y)|^2 \leq m(\mathrm{Lip}(f))^2 |x-y|^2.$$

Step 3: For the proof that the extension map $\bar{f}$ can be chosen in such a way that $\mathrm{Lip}(\bar{f}) = \mathrm{Lip}(f)$ we refer to Kirszbraun's theorem ([FE69], Theorem 3.1.16). □

Theorem 16 (Whitney's extension theorem). *Let A be a subset of $\mathbb{R}^n$ and $f : A \longrightarrow \mathbb{R}^m$ be a Lipschitz function. For every $\epsilon > 0$ there exists a function $\bar{f} \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ such that $\mathcal{L}^n\big(\{x \in A : f(x) \neq \bar{f}(x)\}\big) < \epsilon$.*

Proof: See [FE69], Theorem 3.1.16.

Definition 17. (Jacobian of a linear map) Assume $L : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is linear.

(i) If $m \leq n$, by Polar Decomposition Theorem we can write $L = O \circ S$, where $S : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is a symmetric map and $O : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is an orthogonal map, and we define the **Jacobian** of L to be

$$[\![L]\!] = |\det S|.$$

(ii) If $m \geq n$, by Polar Decomposition Theorem we can write $L = S \circ O^\star$, where $S : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a symmetric map and $O : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is an orthogonal map, and we define the **Jacobian** of L to be

$$[\![L]\!] = |\det S|.$$

Definition 18. Let $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a Lipschitz map. The **Jacobian of** f is

$$Jf(x) \equiv [\![Df(x)]\!] \qquad \text{for } \mathcal{L}^m \text{a.e.} \quad x.$$

Remark 19. Here we have implicitly used Theorem 15 to extend the function from A to the whole $\mathbb{R}^m$ and Theorem 14, which ensures the differentiability of the function f a.e. and so the existence of Jf a.e.

Another technical result that we are going to use to prove the Area Formula and Lemma 40 is:

Lemma 20. *Let $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$, $m \leq n$.*
Let $t > 1$ and the Borel set $A \equiv \{x |\ Df(x)$ exists, $Jf(x) > 0\}$. Then there is a countable collection $\{E_k\}_{k=1}^\infty$ of Borel subsets of $\mathbb{R}^m$ such that

(i) $A = \bigcup_{k=1}^\infty E_k$;

(ii) $f|_{E_k}$ *is injective, for all* $k \in \mathbb{N}$;

(iii) for each $k \geq 1$ there exists a symmetric automorphism $T_k : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ such that

$$Lip\big((f|_{E_k}) \circ T_k^{-1}\big) \leq t, \qquad Lip\big(T_k \circ (f|_{E_k})^{-1}\big) \leq t,$$

$$t^{-m} |\det T_k| \leq Jf|_{E_k} \leq t^m |\det T_k|.$$

Proof: See [EG92], Section 3.3.1, Lemma 3. □

Next we recall the following theorem, which relates the $m-$measure of the image of a subset $A \subset \mathbb{R}^n$ via a Lipschitz function to the $m-$measure of A itself:

Theorem 21. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be Lipschitz, $A \subset \mathbb{R}^n, 0 \leq m < \infty$. Then*

$$\mathcal{H}^m(f(A)) \leq (Lip(f))^m \mathcal{H}^m(A).$$

Proof: See [EG92], Section 2.4.1, Theorem 1. $\square$

In particular, if $n > k$ and $P : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is the canonical projection, for each $A \subset \mathbb{R}^n$ and $0 \leq m < \infty$ we have:

$$\mathcal{H}^m(P(A)) \leq \mathcal{H}^m(A),$$

since $\mathrm{Lip}(P) = 1$.

Finally we remark the following theorem wich gives a link between the Hausdorff measure and a Borel-regular measure:

Theorem 22. *If μ is a Borel-reguler measure on $\mathbb{R}^n$ and $t \geq 0$ is such that if $A \subset B$ and $\Theta^{*,m}(\mu, B, x) \leq t$ for all $x \in A$, then*

$$t\,\mathcal{H}^n(A) \leq \mu(B).$$

Proof: We refer to [SI84], Theorem 3.2-(1). $\square$

2.4 Weak* Convergence of Radon Measures

First of all, we recall the following general construction. If X is a Banach space, it is possible to define the **weak* topology** on its dual X' as the weakest topology on X' which makes continuous the family of applications $(\varphi_x)_{x\in X}$, where for each $x \in X$ the map $\varphi_x : X' \longrightarrow \mathbb{R}$ is defined by $f \mapsto \varphi_x(f) =< f, x >= f(x)$ ($< \cdot, \cdot >$ is the duality between X and X').

In the same way, we define the weak* convergence of measures. Indeed, if we endow the space of continuous functions with compact support $C_c(\mathbb{R}^n; \mathbb{R}^m)$ with the topology of uniform convergence on compact sets (i.e., given a sequence of functions $\{\varphi_j\}_{j\in\mathbb{N}} \subset C_c(\mathbb{R}^n; \mathbb{R}^m)$ we say that $\varphi_j \to \varphi$ uniformly on compact sets if there exists a compact set K so that $\mathrm{supp}(\varphi_j) \subset K$ for every j and $\varphi_j \to \varphi$ uniformly). The space $C_c(\mathbb{R}^n; \mathbb{R}^m)$ endowed with this topology is a Banach space.

Now, let μ be a Radon measure. Then the functional

$$\varphi \to \int_{\mathbb{R}^n} \varphi d\mu$$

is continuous and linear on $C_c(\mathbb{R}^n; \mathbb{R}^m)$. Conversely, we have the following theorem:

Theorem 23 (Riesz' Representation Theorem)**.** *Let $L : C_c(\mathbb{R}^n; \mathbb{R}^m) \to \mathbb{R}$ be a linear functional satisfying*

$$\sup\{L(\varphi) : \varphi \in C_c(\mathbb{R}^m; \mathbb{R}^m), ||\varphi||_\infty \leq 1, supp(f) \subset K\} < \infty$$

for each compact set $K \subset \mathbb{R}^n$. Then there exists a Radon measure μ on $\mathbb{R}^n$ and a $\mu-$measurable function $\sigma : \mathbb{R}^n \to \mathbb{R}^m$ such that

(i) $|\sigma(x)| = 1$ for $\mu-$almost every x, and

(ii) $L(\varphi) = \int \varphi \cdot \sigma d\mu$, *for all* $\varphi \in C_c(\mathbb{R}^n; \mathbb{R}^m)$, *where* $\varphi \cdot \sigma$ *is the usual Euclidean scalar product in* $\mathbb{R}^m$, *that is* $\varphi(x) \cdot \sigma(x) = \sum_{i=1}^{m} \varphi_i(x)\sigma_i(x)$.

Proof: We refer to [EG92], Section 1.8, Theorem 1.

This allows us to endow the space of Radon measures with the topology of the dual space of $C_c(\mathbb{R}^n)$ and to give the following definition:

Definition 24. Let $\{\mu_j\}_{j\in\mathbb{N}}$ be a sequence of Radon measures on $\mathbb{R}^n$. We say that μ_j **converges weakly* to** μ, $\mu_j \underset{k\to\infty}{\overset{*}{\rightharpoonup}} \mu$, if

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} \varphi \, d\mu_j = \int_{\mathbb{R}^n} \varphi \, d\mu$$

for all $\varphi \in C_c(\mathbb{R}^n)$.

We recall also the following theorem:

Theorem 25. *Let* $\{\mu_j\}_{j\in\mathbb{N}}, \mu$ *be Radon measures on* $\mathbb{R}^n$. *The following statements are equivalent:*

(1) $\lim_{j\to\infty} \int_{\mathbb{R}^n} \varphi(x)\, d\mu_j = \int_{\mathbb{R}^n} \varphi(x)\, d\mu$ *for all* $\varphi \in C_c(\mathbb{R}^n)$*;*

(2)
- $\limsup_{j\to\infty} \mu_j(K) \leq \mu(K)$ *for any* $K \subset \mathbb{R}^n$ *compact,*
- $\liminf_{j\to\infty} \mu_j(A) \geq \mu(A)$ *for any* $A \subset \mathbb{R}^n$ *open;*

(3) $\lim_{j\to\infty} \mu_j(B) = \mu(B)$ *for any bounded Borel set* $B \subset \mathbb{R}^n$ *such that* $\mu(\partial B) = 0$.

Remark 26. Sometimes Theorem 25 is called Portmanteau's Theorem.

Proof: (1) $\Rightarrow$ **(2)** Let $K \subset \mathbb{R}^n$ be a compact set and $A \subset \mathbb{R}^n$ be an open set with the property that $K \subset A$. There exists $\varphi \in C_c(\mathbb{R}^n)$ so that $0 \leq f \leq 1$, $\varphi = 1$ on K and $\varphi = 0$ on $\mathbb{R}^n \setminus A$. Then

$$\limsup_{j\to\infty} \mu_j(K) \leq \lim_{j\to\infty} \int_{\mathbb{R}^n} \varphi(x)\, d\mu_j = \int_{\mathbb{R}^n} \varphi(x)\, d\mu \leq \mu(A).$$

In order to obtain the first claim it suffices to recall that $\mu(K) = \inf_{K \underset{\text{open}}{\subset} A} \mu(A)$.

With the same argument we prove also the second statement.

(2) $\Rightarrow$ **(1)** Let $B \subset \mathbb{R}^n$ be a bounded Borel set such that $\mu(\partial B) = 0$. Then

$$\mu(B) = \mu(\mathrm{int}(B)) \leq \liminf_{j\to\infty} \mu_j(\mathrm{int}(B)) \leq \limsup_{j\to\infty} \mu_j(\bar{B}) \underset{\bar{B}\text{compact}}{\leq} \mu(\bar{B}) = \mu(B).$$

Therefore we obtain the existence of the limit and the equality

$$\lim_{j\to\infty} \mu_j(B) = \mu(B).$$

(3)$\Rightarrow$**(1)** Let $\varphi \in C_c(\mathbb{R}^n)$. Without loss of generality we can assume $\varphi \geq 0$. Next we choose $R > 0$ so that $\mu\big(\partial B(0,R)\big) = 0$ and $\mathrm{supp}(\varphi) \subset B(0,R)$. Moreover we fix $\epsilon > 0$ and we take $0 = t_0 < t_1 < \cdots < t_N = 2||\varphi||_\infty$ be such that:

- $t_i - t_{i-1} < \epsilon$, for all $i = 1, \ldots, N$;

- $\mu\big(\varphi^{-1}(\{t_i\})\big) = 0$, for all $i = 1, \dots, N$.

The sets $B_i = \varphi^{-1}\big((t_{i-1}, t_i]\big)$, $\forall\, i = 1, \dots, N$ are bounded, Borel and $\mu(\partial B_i) = 0$ for $i \geq 2$.
Therefore:

$$\sum_{i=2}^{N} t_{i-1}\mu(B_i) \leq \int_{\mathbb{R}^n} \varphi(x)d\mu = \int_{\cup_{i=1}^N B_i} \varphi(x)d\mu = \sum_{i=1}^{N} \int_{B_i} \varphi(x)\, d\mu \leq$$
$$\leq \sum_{i=1}^{N} t_i\mu(B_i) \leq \epsilon\mu(B(0,R)) + \sum_{i=2}^{N} t_i\mu(B_i).$$

With the same argument we get

$$\sum_{i=2}^{N} t_{i-1}\mu_j(B_i) \leq \int_{\mathbb{R}^n} \varphi(x)d\mu_j \leq \epsilon\mu_j(B(0,R)) + \sum_{i=2}^{N} t_i\mu_j(B_i).$$

Next we take the difference

$$\sum_{i=2}^{N} t_{i-1}\mu(B_i) - \sum_{i=2}^{N} t_{i-1}\mu_j(B_i) \leq \int_{\mathbb{R}^n} \varphi(x)d\mu - \int_{\mathbb{R}^n} \varphi(x)d\mu_j \leq$$
$$\leq \Big(\epsilon\mu(B(0,R)) + \sum_{i=2}^{N} t_i\mu(B_i)\Big) - \Big(\epsilon\mu_j(B(0,R)) + \sum_{i=2}^{N} t_i\mu_j(B_i)\Big)$$

which can be rewritten as

$$\sum_{i=2}^{N} t_i\mu_j(B_i) - \sum_{i=2}^{N} t_{i-1}\mu_j(B_i) + \epsilon\mu_j(B(0,R)) \leq \int_{\mathbb{R}^n} \varphi(x)d\mu - \int_{\mathbb{R}^n} \varphi(x)d\mu_j \leq$$
$$\leq \sum_{i=2}^{N} t_i\mu(B_i) - \sum_{i=2}^{N} t_{i-1}\mu(B_i) + \epsilon\mu(B(0,R)).$$

At this point we let $j \to \infty$ and we obtain

$$\limsup_{j\to\infty} \Big| \int_{\mathbb{R}^n} \varphi(x)d\mu - \int_{\mathbb{R}^n} \varphi(x)d\mu_j \Big| \leq \epsilon\mu(B(0,R)) + \sum_{i=2}^{N} (t_i - t_{i-1})\, \mu(B_i)$$
$$\leq \epsilon\mu(B(0,R)) + \epsilon \sum_{i=2}^{N} \mu(B_i) \leq 2\epsilon\mu(B(0,R)).$$

In order to prove that $\limsup = \lim = 0$, it suffices to let $\epsilon \to 0$. $\square$

Theorem 25 gives us other two criterions in order to establish if a sequence $\{\mu_j\}_{j\in\mathbb{N}}$ of Radon measures on $\mathbb{R}^n$ converges weakly* to a Radon measure μ on $\mathbb{R}^n$: one of the three statements of Theorem 25 must hold.

Finally, we recall the theorem which ensures the sequential compactness for the weak notion of convergence:

Theorem 27. *Let $(\mu_k)_{k\in\mathbb{N}}$ be a sequence of outer Radon measures in $\mathbb{R}^n$ such that for any compact set $K \subset \mathbb{R}^n$ we have $\sup_{k\in\mathbb{N}} \mu_k(K) < +\infty$.*
Then there exists a subsequence $(\mu_{k_j})_{j\in\mathbb{N}}$ and a Radon outer measure in $\mathbb{R}^n$ such that

$$\mu_{k_j} \underset{j\to\infty}{\overset{*}{\rightharpoonup}} \mu.$$

Proof: See [EG92], Theorem 2, Section 1.9. $\square$

2.5 Area Formula

In the proof of Theorem 42 the following result will play a central role:

Theorem 28 (Area Formula)**.** *Let $A \subset \mathbb{R}^m$ be a $\mathcal{L}^m$- measurable set and $f : A \longrightarrow \mathbb{R}^n$ Lipschitz, $m \leq n$, then*

$$\int_{\mathbb{R}^n} \mathcal{H}^0\big(A \cap f^{-1}\{y\}\big)\, d\mathcal{H}^m(y) = \int_A Jf\, d\mathcal{L}^m.$$

Remark 29. We note that if $f : A \longrightarrow \mathbb{R}^n$ is also injective we have the identity $\mathcal{H}^m(f(A)) = \int_{\mathbb{R}^n} \mathcal{H}^0\big(A \cap f^{-1}\{y\}\big)\, d\mathcal{H}^m(y)$ and so the area formula can be written as

$$\mathcal{H}^m(f(A)) = \int_A Jf\, d\mathcal{L}^m.$$

Proof of the Area Formula. By Rademacher's Theorem we can assume that $Df(x)$ and $Jf(x)$ exist for all $x \in A$. Moreover, without loss of generality, we can also assume $\mathcal{L}^m(A) < \infty$.

First of all we introduce the following lemma, which gives a sense to the integral $\int_{\mathbb{R}^n} \mathcal{H}^0\big(A \cap f^{-1}\{y\}\big)\, d\mathcal{H}^m(y)$:

Lemma 30. *Let $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a Lipschitz map and $A \subset \mathbb{R}^m$ be $\mathcal{L}^m$-measurable. Then:*

(i) $f(A)$ is $\mathcal{H}^m$- measurable,

(ii) the mapping $y \longmapsto \mathcal{H}^0\big(A \cap f^{-1}\{y\}\big)$ is $\mathcal{H}^m$-measurable on $\mathbb{R}^n$,

(iii) $\int_{\mathbb{R}^n} \mathcal{H}^0\big(A \cap f^{-1}\{y\}\big)\, d\mathcal{H}^m(y) \leq (Lip(f))^m \mathcal{L}^m(A)$.

Case 1. $A \subset \{Jf > 0\}$ Fix $t > 1$ and take Borel disjoint sets $\{E_k\}_{k=1}^{\infty}$ as in Lemma 20 (in particular $A = \cup_{k=1}^{\infty} E_j$).

Then define the family $\mathcal{B}_k = \{Q |\ Q = (a_1, b_1] \times \cdots \times (a_m, b_m], a_i = \frac{c_i}{k}, b_i = \frac{c_i+1}{k}, c_i \in \mathbb{Z}, i = 1, 2, \ldots, m\}$ and set $\forall\, Q_i \in \mathcal{B}_k, \forall\, j = 1, 2, \ldots$

$$F_j^i = E_j \cap Q_i \cap A.$$

It's still true that the sets F_j^i are disjoint and $A = \cup_{i,j=1}^{\infty} F_j^i$, since $E_j = \cup_{i=1}^{\infty} F_j^i$.
Step 1: Claim:

$$\lim_{k\to\infty} \sum_{i,j=1}^{\infty} \mathcal{H}^m(f(F_j^i)) = \int_{\mathbb{R}^n} \mathcal{H}^0\big(A \cap f^{-1}\{y\}\big)\, d\mathcal{H}^m(y).$$

Proof of the Claim: Define

$$g_k \equiv \sum_{i,j=1}^{\infty} \chi_{f(F_j^i)}.$$

Clearly $g_k(y)$ is measurable for all k since it is the characteristic function of a countable union of $\mathcal{H}^m$−measurable sets. Indeed $F_j^i = E_j \cap Q_i \cap A$ is the intersection of three $\mathcal{L}^m$−measurable sets (in fact we have picked A measurable, while E_j are Borel sets, and so $\mathcal{L}^m$− measurable, but also each Q_i, being a cube in $\mathbb{R}^m$, is $\mathcal{L}^m$−measurable) and by (i) of the previous lemma we find that $f(F_j^i)$ is $\mathcal{H}^m$− measurable.

In particular we observe that $g_k(y)$ is exactly the number of sets F_j^i such that $y \in f(F_j^i)$, that is $F_j^i \cap f^{-1}\{y\} \neq \emptyset$. Since $A = \cup_{i,j=1}^{\infty} F_j^i$, $g_k(y) \underset{k\to\infty}{\nearrow} \mathcal{H}^0\big(A \cap f^{-1}\{y\}\big)$. Now, since

$$0 \leq g_1(y) \leq g_2(y) \leq \dots \quad \forall y \in \mathbb{R}^n$$

and $\lim_{k\to\infty} g_k(y) = \mathcal{H}^0\big(A \cap f^{-1}\{y\}\big)$, we can apply the Monotone Convergence Theorem to obtain the result.

Step 2: Observe that we can write

$$\mathcal{H}^m(f(F_j^i)) = \mathcal{H}^m(f|_{E_j} \circ T_j^{-1} \circ T_j(F_j^i))$$

and by (iii) in Lemma 20, we have also $\mathcal{H}^m(f|_{E_j} \circ T_j^{-1} \circ T_j(F_j^i)) \leq t^m \mathcal{L}^m(T_j(F_j^i))$ and

$$\mathcal{L}^m(T_j(F_j^i)) = \mathcal{H}^m(T_j \circ (f|_{E_j})^{-1} \circ f(F_j^i)) \leq t^m \mathcal{H}^m(f(F_j^i)).$$

Lemma 31. *Let $L : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a linear map, $m \leq n$. Then*

$$\mathcal{H}^m(L(A)) = [\![L]\!]\mathcal{L}^m(A),$$

for all $A \subset \mathbb{R}^n$.

Using again (iii) in Lemma 20 and this lemma, we find

$$\begin{aligned} t^{-2m}\mathcal{H}^m(f(F_j^i)) &\leq t^{-m}\mathcal{L}^m(T_j(F_j^i)) = t^{-m}|\det T_j|\mathcal{L}^m(F_j^i) \\ &\leq \int_{F_j^i} Jf\,dx \leq t^m|\det T_j|\mathcal{L}^m(F_j^i) = t^m\mathcal{L}^m(T_j(F_j^i)) \\ &\leq t^{2m}\mathcal{H}^m(f(F_j^i)). \end{aligned}$$

Then we can sum on i and j:

$$t^{-2m}\sum_{i,j=1}^{\infty}\mathcal{H}^m(f(F_j^i)) \leq \int_A Jf\,dx \leq t^{2m}\sum_{i,j=1}^{\infty}\mathcal{H}^m(f(F_j^i)).$$

Now we recall the Claim in Step 1 and we let $k \to \infty$ to find

$$t^{-2m}\int_{\mathbb{R}^n}\mathcal{H}^0\big(A\cap f^{-1}\{y\}\big)\,d\mathcal{H}^m(y) \leq \int_A Jf\,dx \leq t^{2m}\int_{\mathbb{R}^n}\mathcal{H}^0\big(A\cap f^{-1}\{y\}\big)\,d\mathcal{H}^m(y).$$

At this point we get the conclusion just letting $t \to 1^+$.

Case 2: $A \subset \{Jf = 0\}$. Fix $\epsilon > 0$ and write f as $f = p \circ g$, where

$$g : \mathbb{R}^m \longrightarrow \mathbb{R}^n \times \mathbb{R}^m, \qquad g(x) \equiv (f(x), \epsilon x), \text{ for } x \in \mathbb{R}^m$$

and

$$p : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n, \qquad p(y,z) = y, \text{ for } y \in \mathbb{R}^n, z \in \mathbb{R}^m.$$

Step 1: Claim: There exists a constant $C > 0$ such that

$$0 < Jg(x) \leq C\epsilon$$

for all $x \in A$.

Proof of the claim We observe that $g = (f^1, \dots f^n, \epsilon x_1, \dots \epsilon x_m)$ and so

$$Dg(x) = \begin{pmatrix} Df(x) \\ \epsilon\mathbb{I} \end{pmatrix}_{(n+m)\times m}.$$

Now the Binet-Cauchy Formula gives us a link between $Jf(x)^2$ and the $(m \times m)-$ subdeterminants of $Df(x)$, indeed:

Theorem 32 (Binet - Cauchy Formula)**.** *Assume $m \leq n$ and $L : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is linear. Then*

$$[\![L]\!]^2 = \sum_{\lambda \in \Lambda} (\det(P_\lambda \circ L))^2.$$

Therefore $Jg(x)^2 = \{\text{sum of squares of } (n \times n) - \text{subdeterminants of } Dg(x)\} \geq \epsilon^{2m} > 0$, being the sum of the squares of $(m \times m)-$ subdeterminants of $Dg(x)$.
Moreover using again Binet-Cauchy Formula and recalling that $|Df(x)| \leq \mathrm{Lip}(f) < \infty$, we obtain

$$Jg(x)^2 \leq C\epsilon^2 \qquad \text{for each } \; x \in A$$

since $Jg(x)^2$ is sum of $Jf(x)^2$ and squares of terms each involving at least one ϵ.

Step 2 $p : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a projection, then

$$\mathcal{H}^m(f(A)) \leq \mathcal{H}^m(g(A))$$

and, by the first Case, we have

$$\int_{\mathbb{R}^{m+n}} \mathcal{H}^0\big(A \cap g^{-1}\{y,z\}\big)\, d\mathcal{H}^{m+n}(y,z) = \int_A Jg(x)\, dx.$$

Then

$$\begin{aligned}\mathcal{H}^m(f(A)) \leq \mathcal{H}^m(g(A)) &\leq \int_{\mathbb{R}^{m+n}} \mathcal{H}^0\big(A \cap g^{-1}\{y,z\}\big)\, d\mathcal{H}^{m+n}(y,z) \\ &= \int_A Jg(x)\, dx \leq \epsilon C \mathcal{L}^m(A).\end{aligned}$$

Now we let $\epsilon \to 0$ to conclude $\mathcal{H}^m(f(A)) = 0$; in particular, since $\mathrm{supp}\mathcal{H}^0\big(A \cap f^{-1}\{y\}\big) \subset f(A)$,

$$\int_{\mathbb{R}^m} \mathcal{H}^0\big(A \cap f^{-1}\{y\}\big)\, d\mathcal{H}^m = 0.$$

Recalling that in this case $A \subset \{Jf = 0\}$, we find

$$\int_A Jf\, dx = 0 = \int_{\mathbb{R}^m} \mathcal{H}^0\big(A \cap f^{-1}\{y\}\big)\, d\mathcal{H}^m.$$

General case: We write $A = A_1 \cup A_2$ with $A_1 \subset \{Jf > 0\}$ and $A_2 \subset \{Jf = 0\}$ and we apply the first two cases above. $\square$

Lemma 33. *Let $A \subset \mathbb{R}^m$ be a $\mathcal{L}^m-$measurable set and $f : A \longrightarrow \mathbb{R}^n$ be an injective function. Then for each $u : \mathbb{R}^n \longrightarrow \mathbb{R}$*

$$\int_A (u \circ f)\, Jf\, d\mathcal{L}^m = \int_{f(A)} u\, d\mathcal{H}^m. \tag{2.1}$$

Proof: First of all we consider the case in which u is a simple function, i.e. there exist $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ and $E_1, \ldots E_k \subset \mathbb{R}^n$ measurable sets such that

$$u(x) = \sum_{i=1}^k \alpha_i \chi_{E_i}(x)$$

where χ_{E_i} is the characteristic function of E_i.
Then

$$u \circ f(x) = \sum_{i=1}^k \alpha_i \chi_{E_i}(f(x)).$$

and we can rewrite the first member in (2.1) as

$$\begin{aligned}\int_A u(f(x))\,Jf(x)\,d\mathcal{L}^m = \int_A \sum_{i=1}^{k} \alpha_i\chi_{E_i}(f(x))\,Jf(x)\,d\mathcal{L}^m \;=\\ \sum_{i=1}^{k}\alpha_i \int_A \chi_{E_i}(f(x))\,Jf(x)\,d\mathcal{L}^m\\ = \sum_{i=1}^{k}\alpha_i \int_{f^{-1}(E_i)} Jf\,d\mathcal{L}^m\\ \overset{\text{Area form.}}{=} \sum_{i=1}^{k}\alpha_i \int_{E_i} \mathcal{H}^0\big(f^{-1}(E_i)\cap f^{-1}(y)\big)\,d\mathcal{H}^m(y)\\ = \sum_{i=1}^{k}\alpha_i\mathcal{H}^m(E_i).\end{aligned}$$

While the second member is:

$$\int_A u(y)\,d\mathcal{H}^m(y) \;=\; \int_{f(A)} \sum_{i=1}^{k} \alpha_i\,\chi_{E_i}(y)\,d\mathcal{H}^m(y) \;=\; \sum_{i=1}^{k}\alpha_i\mathcal{H}^m(E_i)$$

and we have found the result in the case in which u is a simple function.
In the general case $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a measurable function: then it suffices to recall that there exist a sequence of simple functions $\{\phi_j\}$ such that $0 \le |\phi_1| \le |\phi_2| \le \cdots \le |f|$, $\phi_j \to u$ pointwise and $\phi_j \to u$ uniformely on any set on which u is bounded.
Therefore, observing that u is even an integrable function and has the property that $|\phi_j| \le u$ a.e. for all j, we obtain the result using the Dominated Convergence Theorem:

$$\int_{f(A)} u\,d\mathcal{H}^m \;=\; \lim_{j\to\infty}\int_{f(A)} \phi_j\,d\mathcal{H}^m \;=\; \int_A(\phi_j\circ f)\,Jf\,d\mathcal{L}^m \;=\; \int_A(u\circ f)\,Jf\,d\mathcal{L}^m. \quad \square$$

Chapter 3

Rectifiable Sets and Tangent Measures

The principal sources for this chapter are [AFP00], [DL08], [MA95], [MO09] and [SI84]. In particular, the ideas used in Section 3.2.3 are taken from the proof of Theorem 11.8 in [SI84].

3.1 Rectifiable Sets

Let $1 \leq m \leq n-1$ be an integer. We consider $\mathbb{R}^n$ equipped with the Euclidean metric.

Definition 34. We say that a set $B \subset \mathbb{R}^n$ is m-rectifable if there exist finitely or countably many Borel measurable sets $A_i \subset \mathbb{R}^m$ and Lipschitz functions $f_i : A_i \longrightarrow \mathbb{R}^n$ such that

$$\mathcal{H}^m \Big[B \ominus \bigcup_{i=1}^{\infty} f_i(A_i) \Big] = 0,$$

where $\ominus$ denotes the symmetric difference, i.e., for any two sets A, B, $A \ominus B = (A \setminus B) \cup (B \setminus A)$.

Remark 35. We observe that, using Whitney's extension theorem (Theorem 16), we could take C^1 functions instead of Lipschitz functions in Definition 34.

Definition 36. A measure μ is a $m-$*dimensional rectifiable measure* if there exists a $m-$dimensional rectifiable set B and a Borel function f such that $\mu = f\mathcal{H}^m \llcorner B$.

We restrict our argumentation to subsets B which are Borel mesurable and such that $\mathcal{H}^m(B \cap K) < \infty$, for all $K \subset \mathbb{R}^n$ compact. In such a way $\mathcal{H}^m \llcorner B$ is a Radon measure in $\mathbb{R}^n$.

Definition 37. If μ is a Radon measure and $x \in \mathbb{R}^n$, an m**-tangent measure of** μ **at** x is a Radon measure ν with the property that there exists a sequence $r_j \searrow 0$ such that

$$\nu = \lim_{j \to \infty} r_j^{-m} \left(T_{x, r_j} \right)_* \mu,$$

where $T_{x,r}(y) = \dfrac{y-x}{r}$.

Example 38. We pick $a \in \mathbb{R}^2$ and $r > 0$ and we consider the circle $\partial B(a,r)$ in $\mathbb{R}^2$, which is a C^1 submanifold of $\mathbb{R}^2$. If we choose $\mu = \mathcal{H}^1 \llcorner \partial B(a,r)$ to be the measure, then for each point $y \in \partial B(a,r)$

$$r_j^{-1}(T_{y,r_j})_*\mu = \mathcal{H}^1 \llcorner \left(\frac{\partial B(a,r_j) - y}{r_j} \right)$$

where r_j is a sequence tending to 0. We observe that

$$\frac{\partial B(a,r_j) - y}{r_j} = \left\{ z \in \mathbb{R}^2 : r_j z + y \in \partial B(a,r_j) \right\}$$

and, when r_j tends to zero, this is exactly the tangent line to $\partial B(a,r)$ at the point y. Therefore, we have proved that every tangent measure is equal to $\mathcal{H}^1 \llcorner \{y + tv : t \in \mathbb{R},\ v \in \mathbb{S}^1$ is such that $v \cdot (y-a) = 0\}$.

Notation: We denote the set of all the m-tangent measure of μ at the point x by $\mathrm{Tan}^{(m)}(\mu, x)$.

Proposition 39. *If $B \subset \mathbb{R}^n$ is $m-$rectifiable, then B is $\mathcal{H}^m-$measurable.*

Proof: First of all we observe that saying that B is m-rectiable is equivalent to say that there exist a subset $M_0 \subset \mathbb{R}^n$, $\mathcal{H}^m(M_0) = 0$, finitely or countably many Borel measurable sets $A_i \subset \mathbb{R}^m$ and Lipschitz functions $f_i : A_i \longrightarrow \mathbb{R}^n$ such that

$$B = M_0 \cup \left(\bigcup_{i=1}^{\infty} f_i(A_i) \right).$$

We pick an $i \in \mathbb{N}$ and we note $A = A_i$ and $f = f_i$: $A \subset \mathbb{R}^m$ is $\mathcal{L}^m-$measurable and $\mathcal{L}^m(A) < \infty$. Since the measure $\mathcal{L}^m$ is Radon, then for all j there exists C_j, compact subset of A, with the property that $\mathcal{L}^m(A \setminus C_j) < 1/j$. Moreover we can choose the sequence of sets $\{C_j\}_{j\in\mathbb{N}}$ so that it is increasing, $C_j \subset C_{j+1}$.

Next we set $A_0 = \bigcup_{j=1}^{\infty} C_j$; obviously, $\mathcal{L}^m(A \setminus A_0) = \lim_{j\to\infty} \mathcal{L}^m(A \setminus C_j) = 0$. We observe that $f(C_j)$ is still compact, being an image of a compact set by a continuous function, and that $f(A_0) = \bigcup_{j\in\mathbb{N}} f(C_j)$: therefore, each $f(C_j)$ is $\mathcal{H}^m-$measurable and $f(A_0)$, being a countable union of measurable sets, is $\mathcal{H}^m-$measurable.

Now we recall that $f(A) \setminus f(A_0) \subseteq f(A \setminus A_0)$

$$\mathcal{H}^m\big(f(A) \setminus f(A_0)\big) \le \mathcal{H}^m\big(f(A \setminus A_0)\big) = \big(\mathrm{Lip} f\big)^m \mathcal{L}^m\big(A \setminus A_0\big) = 0,$$

that is, $\mathcal{H}^m(f(A) \setminus f(A_0)) = 0$.
In conclusion, since $f(A_0)$ is $\mathcal{H}^m-$measurable, so it is also $f(A)$. $\square$

It's useful to recall the following two results: the first one, Lemma 40, is a slightly different characterization of rectifiable sets that uses as A_i compact sets and that shows that the family $\{f_i(A_i)\}_{i\in\mathbb{N}}$ can be disjoint (we will use it in subsection 3.2.1). Therefore Lemma 41 relates property (ii) in Theorem 42 to the existence of a finite density, which is equal to 1 (the converse can also be proved, we refer to [AFP00], Theorem 2.63; actually this last Theorem proves something more: the rectifiability of a set is equivalent to the condition that the density is equal to 1 $\mathcal{H}^m-$almost everywhere).

Lemma 40. *If B is m-rectifiable, then there exists a family of sets $\{A_i\}_{i\in\mathbb{N}}$, $A_i \subset \mathbb{R}^m$ and a family of functions $\{f_i\}_{i\in\mathbb{N}}$ with*

- *A_i a compact set;*
- *$f_i : A_i \longrightarrow \mathbb{R}^n$ a bilipschitz map (i.e. Lipschitz, injective and such that $f_i^{-1}|_{f_i(A_i)}$ is Lipschitz too) with the property that $\max\{Lipf_i, Lipf_i^{-1}\} < 1+\epsilon$ for $\epsilon > 0$ fixed;*
- *$f_i(A_i)$ are pairwise disjoint,*

and such that

$$\mathcal{H}^m\Big[B \ominus \bigcup_i f_i(A_i)\Big] = 0.$$

Proof. By Lemma 20 we know that if $A_i \subset \{Jf_i > 0\}$ then there exist Borel sets $A_i \subset \mathbb{R}^m$ and $f_i : A_i \longrightarrow \mathbb{R}^n$ Lipschitz such that

$$\mathcal{H}^m\Big[B \ominus \bigcup_{i=1}^{\infty} f_i(A_i)\Big] = 0.$$

Claim: f_i maps $A_i \cap \{Jf_i = 0\}$ onto a $\mathcal{H}^m-$ zero measure set.

Proof: We write $f_i = p \circ g_i$, where, for $\epsilon > 0$ fixed,

$$g_i : A_i \longrightarrow \mathbb{R}^n \times \mathbb{R}^m, \qquad g_i(x) \equiv \big(f_i(x), \epsilon x\big) \qquad \text{for } x \in A_i$$

and

$$p : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n, \qquad \text{for } y \in \mathbb{R}^n,\ z \in \mathbb{R}^m.$$

We observe that there exists $C_i > 0$ such that for all $x \in A_i$ we have

$$0 < Jg_i(x) \le C_i\epsilon.$$

In fact, if we write $g_i = (f_i^1, \dots, f_i^n, \epsilon x_1, \dots, \epsilon x_m)$, we find that

$$Dg_i(x) = \begin{pmatrix} Df_i(x) \\ \epsilon\mathbb{I} \end{pmatrix} \in \mathcal{M}_{(m+n)\times m}.$$

By Binet-Cauchy formula

$$[\![L]\!]^2 = \sum_{\lambda\in\Lambda} (\det(P_\lambda \circ L))^2,$$

we find the inequality $Jg_i(x)^2 \ge \epsilon^{2n} > 0$, since $Jg_i(x)^2$ is the sum of the squares of $(n\times n)-$ subdeterminants of $Dg_i(x)$.
Again by Binet-Cauchy Formula and by the inequality $|Df_i(x)| \le \text{Lip}(f_i) < \infty$, we obtain

$$Jg_i(x)^2 \le C_i\epsilon^2 \qquad \text{for each } x \in A,$$

since $Jg(x)^2$ is sum of $Jf(x)^2$ and squares of terms each involving at least one ϵ.

At this point,

$$\mathcal{H}^m(f_i(A_i)) \le \mathcal{H}^m(g_i(A_i)) \le \int_{\mathbb{R}^{m+n}} \mathcal{H}^0\left(A_i \cap g_i^{-1}\{y,z\}\right) d\mathcal{H}^m(y,z)$$
$$\overset{\text{Area Formula}}{=} \int_{A_i} Jg(x)\,dx \ \le\ \epsilon C_i \mathcal{L}^m(A_i).$$

and letting $\epsilon \to 0$ we find $\mathcal{H}^m(f_i(A_i)) = 0$. $\square$

Now we want to prove that there exist measurable subsets C_j^i of A_i s.t. $f_i|_{C_j^i}$ is bilipschitz for all j. To do this, we use Lemma 20: indeed, this lemma ensures that there exist a family of Borel measurable subsets $\{C_j^i\}_{j\in\mathbb{N}}$ of A_i such that $\mathcal{L}^m\big(A_i \setminus \bigcup_{j=1}^\infty C_j^i\big) = 0$, $f_{i,j} = f_i|_{C_j^i}$ is injective and with Lipischitz inverse. Actually we observe that in the hypothesis of Lemma 20 the function f is defined in the whole space $\mathbb{R}^m$: in order to obtain this hypothesis we have to use Theorem 15 to extend the functions f_i to Lipschitz functions $\hat{f}_i : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ such that $\mathrm{Lip}\hat{f}_i = \mathrm{Lip} f_i$ and $f_i = \hat{f}_i|_A$).

Then, if we set $E_i = A_i \setminus \bigcup_{j=1}^\infty C_j^i$, we have $\mathcal{H}^m(f_i(E_i)) = 0$, since the Hausdorff measure is absolutely continuous with respect to the Lebesgue measure and for all E_i

$$\mathcal{H}^m(f_i(E_i)) \le (\mathrm{Lip} f_i)^m\, \mathcal{H}^m(E_i) \le (\mathrm{Lip} f_i)^m\, \mathcal{L}^m(E_i) = 0.$$

We note that, if $C_k^i \cap C_j^i \neq \emptyset$, with $k \neq j$, we can consider $\widetilde{c}_k^i = C_k^i$ and $\widetilde{c}_j^i = C_j^i \setminus (C_k^i \cap C_j^i)$ and so it's possible to take these subsets C_j^i in such a way they are pairwise disjoint and also to require empty intersection with the set $f_i^{-1}(f_i(E_i))$ (which has Hausdorff measure equal to zero since $f^{-1}|_{C_j^i}$ is a Lipschitz function).

In addition, we can find a countable family of subsets $D_{j,k}^i$ of these sets C_j^i so that the function f_i can be approximated in each $D_{j,k}^i$ by a constant function, i.e., so that its Lipschitz constant and the Lipschitz constant of its inverse $f_i^{-1}|_{D_{j,k}^i}$ are both near 1 (and then we have found the required condition $\max\{\mathrm{Lip}\, f_i,\ \mathrm{Lip}\, f_i^{-1}\} \le 1+\epsilon$).

In particular, we can prove the following result, which is pretty similar to Lemma 20:

Claim: There exists a countable family of Borel subsets $\{D_{j,k}^i\}_{k=1}^\infty$ of $\mathbb{R}^m$ such that for all $\epsilon \in (0,1)$:

1) $C_j^i = \cup_{k=1}^\infty D_{j,k}^i$;
2) $f_i|_{D_{j,k}^i}$ is injective for all k;
3) for each $k \ge 1$ there exists a symmetric automorphism $T_k : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ such that

$$\mathrm{Lip}\big((f_i|_{D_{j,k}^i}) \circ T_k^{-1}\big) \le 1+\epsilon, \qquad \mathrm{Lip}\big(T_k \circ (f|_{D_{j,k}^i})^{-1}\big) \le 1+\epsilon,$$

Proof: First of all we note that the following inequality is true:

$$\frac{1}{1+\epsilon} + \frac{\epsilon}{2} < 1 < 1+\epsilon - \frac{\epsilon}{2}.$$

Choose C, countable dense subset of C_j^i, and $\mathbf{S}$, countable dense subset of symmetric automorphisms of $\mathbb{R}^m$.
For each $c \in C$, $T \in \mathbf{S}$ and for each $n \in \mathbb{N}^*$ we define $E(c,T,n)$ as the set of all points $x \in C_j^i \cap B(c, 1/n)$ satistying $\forall v \in \mathbb{R}^m$

$$\left(\frac{1}{1+\epsilon} + \frac{\epsilon}{2}\right)|Tv| \le |Df(x)v| \le \left(1+\epsilon - \frac{\epsilon}{2}\right)|Tv| \tag{3.1}$$

and, for all $a \in B(b, 2/n)$,

$$|f_i(a) - f_i(b) - Df_i(b)\cdot(a-b)| \le \frac{\epsilon}{2}|T(a-b)|. \tag{3.2}$$

Summing up these two considitions we find the estimate

$$\frac{1}{1+\epsilon}|T(a-b)| \le |f_i(a) - f_i(b)| \le (1+\epsilon)|T(a-b)| \tag{3.3}$$

for $b \in E(c, T, n)$, $a \in B(x, 2/n)$.

We can now relabel the countable collection $\{E(c, T, n) : c \in C, T \in \mathbf{S}, \mathbf{n} \in \mathbb{N}^*\}$ as $\{D^i_{j,k}\}_{k=1}^{\infty}$. Indeed, we note that each $E(c, T, i)$ is a Borel set since Df is Borel measurable and we now see that $C^i_j = \bigcup_{k=1}^{\infty} D^i_{j,k}$: let $b \in C^i_j$ and decompose $Df_i(b) = O \circ S$ where O is an orthogonal map and S is a symmetric map. Then we choose $T \in \mathbf{S}$ such that

$$\mathrm{Lip}(T \circ S^{-1}) \le \left(\frac{1}{1+\epsilon} + \frac{\epsilon}{2}\right)^{-1}, \qquad \mathrm{Lip}(S \circ T^{-1}) \le \frac{1}{1+\epsilon} - \frac{\epsilon}{2}.$$

Next pick $n \in \mathbb{N}^*$ and $c \in C$ so that $|b - c| < 1/n$,

$$|f_i(a) - f_i(b) - Df_i(b) \cdot (a-b)| \le \frac{\epsilon}{\mathrm{Lip}(T^{-1})}|a-b| \le \epsilon|T(a-b)|$$

for all $a \in B(b, 2/n)$ and so $b \in E(c, T, n)$.

At this point, for any set $D^i_{j,k}$ we have $D^i_{j,k} = E(c, T, n)$ for a suitable $c \in C$, $T \in \mathbf{S}$, $\mathbf{n} \in \mathbb{N}^*$. Let $T_k = T$. Using (3.3) we find that for all $b \in D^i_{j,k}$, $a \in B(b, 2/n)$

$$\frac{1}{1+\epsilon}|T_k(a-b)| \le |f_i(a) - f_i(b)| \le (1+\epsilon)|T_k(a-b)|.$$

Recalling that $D^i_{j,k} \subset B(c, 1/n) \subset B(b, 2/n)$, we obtain that this inequality is still true for all $a, b \in D^i_{j,k}$: this implies that $f_i|_{D^i_{j,k}}$ is one-to-one and also that

$$\mathrm{Lip}\big((f_i|_{D^i_{j,k}}) \circ T_k^{-1}\big) \le 1+\epsilon, \qquad \mathrm{Lip}\big(T_k \circ (f|_{D^i_{j,k}})^{-1}\big) \le 1+\epsilon. \quad \square$$

Now, we are left to prove that:

1) C^i_j can be chosen to be compact,

2) $f_{i,j}(C^i_j) \cap f_{l,k}(C^i_k) = \emptyset$, $\forall i, j, k, l$.

Consider a finite set $J \subset \mathbb{N}$ and define

$$P^i_J = \bigcap_{j \in J} f_i(C^i_j) \setminus \bigcup_{j \in J} f_i(C^i_j).$$

If we set $\mathfrak{J} = \{J \subset \mathbb{N} \,|\, P^i_J \neq \emptyset\}$, we find that $\{P^i_J \,|\, J \in \mathfrak{J}\}$ is a partition of $\bigcup_{j=1}^{\infty} f_i(A_j)$.

We fix a $j \in \mathbb{N}$ and we consider all the subsets $J \in \mathfrak{J}$ containing j: $f_i|_{C^i_j \cap f_i^{-1}(P^i_J)}$ is still bilipschitz onto its image $f_i|_{C^i_j \cap f_i^{-1}(P^i_J)}(C^i_j) = P^i_J$ (being the restrinction of the bilipschitz map $f_i|_{C^i_j}$).

Then we choose a compact set K^i_J in P^i_J and we note that $D^i_J = f_i^{-1}(K^i_J)$ is still a compact subset of A_i, since f_i is a continuous map. Moreover $f_i|_{D^i_J} : D^i_J \longrightarrow K^i_J$ is still a bilipschitz map and we have $f_i(D^i_J) = f_i(D^i_K)$ or $f_i(D^i_J) \cap f_i(D^i_K) \neq \emptyset$, for all $K, J \in \mathfrak{J}$.

At this point, we can get the conclusion, since the sets K_J^i can be chosen to cover half Hausdorff measure of $f_i(A_i)$ and we can argue as above for $f_i(A_i) \setminus K_J^i$. Therefore we have found a covering for $\mathcal{H}^m$- almost all $f_i(A_i)$. $\square$

Lemma 41. *If for all x there exists $W_x \in \mathbb{G}(m, \mathbb{R}^n)$ such that* $\mathrm{Tan}^{(m)}(\mu, x) = \{\mathcal{H}^m \llcorner W_x\}$*, then*

$$\Theta^m(\mu, x) = 1.$$

Proof: We have

$$r^{-m}\big(T_{x,r}\big)_*\mu \rightharpoonup \mathcal{H}^m \llcorner W_x$$

in the weak-$*$ topology.

We take an open ball $U(x, (1-\epsilon)r)$ of center in x and radius $(1-\epsilon)r$, with $r > 0$ and $\epsilon \in (0,1)$; by (ii) in Theorem 25, we have:

$$\begin{aligned}(1-\epsilon)^m \alpha(m) &= \big(\mathcal{H}^m \llcorner W_x\big)\big(U(0, 1-\epsilon)\big)\big) \\ &\leq \liminf_{r \to 0^+} r^{-m}\big(T_{x,r}\big)_*\big(\mathcal{H}^m \llcorner A\big)\big(U(0, 1-\epsilon)\big) \\ &= \liminf_{r \to 0^+} r^{-m}\big(\mathcal{H}^m \llcorner A\big)\big(U(x, (1-\epsilon)r)\big).\end{aligned}$$

Therefore,

$$(1-\epsilon)^m \leq \liminf_{r \to 0^+} \frac{\big(\mathcal{H}^m \llcorner A\big)\big(B(x,r]\big)}{\alpha(m) r^m} \leq \limsup_{r \to 0^+} \frac{\big(\mathcal{H}^m \llcorner A\big)\big(B(x,r]\big)}{\alpha(m) r^m} \equiv 1.$$

Now we let $\epsilon \to 0^+$, then $U(x, (1-\epsilon)r) \nearrow U(x,r)$, $\big(\mathcal{H}^m \llcorner A\big)\big(U(x,r)\big) \leq \big(\mathcal{H}^m \llcorner A\big)\big(B(x,r]\big)$ and, using Theorem 25 -(ii),

$$\begin{aligned}\limsup_{r \to 0^+} \frac{\big(\mathcal{H}^m \llcorner A\big)\big(B(x,r]\big)}{r^m} &= \limsup_{r \to 0^+} r^{-m}\big(T_{x,r}\big)_*\big(\mathcal{H}^m \llcorner A\big)\big(B(0,1]\big) \\ &\leq \big(\mathcal{H}^m \llcorner W_x\big)\big(B(0,1]\big) = \alpha(m),\end{aligned}$$

and so

$$\Theta^m(\mu, x) = 1. \quad \square$$

3.2 A criterion for rectifiability

Our goal is to prove the following theorem:

Theorem 42. *Let $B \subseteq \mathbb{R}^n$ be a Borel set such that $\mu = \mathcal{H}^m \llcorner B$ is a Radon measure, then the following statements are equivalent:*

(i) *B is m-rectifiable;*

(ii) *For μ–a.e. $x \in \mathbb{R}^n$ there exists a m-dimensional plane $W_x \in \mathbb{G}(m, \mathbb{R}^n)$ such that*

$$\mathrm{Tan}^{(m)}(\mu, x) = \Big\{\mathcal{H}^m \llcorner W_x\Big\}$$

Remark 43. With $\mathbb{G}(m, \mathbb{R}^n)$ we refer to the Grassman space, i.e. the space of all the m-dimensional subspaces (or m-planes) of $\mathbb{R}^n$.

3.2.1 Some properties of the function $\varphi(x) = W_x$

First of all we recall that the support of a measure μ is (the closure of) the set:

$$\operatorname{supp}\mu = \big\{x \in \mathbb{R}^n : \text{ for all } V_x \in \mathcal{B}(\mathbb{R}^n), \text{ neighbourhood of } x, \text{ we have } \mu(V_x) > 0\big\}.$$

We consider a rectifiable measure μ and the $\mu-$almost everywhere defined function φ:

$$\begin{aligned} \operatorname{supp}\mu &\xrightarrow{\varphi} \mathbb{G}(m, \mathbb{R}^n) \\ x &\mapsto \varphi(x) = W_x, \end{aligned} \tag{3.4}$$

which associates to each point in the support of the measure μ the plane W_x such that

$$r_j^{-m}(T_{x,r_j})_*\mu \rightharpoonup \mathcal{H}^m \llcorner W_x$$

where $\{r_j\}_{j\in\mathbb{N}}$ is a sequence which tends to 0.

Remark 44. We recall that, since suppμ is $m-$rectifiable, there exist a countable family of Borel sets $\{A_i\}_{i\in\mathbb{N}}$ and a countable family of Lipschitz functions $\{f_i : A_i \to \mathbb{R}^n\}_{i\in\mathbb{N}}$ such that

$$\mathcal{H}^m\Big(\operatorname{supp}\mu \ominus \bigcup_{i=1}^{\infty} f_i(A_i)\Big).$$

According to Lemma 40, the functions f_i can be chosen to be bilipschitz (and so f_i and f_i^{-1} are both continuous) and we can take the sets A_i with the property that $f_i(A_i)$ are pairwise disjoint.

This means that each $x \in \operatorname{supp}\mu$ belongs to one $f_i(A_i)$, i.e., that there exists a point $\xi_x \in A_i$ such that $x = f_i(\xi)$.

Moreover, we recall that W_x is the image of the differential of f_i, evaluated in ξ_x:

$$W_x = Df_i(\xi_x)[\mathbb{R}^m].$$

Proposition 45. *The function φ is $\mu-$measurable.*

Proof: For what we have seen in Remark 44, it suffices to prove that for each $i \in \mathbb{N}$ the function $Df_i : A_i \longrightarrow \mathbb{R}^n$ is measurable. This is equivalent to say that all the entries of the matrix $Df_i(\cdot)$ are measurable.

We note that each function f_i is defined in $A_i \subset \mathbb{R}^m$, but it can be extended to the whole $\mathbb{R}^m$ using Theorem 15. Then

$$f_i : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

and, for each $x \in \mathbb{R}^m$, $f_i(x) = (f_i^1(x), \dots, f_i^n(x))$, where $f_i^j : \mathbb{R}^m \longrightarrow \mathbb{R}$ is a Lipschitz map for all $j \in \{1, \dots, n\}$.

At this point it suffices to recall that for almost all $\xi \in \mathbb{R}^m$ we have

$$\frac{\partial f_i^j}{\partial \xi_k}(\xi) = \lim_{t\to 0} \frac{f_i^j(\xi + te_k) - f_i^j(\xi)}{t}.$$

But each f_i^j is a Lipschitz function, thus, in particular, it is a continuous funcion in ξ. Therefore, $\dfrac{\partial f_i^j}{\partial \xi_k}(\xi)$ is Borel measurable and we get the conclusion. $\square$

We observe that if suppμ is a submanifold of $\mathbb{R}^n$ of class C^1, then the function φ defined in (3.4) is continuous.

3.2.2 Proof of $(i) \Rightarrow (ii)$

This part of the proof of Theorem 42 is based on the Area Formula.
By Lemma 40, we know that $\mathcal{H}^m$−a.e. $x \in B$ belongs to one and only one $f_i(A_i)$ since $\mathcal{H}^m\left[B \ominus \bigcup_i f_i(A_i)\right] = 0$ and that we can require that the sets A_i are compact, while $f_i(A_i)$ are pairwise disjoint.
Therefore, in order to prove that for all $u \in C_c(\mathbb{R}^n)$

$$r^{-m} \int_{B\subset\mathbb{R}^n} u\, d(T_{a,r})_* \mu(x) \underset{r\to 0}{\longrightarrow} \int_{B\subset\mathbb{R}^n} u\, d(\mathcal{H}^m \llcorner W_a)(x),$$

it suffices to prove the result for sets of the form $B = f_i(A_i)$, since the integral over the set having zero $\mathcal{H}^m$−measure is zero and the integral over the disjoint union $\bigcup_{i=1}^{\infty} f_i(A_i)$ is just the sum of the integrals over each $f_i(A_i)$.

We denote with ξ the variables in $A_i \subset \mathbb{R}^m$ and with x the points belonging to $f_i(A_i) \subset \mathbb{R}^n$. Then we consider a point $a \in f_i(A_i)$, that we can suppose being the image of 0 by f_i, i.e.

$$\begin{aligned} A_i &\to f_i(A_i) = B \\ 0 &\longmapsto a = f_i(0). \end{aligned}$$

In particular, we assume that:

- f_i is differentiable at 0;
- 0 is a Lebesgue point of $J_m f_i$ with respect to the Lebesgue measure;
- 0 is a point of $\mathcal{H}^m$−density equals to 1 in A_i, i.e., $\lim_{r\to 0} \dfrac{\mathcal{H}^m(B(0,r)\cap A_i)}{\alpha(m) r^m}$.

We recall that if μ is a Radon measure on $\mathbb{R}^n$, $1 \le p < \infty$, a point $x \in \mathbb{R}^n$ is called **Lebesgue point** of $f \in L^p_{loc}(\mathbb{R}^n, \mu)$ with respect to μ if

$$\lim_{r\to 0} \fint_{B(x,r)} |f - f(x)|^p \, d\mu = 0.$$

Finally, we set $\mu = \mathcal{H}^m \llcorner f_i(A_i)$ and $W_a = Df_i(0)\big[\mathbb{R}^m\big] \in \mathbb{G}(m, \mathbb{R}^n)$.

Then, if we take $u \in C_c(\mathbb{R}^n)$, we recall that our aim is to prove that

$$\begin{aligned} r^{-m} \int_{B \subset \mathbb{R}^n} u \, d(T_{a,r})_* \mu(x) \underset{r \to 0}{\longrightarrow} & \int_{B \subset \mathbb{R}^n} u \, d(\mathcal{H}^m \llcorner W_a)(x) \\ & \overset{Lemma\,33}{\underset{B = f_i(A_i)}{=}} \int_{A_i \subset \mathbb{R}^m} u[Df_i(0)(\xi)] \cdot J(Df_i(0)) \, d\mathcal{L}^m(\xi). \\ & \overset{J(Df_i(0)) = Jf_i(0)}{=} \int_{A_i \subset \mathbb{R}^m} u[Df_i(0)(\xi)] \cdot Jf_i(0) \, d\mathcal{L}^m(\xi). \end{aligned}$$

First of all, recalling the definition of pushforward of a measure and using again Lemma 33, we can rewrite the first integral above as:

$$r^{-m} \int_B \big(u \circ T_{a,r}\big) \, d\mathcal{H}^m \llcorner f_i(A_i)(x) \; = \; r^{-m} \int_{A_i} \big(u \circ T_{a,r} \circ f_i\big) \, Jf_i \, d\mathcal{L}^m(\xi).$$

In addiction, we remark the following facts:

1) since u is a continuous function with compact support, there exists $\max |u| < \infty$ (by Weiestrass Theorem) and u is uniformly continuous, that is, for all $\epsilon > 0$, there exists $\delta > 0$ such that for each $x, y \in \operatorname{supp}(u)$, $|x - y| < \delta$, then $|u(x) - u(y)| < \epsilon$. In particular, the uniform continuity of u can be written also using the oscillation of u
$$|u(x_1) - u(x_2)| \leq \operatorname{osc}_u(|x_1 - x_2|)$$
where $\operatorname{osc}_u \; = \; \sup_{y \in \mathbb{R}^n} \; u(y) - \inf_{y \in \mathbb{R}^n} \; u(y)$.

2) f_i is differentiable at the point 0, that is,
$$\lim_{\xi \to 0} \frac{\big|f_i(\xi) - f_i(0) - Df_i(0)(\xi)\big|}{|\xi|} \; = \; 0$$
and this is equivalent to say that $\big|f_i(\xi) - f_i(0) - Df_i(0)(\xi)\big|$ is $o(|\xi|)$, i.e.,
$$\big|f_i(\xi) - f_i(0) - Df_i(0)(\xi)\big| \sim o\big(|\xi|\big).$$

3) Actually, we are interested in the support of $u[Df_i(0)(\xi)]$. For the previous point, this is equivalent to find the set of $\xi \in \mathbb{R}^m$ where $u\left[\dfrac{f_i(\xi) - f_i(0)}{r}\right] \neq 0$. Therefore, since $\operatorname{supp}(u) \subseteq B_R = U(0, R)$ for some $R > 0$, it has to occur $\left|\dfrac{f_i(\xi) - f_i(0)}{r}\right| < R$.

4) If 0 is a point of $\mathcal{H}^m-$density 1 in A_i, then it's a point of density 1 also for the set $\dfrac{1}{r} A_i = \left\{\dfrac{\xi}{r} : \xi \in A_i\right\}$, for all $r > 0$. Indeed:

$$\begin{aligned} \lim_{r \to 0} \frac{\mathcal{H}^m\big(B(0,r) \cap \frac{1}{r} A_i\big)}{\alpha(m) r^m} & = \lim_{r \to 0} \frac{\frac{1}{r^m} \mathcal{H}^m\big(B(0, r^2) \cap A_i\big)}{\alpha(m) r^m} \\ & \overset{r^2 = s}{=} \lim_{s \to 0} \frac{\mathcal{H}^m\big(B(0,s) \cap A_i\big)}{\alpha(m) s^m} = 1. \end{aligned}$$

For the first of the points above we have

$$\left|u\left(\frac{f_i(\xi)-f_i(0)}{r}\right)-u\left(\frac{Df_i(0)(\xi)}{r}\right)\right|\le \operatorname{osc}_u\left(\left|\frac{f_i(\xi)-f_i(0)}{r}-\frac{Df_i(0)(\xi)}{r}\right|\right),$$

while, $\left|\frac{f_i(\xi)-f_i(0)}{r}-\frac{Df_i(0)(\xi)}{r}\right|$ is $o(|\xi|)$ only if $|\xi|\le C_r$, for some $C_r>0$ depending on r, that is, $\xi\in B_{C_r}=U(0,C_r)$.

Hence:

$$\begin{aligned} r^{-m}\int_B \Big(u\circ T_{a,r}\Big)\, d\mathcal{H}^m\llcorner f_i(A_i)(x) \;&=\; r^{-m}\int_{A_i\cap B_{C_r}} u\left[\frac{f_i(\xi)-a}{r}\right] Jf_i(\xi)\, d\mathcal{L}^m(\xi)\\ &\overset{a=f_i(0)}{=}\; r^{-m}\int_{A_i\cap B_{C_r}} u\left[\frac{f_i(\xi)-f_i(0)}{r}\right]Jf_i(\xi)d\mathcal{L}^m(\xi)\end{aligned}$$

Now, we can suppose that $C_r\sim r$ and we rewrite the last integral using the characteristic function of A_i:

$$\begin{aligned}\int_{A_i\cap B_r} u\left[\frac{f_i(\xi)-f_i(0)}{r}\right]Jf_i(\xi)d\mathcal{L}^m(\xi) \;&=\; \int_{B_r} u\left[\frac{f_i(\xi)-f_i(0)}{r}\right]\chi_{A_i}(\xi)Jf_i(\xi)d\mathcal{L}^m(\xi)\\ &= \int_{B_r} u\left[\frac{f_i(\xi)-f_i(0)}{r}\right]\chi_{A_i}(\xi)\big(Jf_i(\xi)-Jf_i(0)+Jf_i(0)\big)d\mathcal{L}^m(\xi).\end{aligned}$$

Then we are interested in computing the following sum of integrals:

$$\begin{aligned}&\lim_{r\to 0}\fint_{B(0,r)} u\left[\frac{f_i(\xi)-f_i(0)}{r}\right]\chi_{A_i}(\xi)\big(Jf_i(\xi)-Jf_i(0)\big)d\mathcal{L}^m(\xi)+\\ &\qquad+\lim_{r\to 0}\fint_{B(0,r)} u\left[\frac{f_i(\xi)-f_i(0)}{r}\right]\chi_{A_i}(\xi)Jf_i(0)d\mathcal{L}^m(\xi).\end{aligned}$$

For the first one of these two integrals we recall that 0 is a Lebesgue point for Jf_i and that A_i is compact:

$$\begin{aligned}&\lim_{r\to 0}\left|\fint_{B_r} u\left[\frac{f_i(\xi)-f_i(0)}{r}\right]\chi_{A_i}(\xi)\big(Jf_i(\xi)-Jf_i(0)\big)d\mathcal{L}^m(\xi)\right|\\ &\le \lim_{r\to 0}\fint_{B_R}\left|u\left[\frac{f_i(\xi)-f_i(0)}{r}\right]\chi_{A_i}(\xi)\big(Jf_i(\xi)-Jf_i(0)\big)\right|d\mathcal{L}^m(\xi)\\ &\le \mathcal{L}^m(A_i)\sup_{A_i}|u|\;\lim_{r\to 0}\fint_{B_r}\big|Jf_i(\xi)-Jf_i(0)\big|d\mathcal{L}^m(\xi)\\ &= C\;\lim_{r\to 0}\fint_{B_r}\big|Jf_i(\xi)-Jf_i(0)\big|d\mathcal{L}^m(\xi)=0.\end{aligned}$$

As for the second term in the sum, first of all we apply the changement of variables $y=\dfrac{\xi}{r}$, possible for the fourth of the points above:

$$\begin{aligned}&\lim_{r\to 0}\frac{1}{r^m}\int_{B_r\cap A_i} u\left[\frac{f_i(\xi)-f_i(0)}{r}\right]Jf_i(0)d\mathcal{L}^m(\xi)\\ &\overset{dy=r^{-m}d\xi}{=}\;\lim_{r\to 0}\int_{B(0,1)\cap\frac{1}{r}A_i} u\left[\frac{f_i(ry)-f_i(0)}{r}\right]Jf_i(0)d\mathcal{L}^m(y).\end{aligned}$$

Now we note that, according to the second and the third of the points above and for the continuity of u, we have

$$\lim_{r\to 0} u\left[\frac{f_i(ry)-f_i(0)}{r}\right] = u\big(Df_i(0)(\xi)\big)$$

and $\left|u\left[\frac{f_i(ry)-f_i(0)}{r}\right]Jf_i(0)\right| \le \sup_{B(0,1)} |u||Jf_i(0)| \in L^\infty(B(0,1)) \subset L^1(B(0,1))$.

Therefore we can apply Dominated Convergence Theorem of Lebesgue and conclude that

$$\lim_{r\to 0}\frac{1}{r^m}\int_{B_r\cap A_i} u\left[\frac{f_i(\xi)-f_i(0)}{r}\right]Jf_i(0)d\mathcal{L}^m(\xi) \;=\; \int_{A_i} u(Df_i(0)(\xi))Jf_i(0)d\mathcal{L}^m(\xi). \quad \square$$

Example 46. In $\mathbb{R}^2$ we consider a set B which is 1-rectifiable ($\mathcal{H}^1(B)<\infty$)

$$B = \bigcup_{i=1}^{\infty} \partial B\Big(a_i, \frac{1}{2^i}\Big),$$

where $\{a_i\}_i$ is a dense subset in $\mathbb{R}^2$.

This example shows that the structure of a rectifiable set can be very strange: in this case, the set B is a union of pieces of circumference and it has the property that its closure is the whole space $\mathbb{R}^2$.

By Lemma 40, we have the existence of compact sets $A_i \subset \mathbb{R}^1$ and bilipschitz functions $f_i : A_i \longrightarrow \partial B\big(a_i, 1/2^i\big) \subset \mathbb{R}^2$ and such that the sets $f_i(A_i)$ are pairwise disjoint. Then we can define the measure $\mu = \sum_i \mu_i = \sum_i \mathcal{H}^1 \llcorner f_i(A_i)$ and we find

$$r^{-1}\,(T_{a,r})_*\mu(B(0,1)) \;=\; \frac{\mu(T_{a,r}^{-1}(B(0,1)))}{r} \;=\; \frac{\mu(B(a,r))}{r}. \quad \square$$

3.2.3 Proof of $(ii) \Rightarrow (i)$

First of all we take $R>0$ such that $\mu(\partial B(0,R))=0$ and we replace μ with $\mu \llcorner B(0,R)$: in such a way we obtain a finite measure.

We want to prove that if for $\mu-$almost every point $x\in\mathbb{R}^n$ there exists a $m-$dimensional plane W_x such that

$$\mathrm{Tan}^m(\mu,x) \;=\; \big\{\mathcal{H}^m \llcorner W_x\big\},$$

where $\mu = \mathcal{H}^m \llcorner B$, then B is $m-$rectifiable, that is the existence of a countable family of sets $F_i = \mathrm{graph}(f_i)$ with f_i Lipschitz maps such that:

$$\mathcal{H}^m\Big(B \setminus \bigcup_{i=1}^{\infty} F_i\Big) = 0.$$

In order to do this we argue following these steps:

Plan of the proof:

Step 1: We take a set $F \subset \mathbb{R}^n$ with $\mu(\mathbb{R}^n \setminus F) \leq \frac{1}{4}\mu(\mathbb{R}^n)$ and such that for each $x \in F$ there exists a $m-$dimensional plane W_x such that $\mathrm{Tan}^{(m)}(\mu, x) = \mathcal{H}^m \llcorner W_x$.
Then in this set we define two sequences of measurable functions, $\{f_k\}_{k\geq 1}$ and $\{q_k\}_{k\geq 1}$, with the property that

$$\lim_{k\to\infty} f_k(x) \geq \theta_0 \in (0,1]$$

and

$$\lim_{k\to\infty} q_k(x) = 0$$

for all $x \in F$.

Step 2: By the measurability of the functions f_k and q_k we can use Egorov's theorem in order to find E, a $\mu-$measurable subset of F such that the limits above hold uniformely for all $x \in E$ and with the property that $\mu(F \setminus E) \leq \frac{1}{4}\mu(\mathbb{R}^n)$.

Step 3: Therefore we choose $N = N(n,m)$ $(n-m)-$dimensional subspaces of $\mathbb{R}^n$, $\pi_1, \dots \pi_N$ such that for each π, $(n-m)$-dimensional subspace of $\mathbb{R}^n$, there exists $j \in \{1, \dots N\}$ which satisfies the condition $\mathrm{dist}(\pi, \pi_j) < 1/16$.
Using these planes we are able to write the set E as a union of N subsets E_j, where E_j is the set of points whose tangent space W_x has distance less than $1/16$ from the plane π_j (we observe that the tangent space exists for all $x \in E$, being a subset of F).

Step 4: Now we construct a Lipschitz function $f_j : \mathbb{R}^m \to \mathbb{R}^{n-m}$ such that for each $x \in E_j$ there is a neighbourhood of x in E_j which is entirely conteined in the graph of f_j (possibly applying an orthogonal transormation q_j to $\mathrm{graph}(f_j)$).
In particular, $E \subset \bigcup_{i=1}^M q_i(\mathrm{graph} f_i)$.

Step 5: Replacing μ by $\mu \llcorner \left(\mathbb{R}^n \setminus \bigcup_{i=1}^M q_i(\mathrm{graph}(f_i))\right)$ and arguing as above, we find countably many Lipschitz graphs which cover $\mathcal{H}^m-$almost everywhere B: we have proved that B is $m-$rectifiable.

Proof: We take up again the five steps presented above.

Step 1: For any $(n-m)-$dimensional subspace $\pi \subset \mathbb{R}^n$ and any $\alpha \in (0,1)$, we denote with p_π the orthogonal projection of $\mathbb{R}^n$ onto π and with $X_\alpha(\pi, x)$ the cone defined by

$$X_\alpha(\pi, x) = \{y \in \mathbb{R}^n : |p_\pi(y-x)| \geq \alpha|y-x|\}.$$

Moreover, for two $(n-m)-$ dimensional subspaces π, π' we define the distance between π, π' by

$$\mathrm{dist}\,(\pi, \pi') = \sup_{|x|=1} |p_\pi(x) - p_{\pi'}(x)|.$$

Choose $\theta_0 \in (0,1]$ and a Borel-measurable subset $F \subset \mathbb{R}^n$ such that

$$\mu(\mathbb{R}^n \setminus F) \leq \frac{1}{4}\mu(\mathbb{R}^n) \tag{3.5}$$

and such that for each $x \in F$ there exists $W_x \in \mathbb{G}(m, \mathbb{R}^n)$ such that $\mathrm{Tan}^{(m)}(\mu, x) = \mathcal{H}^m \llcorner W_x$. Thus in particular for $x \in F$ we have

$$\theta_0 \leq \lim_{j\to\infty} \frac{\mu(B(x, r_j))}{\alpha(m) r_j^m} = \frac{\mathcal{H}^m \llcorner W_x(B(0,1))}{\alpha(m)} = 1$$

and

$$\lim_{j\to\infty} \frac{\mu\big(X_{\frac{1}{2}}(\pi_x, x) \cap B(x, r_j)\big)}{\alpha(m) r_j^m} = 0,$$

where $\pi_x = (W_x)^\perp$. We recall that if X is a metric space and f is a function defined on X, the limit $\lim\limits_{x\to 0} f(x)$ exists if and only if $\lim_{j\to\infty} f(x_j)$ exists for any sequence $\{x_j\} \subset X$ with the property that $x_j \overset{j\to\infty}{\longrightarrow} 0$, and these two limits coincide. Then for each $r_j \searrow 0$

$$\lim_{r_j\to\infty} \frac{\mu\big(X_{\frac{1}{2}}(\pi_x, x) \cap B(x, r_j)\big)}{\alpha(m) r_j^m} = \Theta^m\big(\mu \llcorner X_{\frac{1}{2}}(\pi_x, x), x\big) \overset{\text{def.}}{=} \lim_{r\to 0} \frac{\mu\big(X_{\frac{1}{2}}(\pi_x, x) \cap B(x, r)\big)}{\alpha(m) r^m}.$$

If $y \in W_x$, $y \neq x$, then $p_{\pi_x}(y - x) = 0 \not\geq \frac{1}{2}|x - y|$ and so $y \notin X_{\frac{1}{2}}(\pi, x)$, while $\mathrm{Tan}^{(m)}(x, \mu) = \{\mathcal{H}^m \llcorner W_x\}$ (from which we deduce that $\Theta^m\big(\mu \llcorner X_{\frac{1}{2}}(\pi_x, x), x\big) = 0$).

For $k = 1, 2, \ldots$ and $x \in F$ we define

$$f_k(x) = \inf_{0<\rho<\frac{1}{k}} \frac{\mu(B(x, \rho])}{\alpha(m)\rho^n}$$

and

$$q_k(x) = \sup_{0<\rho<\frac{1}{k}} \frac{\mu\big(X_{\frac{1}{2}}(\pi_x, x) \cap B(x, \rho)\big)}{\alpha(m)\rho^m}.$$

Claim: f_k and q_k are both measurable.

Proof: f_k *is measurable:* By the density of $\mathbb{Q}$ in $\mathbb{R}$, we can assume $\rho \in \mathbb{Q}$, i.e.,

$$f_k(x) = \inf_{0<\rho<\frac{1}{k}, \rho\in\mathbb{Q}} \frac{\mu(B(x, \rho])}{\alpha(m)\rho^n}.$$

We fix $\rho \in \mathbb{Q}$ and we consider

$$f_\rho(x) = \mu\big(B(x, \rho]\big).$$

Since $B(x, \rho]$ is closed, the function $x \mapsto f_\rho(x)$ is upper semi-continuous. Indeed if $x_j \to x$ in $\mathbb{R}^n$, then for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_j \in B(x, \epsilon)$ if $j \geq N$.
Now μ is a Radon measure and

$$\mu\big(B(x_j, \rho]\big) \leq \mu\big(B(x, \rho + \epsilon]\big) \leq \mu\big(B(x, \rho]\big) + \eta(\epsilon),$$

where η is a function such that $\eta(\epsilon) \overset{\epsilon}{\longrightarrow} 0$. Therefore, if $x_j \to x$, $\limsup\limits_{j\to\infty} f_\rho(x_j) = f_\rho(x)$.
At this point we have also the measurability of f_k since any upper semi-continuous function is (Borel) measurable. Indeed the upper semi-continuity of the function implies that the set $U = \{x : f_k(x) < t\}$ is open (and this is equivalent to the measurability of f_k): we fix $x \in U$ and $\epsilon = t - f_k(x)$; using the definition of semi-continuity, we find that there exists a $\delta > 0$ such that for all $y \in B(x, \delta)$, $f_k(y) < f_k(x) + \epsilon$ and so $y \in U$, i.e. U is open.

q_k *is measurable*: In order to prove the measurability of g_k the argument is slightly different.
We just consider $q_k(x) = \sup_{0<\rho<\frac{1}{k}, \rho\in\mathbb{Q}} \mu\big(X_{\frac{1}{2}}(\pi_x, x) \cap B(x, \rho)\big)$ and we observe that the set $X_{\frac{1}{2}}(\pi_x, x)$ is closed.
The μ–a.e. defined function

$$\begin{aligned} \operatorname{supp}\mu &\to \mathbb{G}(m, \mathbb{R}^n) \\ x &\mapsto \pi_x, \end{aligned} \tag{3.6}$$

where the Grassmannian space is equipped with the topology given by the metric $||p_{\pi_1} - p_{\pi_2}|| = \mathrm{dist}(\pi_1, \pi_2) = \sup_{|x|=1} |p_\pi(x) - p_{\pi'}(x)|$ for all $\pi_1, \pi_2 \in \mathbb{G}(m, \mathbb{R}^n)$, is μ–a.e. (Borel) measurable, according to Proposition 45.
We recall that we have the following hypothesis:

$$\mathrm{Tan}^{(m)}(\mu, x) = \{\mu_x\} = \{\mathcal{H}^m \llcorner W_x\}$$

that is, for all $\varphi \in C_c(\mathbb{R}^m)$ and for all sequence $r_j \searrow 0$

$$\int_{\mathbb{R}^n} \varphi(y)\, d\mu_x(y) \;=\; \lim_{j\to\infty} r_j^{-m} \int_{\mathbb{R}^n} \varphi\Big(\frac{y-x}{r_j}\Big)\, d\mu(y)$$

i.e., if we pose as usual $\mu_{x,r} = r^{-m}\big((T_{x,r})_\sharp \mu\big)$,

$$\mu_{x,r} \underset{r\to 0}{\rightharpoonup} \nu_x = \mathcal{H}^m \llcorner W_x.$$

Then the map from $\mathbb{R}^n$ to $\mathbb{R}$ defined as the composition of the functions

$$\begin{aligned} \mathbb{R}^n \;&\to\; C_c(\mathbb{R}^n)^* \;\to\; \mathbb{R} \\ x \;&\mapsto\; \nu_x \overset{\mathrm{ev}_\varphi}{\mapsto} \int \varphi\, d\nu_x, \end{aligned} \tag{3.7}$$

where $\varphi \in C_c$, is measurable with respect to the weak-* topology.

Lemma 47. *If $\varphi \in C_c(\mathbb{R}^n)$ then the function*

$$\begin{aligned} \mathbb{R}^n \;&\longrightarrow\; \mathbb{R} \\ x \;&\mapsto\; \int \varphi\, d\nu_x \end{aligned} \tag{3.8}$$

is Borel measurable.

Proof: We recall that

$$\int \varphi\, d\nu_x \;=\; \lim_{j\to\infty} \int \varphi\, d\mu_{x,r_j}$$

and that

$$\int \varphi\, d\nu_{x,r_j} \;=\; r_j^{-m} \int \varphi\left(\frac{y-x}{r_j}\right) d\mu(y).$$

The function

$$x \;\mapsto\; r_j^{-m} \int \varphi\left(\frac{y-x}{r_j}\right) d\mu(y)$$

is Borel measurable, since it's continuous.
Then the function in the statement of the lemma is Borel measurable too, being the limit of a sequence of measurable functions. $\square$

At this point we use Pettis' measurability theorem in order to ensure that the function $x \mapsto \nu_x$ is measurable. The notions that we are going to use are that one in Definition 4 and the following one:

Definition 48. Let $(X, \mathcal{M})$ be a measurable space and Y be a Banach space. Then $f : X \to Y$ is said to be **weakly measurable** if, for every continuous linear functional $g \in Y^*$, the function

$$\begin{aligned} g \circ f : X &\to \mathbb{R} \\ x &\mapsto g(f(x)) \end{aligned} \tag{3.9}$$

is a measurable function with respect to $\mathcal{M}$ and the Borel $\sigma-$algebra on $\mathbb{R}$.

Theorem 49 (Pettis' Theorem)**.** *A function $f : X \to B$ defined in a space with measure $(X, \mathcal{A}, \mu)$ and taking values in a Banach separable space B is (strongly) measurable (with respect to $\mathcal{A}$ on X and the Borel $\sigma-$ algebra on B) if and only if it is weakly measurable.*

Proof: See [PE38], Theorem 1.1. □

Actually the space $C_c(\mathbb{R}^n)^*$ is not a Banach space (it's just a separable Fréchet space) but in our case it suffices to consider the following subset of $C_c(\mathbb{R}^n)^*$:

$$Z = \big\{\mathcal{H}^m \llcorner W \,:\, W \in \mathbb{G}(m, \mathbb{R}^n)\big\}.$$

In Z there are two different topologies: the topology given by the restriction of the norm naturally defined in the dual space $C_c(\mathbb{R}^n)^*$, (Z, dist), and the weak-* topology, $(Z, \sigma < C_c(\mathbb{R}^n)^*, C_c(\mathbb{R}^n) >)$. We would like to prove that these two topologies are equivalent. Obviously the second one is weaker than the first one and so, in order to get the conclusion, we want to prove that the topology genereted by the norm is contained in the weak-* topology. We observe that Z endowed with the topology generated by dist is compact, but Z is compact also with the topology given by $\sigma(C_c(\mathbb{R}^n)^*, C_c(\mathbb{R}^n))$, which is weaker than the first one: we have proved that these two topologies coincide, i.e. they have the same open sets, since the whole space is compact not only if we consider the strongest topology (Z, dist), but even in the weak-* topology.

Now we can go on in the proof of the measurability of the function $x \mapsto \mu\big(X_{\frac{1}{2}}(\pi_x, x) \cap B(x,\rho)\big)$: to this purpose, we consider a sequence of functions $\varphi_j(x,y) \searrow 1_{X_{\frac{1}{2}}(\mu,x)}(y)\, 1_{B(x,r)}(y)$, for all $y \in \mathbb{R}^n$ such that for all x the function $\varphi_j(x,\cdot) \in C_c(\mathbb{R}^n)$ and

$$\int \varphi_j(x,y)\, d\mu(y) \underset{j\to\infty}{\longrightarrow} \mu\big(X_{\frac{1}{2}}(\pi_x, x) \cap B(x,\rho)\big).$$

Then it suffices to prove that for all $j \in \mathbb{N}$ the function

$$x \mapsto \int \varphi_j(x,y)\, d\mu(y)$$

is Borel measurable. To reach this conclusion it's enough to observe that this function is actually continuous. □

Step 2: From Step 1 we have

$$\lim_{k\to\infty} f_k(x) \geq \theta_0 \qquad \text{and} \qquad \lim_{k\to\infty} q_k(x) = 0 \quad \forall x \in F$$

and, since f_k and q_k are both measurable, we are allowed to use Egorov's theorem which ensures that there exists a $\mu-$ measurable subset $E \subset F$ such that

$$\mu(F \setminus E) \leq \frac{1}{4}\mu(\mathbb{R}^n) \tag{3.10}$$

and the limits above hold unifomely for $x \in E$, i.e., for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\frac{\mu(B(x,\rho))}{\alpha(m)\rho^n} \geq \theta_0 - \epsilon \tag{3.11}$$

and

$$\frac{\mu\big(X_{\frac{1}{2}}(\pi_x, x) \cap B(x,\rho)\big)}{\alpha(m)\rho^m} \leq \epsilon. \tag{3.12}$$

$x \in E,\ 0 < \rho < \delta.$

Step 3: At this point, we can choose $(n-m)-$dimensional subspaces $\pi_1, \ldots, \pi_N$ of $\mathbb{R}^n$, where $N \in \mathbb{N}$ depends on n and m, such that for each $(n-m)-$dimensional subspace π of $\mathbb{R}^n$ there exists $j \in \{1, \ldots, N\}$ such that $d(\pi, \pi_j) < \frac{1}{16}$.

Then we define the subsets $E_1, \ldots, E_N$ by:

$$E_j = \{x \in E : \operatorname{dist}(\pi_j, \pi_x) < \frac{1}{16}\}$$

and we clearly find $E = \bigcup_{j=1}^{N} E_j$.

Claim: If $\epsilon = \theta_0/16^n$ and if $\delta > 0$ is such that (3.11) and (3.12) hold, then

$$X_{\frac{3}{4}}(\pi_j, x) \cap E_j \cap B\Big(x, \frac{\delta}{2}\Big) = \{x\}, \qquad \text{for all}\, x \in E_j,\quad j = \ \ldots, N. \tag{3.13}$$

Proof of the claim: By contradiction, we suppose that there exists $j \in \{1, \ldots, N\}$ such that $x \in E_j$ and $y \in X_{\frac{3}{4}}(\pi_j, x) \cap E_j \cap B\Big(x, \frac{\delta}{2}\Big)$, i.e., there exists $\rho \in (0, \delta/2)$ such that $y \in X_{\frac{3}{4}}(\pi_j, x) \cap E_j \cap \partial B(x, \rho)$. Then, by (3.12), since $x \in E$ and $2\rho < \delta$, we have

$$\mu\big(X_{\frac{1}{2}}(\pi_x, x) \cap B(x, 2\rho)\big) < \epsilon\, \alpha(m)\, (2\rho)^m.$$

Again, observing that $B(y, \rho/8) \subset \big(X_{\frac{1}{2}}(\pi_x, x) \cap B(x, 2\rho)\big)$, we find, applying (3.11):

$$\mu\big(X_{\frac{1}{2}}(\pi_x, x) \cap B(x, 2\rho)\big) \geq \mu\big(B(y, \frac{\rho}{8})\big) \geq \theta_0\, \alpha(m) \left(\frac{\rho}{8}\right)^m$$

which contradicts (3.12), recalling that $\epsilon = \theta_0/16^m$. $\square$

Step 4: From (3.13) we want to find a Lipschitz function f such that the condition

$$E_j \cap B\Big(x, \frac{\delta}{2}\Big) \subset q(\operatorname{graph} f), \tag{3.14}$$

holds for all $x \in E_j$. Here q is an orthogonal transformation in $\mathbb{R}^n$ with the property that $q(\pi_j) = \mathbb{R}^{n-m}$ (as consequence of the claim above).

We prove the following lemma:

Lemma 50 (Bow-tie Lemma)**.** *Let $f : \mathbb{R}^m \longrightarrow \mathbb{R}^{n-m}$ be a function (and so graph$(f) \subset \mathbb{R}^n$).*
Then f is Lipschitz with constant $L > 0$ if and only if for all $x \in \mathbb{R}^m$

$$\mathit{graph}(f) \subset \big(x, f(x)\big) + K_L \tag{3.15}$$

where $K_L \subset \mathbb{R}^n$ is the cone defined by

$$K_L = \Big\{ (x, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m} : |y| \leq L|x| \Big\}.$$

Proof: $'' \Rightarrow ''$ If f is a Lipschitz function with constant L,

$$|f(x) - f(y)| \leq L|x-y|, \qquad \text{for all } x, y \in \mathrm{Dom}(f).$$

This is equivalent to say that the absolute value of the slopes of all secant lines is bounded by L.
Hence, for all $x \in \mathbb{R}$ the graph of the function lies completely in $\big(x, f(x)\big) + K_L$.
$'' \Leftarrow ''$ If for all $x \in \mathbb{R}$ $\mathrm{graph}(f) \subset \big(x, f(x)\big) + K_L$, then

$$|f(x) - f(y)| \leq L|x-y| \qquad \text{for all } x, y \in \mathbb{R}$$

and f is Lipschitz. $\square$

Condition (3.15) is expressed in (3.13): in fact,

$$X_{\frac{3}{4}}(\pi_j, x) \cap E_j \cap B\Big(x, \frac{\delta}{2}\Big) = \{x\} \implies E_j \cap B\Big(x, \frac{\delta}{2}\Big) \subset \Big(\mathbb{R}^n \setminus X_{\frac{3}{4}}(\pi_j, x)\Big) \cup \{x\}$$

and the set $\Big(\mathbb{R}^n \setminus X_{\frac{3}{4}}(\pi_j, x)\Big) \cup \{x\}$ is of the form K_L .

Moreover, by the definition of $X_{\frac{3}{4}}(\pi_j, x)$, we observe that the tangent planes of graph(f) are $(n-m)-$dimensional subspaces of $\mathbb{R}^n$ and so the function f is $\mathbb{R}^{n-m}$ valued and defined in a subset of $\mathbb{R}^m$.
Therefore we can find a Lipschitz function $f = (f^1, \dots, f^{n-m})$ defined in the whole space $\mathbb{R}^m$ (possibly using the Extension Theorem for Lipschitz functions in order to extend f to $\mathbb{R}^m$, and so $f : \mathbb{R}^m \longrightarrow \mathbb{R}^{n-m}$) such that if we take $x \in E_j$, we have

$$E_j \cap B\Big(x, \frac{\delta}{2}\Big) \subset q(\mathrm{graph}\, f)$$

as required in (3.14).

At this point, we observe that we can choose the Lipschitz functions $f_1, \dots, f_M$ and the orthogonal transformations $q_1, \dots, q_M$ such that

$$E \subset \bigcup_{i=1}^{M} q_i(\mathrm{graph}\, f_i)$$

and we recall that we can do this since $j \in \{1, \dots, N\}$ and $x \in E_j$ are arbitrary.

Step 5: Since we have chosen the set E such that (3.5) and (3.10) hold, we have

$$\mu\bigg(\mathbb{R}^n \setminus \bigcup_{i=1}^{M} q_i(\mathrm{graph}\, f_i)\bigg) \leq \mu\big(\mathbb{R}^n \setminus E\big) = \mu\big(\mathbb{R}^n \setminus F\big) + \mu\big(F \setminus E\big) \leq \frac{1}{2}\mu(\mathbb{R}^n).$$

In order to cover the whole set $B \subset \mathbb{R}^n$, we can argue as we have done until now, but we need to replace μ by $\mu \llcorner \bigg(\mathbb{R}^n \setminus \bigcup_{i=1}^{M} q_i(\mathrm{graph}\,(f_i))\bigg)$.
In such a way we find that there exist countably man Lipschitz graphs $F_i = \mathrm{graph}\,(f_i)$, where $f_i : \mathbb{R}^m \longrightarrow \mathbb{R}^{n-m}$ such that $\mu\bigg(\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} F_i\bigg) = 0$.

Using Theorem 22 with $t = 1$ (by Lemma 41), we deduce $\mathcal{H}^m\bigg(B \setminus \bigcup_{i=1}^{\infty} F_i\bigg) = 0$ and so B is $m-$ rectifiable.

Chapter 4

Rectifiability of the reduced boundary

In this chapter we use the terminology presented in [EG92]: the following theorems are taken from the same book, except for the Isopermetric Inequality that can be found in [AFP00]. Indeed as far as BV functions and sets of finite perimeter described in the first section we refer to sections 5.1 and 5.2 of [EG92], while for the speech about the reduced boundary we refer to section 5.7 (and, in particular, to subsectios 5.7.1 and 5.7.2).

4.1 BV Functions and Sets of Finite Perimeter

Let $U \subset \mathbb{R}^n$ an open set.

Definition 51. We say that a function $f \in L^1(U)$ has *bounded variation* in U if

$$\sup\left\{ \int_U f \operatorname{div}\varphi \, dx | \varphi \in C^1_c(U;\mathbb{R}^n),\ |\varphi| \le 1 \right\} < \infty$$

and we write $f \in BV(U)$.

Definition 52. An $\mathcal{L}^n$-measurable subset $E \subset \mathbb{R}^n$ has *finite perimeter* in U if

$$1_E \in BV(U).$$

It's useful to introduce also local versions of the above concepts:

Definition 53. A function $f \in L^1_{\text{loc}}(U)$ has *locally bounded variation* in U ($f \in BV_{\text{loc}}(U)$) if for each open set with compact closure $V \subset\subset U$ (i.e. open bounded set),

$$\sup\left\{ \int_V f \operatorname{div}\varphi \, dx | \varphi \in C^1_c(V;\mathbb{R}^n),\ |\varphi| \le 1 \right\} < \infty.$$

Definition 54. An $\mathcal{L}^n$-measurable subset $E \subset \mathbb{R}^n$ has *locally finite perimeter* in U if

$$1_E \in BV_{\text{loc}}(U).$$

Then we can introduce the following important theorem which ensures that the weak first partial derivatives of a BV function are actually Radon measures:

Theorem 55 (Structure Theorem for $BV_{\rm loc}$ Functions)**.** *Let $f \in BV_{loc}(U)$. Then there exists a Radon measure μ on U and a $\mu-$measurable function $\sigma : U \to \mathbb{R}^n$ such that*

(i) $|\sigma(x)| = 1 \;\; \mu-a.e.$

(ii) $\displaystyle\int_U f\, div\varphi\, dx \;=\; -\int_U \varphi \cdot \sigma\, d\mu,$

for all $\varphi \in C^1_c(U;\mathbb{R}^n)$.

Proof: See [EG92], Section 5.1, Theorem 1. □

If $f \in BV_{\rm loc}(U)$ we will denote the measure $\mu \equiv ||Df||$ and we call it *variation measure of* f; in particular, if $f = 1_E$ where E is a set of locally finite perimeter in U, we will write $\mu \equiv ||\partial E||$ (*perimeter measure of* E) and $n_E \equiv -\sigma$.
Hence, with this notation, for all $\varphi \in C^1_c(U;\mathbb{R}^n)$

$$\int_E {\rm div}\varphi\, dx \;=\; \int_U \varphi \cdot n_E\, d||\partial E||.$$

Remark 56. If $f \in BV_{\rm loc}(U) \cap L^1(U)$, then $f \in BV(U)$ if and only if $||Df||(U) < \infty$ and we define

$$||f||_{BV(U)} \equiv ||f||_{L^1(U)} + ||Df||(U).$$

Moreover we recall the following fact that we will use in the proof of Theoreom 65:

Theorem 57 (Lower Semicontinuity of Variation Measure)**.** *Suppose $f_k \in BV(U)$ $(k = 1, \dots)$ and $f_k \longrightarrow f$ in $L^1_{loc}(U)$. Then*

$$||Df||(U) \;\le\; \liminf_{k\to\infty} \, ||Df_k||(U).$$

Proof: We refer to [EG92], Section 5.2.1, Theorem 1. □

And finally the important inequality:

Theorem 58 (Isoperimetric Inequality)**.** *For any $\mathcal{L}^n-$measurable set $E \subset \mathbb{R}^n$, $n \ge 2$, there holds*

$$\min\left\{\mathcal{L}^n(E), \mathcal{L}^n(\mathbb{R}^n \setminus E)\right\}^{\frac{n-1}{n}} \;\le\; \frac{1}{n\alpha(n)^{\frac{1}{n}}}\, ||\partial E||(\mathbb{R}^n)$$

and there is equality if and only if E or its complement is equivalent to a ball.

Proof: See [AFP00], Theorem 3.46 or [FE69], Theorem 3.2.43 . □

4.2 The reduced boundary

Let E be a set of locally finite perimeter in $\mathbb{R}^n$.

Definition 59. We say that a point $x \in \mathbb{R}^n$ is in the *reduced boundary* of E ($x \in \partial^\star E$) if

(i) $||\partial E||(B(x,r)) > 0$ for all $r > 0$,

(ii) $\lim_{r\to 0} \fint_{B(x,r)} n_E \, d||\partial E|| = n_E(x)$,

(iii) $|n_E(x)| = 1$.

Theorem 60 (Lebesgue-Besicovitch Differentation Theorem)**.** *Let μ be a Radon measure on $\mathbb{R}^n$ and $f \in L^1_{loc}(\mathbb{R}^n, \mu)$. Then*

$$\lim_{r\to 0} \fint_{B(x,r)} f d\mu = f(x)$$

for μ almost every $x \in \mathbb{R}^n$.

Remark 61.

$$||\partial E||(\mathbb{R}^n \setminus \partial^\star E) = 0.$$

Proof: First of all we observe that if (ii) in Definition 59 is true, then also (i) has to hold.

Actually, if $x \in \mathbb{R}^n \setminus \partial^\star E$ and (ii) in Definition 59 doesn't hold, we get a contradiction with Theorem 60, which states that we can differentiate with respect to any Radon measure, and $||\partial E||$ is a Radon measure.

But also if $x \in \mathbb{R}^n \setminus \partial^\star E$ and (iii) in Definition 59 isn't true, we have a contradiction with Riesz Theorem (Theorem 23-(i)) which ensures that $|n_E(x)| = 1$ for $||\partial E||-$almost every $x \in \mathbb{R}^n$.

Therefore, we can conclude that $||\partial E||(\mathbb{R}^n \setminus \partial^\star E) = 0$. $\square$

Lemma 62. *Let $\varphi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$. Then for each $x \in \mathbb{R}^n$ and for $\mathcal{L}^1$ a.e. $r > 0$,*

$$\int_{E\cap B(x,r)} div\varphi \, dy = \int_{B(x,r)} \varphi \cdot n_E \, d||\partial E|| + \int_{E\cap\partial B(x,r)} \varphi \cdot n \, d\mathcal{H}^{n-1},$$

where n is the outward unit normal to $\partial B(x,r)$.

Proof: We take a smooth function $h : \mathbb{R}^n \longrightarrow \mathbb{R}$, then

$$\int_E \operatorname{div}(h\varphi) \, dy = \int_E h\operatorname{div}\varphi \, dy + \int_E Dh \cdot \varphi \, dy$$

and so

$$\int_{\mathbb{R}^n} h\varphi \cdot n_E \, d||\partial E|| = \int_E h\operatorname{div}\varphi \, dy + \int_E Dh \cdot \varphi \, dy.$$

In particular this is true if we consider

$$h_\epsilon(y) \equiv g_\epsilon(|y - x|)$$

with $g_\epsilon : [0,\infty) \longrightarrow [0,1]$

$$g_\epsilon(s) \equiv \begin{cases} 1 & \text{if } 0 \le s \le r, \\ 0 & \text{if } s \ge r + \epsilon, \\ \frac{1}{\epsilon}(r + \epsilon - s) & \text{if } r \le s \le r + \epsilon. \end{cases}$$

Actually, the function h_ϵ is just Lipschitz continuous, not C^1, but we can mollify it. Since

$$g'_\epsilon(s) = \begin{cases} 0 & \text{if } 0 \le s < r \text{ or } s > r+\epsilon \\ -\dfrac{1}{\epsilon} & \text{if } r < s < r+\epsilon \end{cases}$$

we have

$$Dh_\epsilon = \begin{cases} 0 & \text{if } |y-x| < r \text{ or } |y-x| > r+\epsilon \\ -\dfrac{1}{\epsilon}\dfrac{y-x}{|y-x|} & \text{if } r < |y-x| < r+\epsilon. \end{cases}$$

Then we get

$$\int_{\mathbb{R}^n} h_\epsilon \varphi \cdot n_E \, d||\partial E|| = \int_E h_\epsilon \operatorname{div}\varphi \, dy - \frac{1}{\epsilon}\int_{E\cap\{y|r<|y-x|<r+\epsilon\}} \varphi \cdot \frac{y-x}{|y-x|}\, dy.$$

We let $\epsilon \searrow 0$ and we obtain for $\mathcal{L}^1-$a.e. $r > 0$

$$\int_{B(x,r]} \varphi \cdot n_E \, d||\partial E|| = \int_{E\cap B(x,r)} \operatorname{div}\varphi \, dy - \int_{E\cap\partial B(x,r)} \varphi \cdot n \, d\mathcal{H}^{n-1}. \quad \square$$

Lemma 63. *Let $x \in \partial^\star E$. Then there exist positive constants C_1, C_2, C_3 depending only on n, such that*

(i) $\liminf_{r\to 0} \dfrac{\mathcal{L}^n(B(x,r)\cap E)}{r^n} > C_1 > 0,$

(ii) $\liminf_{r\to 0} \dfrac{\mathcal{L}^n(B(x,r)\setminus E)}{r^n} > C_2 > 0,$

(iii) $\limsup_{r\to 0} \dfrac{||\partial(E\cap B(x,r))||(\mathbb{R}^n)}{r^{n-1}} \le C_3.$

Proof: (i) We choose $\varphi \in C^1_c$, $|\varphi| \le 1$, then, according to Lemma 62, for $\mathcal{L}^1$ a.e. $r > 0$

$$\begin{aligned}\int_{E\cap B(x,r)} \operatorname{div}\varphi \, dy &= \int_{B(x,r)} \varphi \cdot n_E \, d||\partial E|| + \int_{E\cap\partial B(x,r)} n \cdot \varphi \, d\mathcal{H}^{n-1} \\ &\le ||\partial E||(B(x,r)) + \mathcal{H}^{n-1}(\partial B(x,r)\cap E)\end{aligned}$$

we obtain

$$||\partial(E\cap B(x,r))||(\mathbb{R}^n) \le ||\partial E||(B(x,r)) + \mathcal{H}^{n-1}(E\cap\partial B(x,r)). \tag{4.1}$$

Next we choose $\varphi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$\varphi \equiv n_E(x) \text{ on } B(x,r).$$

Then

$$\int_{B(x,r)} n_E(x) \cdot n_E \, d||\partial E|| = -\int_{E\cap\partial B(x,r)} n_E(x) \cdot n \, d\mathcal{H}^{n-1}. \tag{4.2}$$

Moreover, since $x \in \partial^\star E$,

$$\lim_{r\to 0} n_E(x) \cdot \fint_{B(x,r)} n_E \, d||\partial E|| = |n_E(x)|^2 = 1$$

and (4.2), for $\mathcal{L}^1$−a.e. $r > 0$ small enough, $0 < r < r_0 = r_0(x)$, we have

$$\frac{1}{2} \le \left| \frac{1}{||\partial E||(B(x,r))} n_E(x) \cdot \int_{E\cap\partial B(x,r)} n \, d\mathcal{H}^{n-1} \right| \le \frac{\mathcal{H}^{n-1}(E \cap \partial B(x,r))}{||\partial E||(B(x,r))} \tag{4.3}$$

This, together with (4.1), leads to

$$||\partial(E \cap B(x,r))||(\mathbb{R}^n) \le 3\mathcal{H}^{n-1}(\partial B(x,r) \cap E) \tag{4.4}$$

for a.e. $0 < r < r_0$.

Next, we consider the function

$$\begin{aligned} g(r) &= \mathcal{L}^n(E \cap B(x,r)) \\ &= \int_0^r \left(\int_{E\cap\partial B(x,r)} d\mathcal{H}^{n-1} \right) ds \\ &= \int_0^r \mathcal{H}^{n-1}(\partial B(x,s) \cap E) \, ds \end{aligned}$$

and the function g is absolutely continuous with

$$g'(r) = \mathcal{H}^{n-1}(\partial B(x,r) \cap E) \qquad \text{for a.e. } r > 0.$$

Then, by the Isoperimetric Inequality (Theorem 58):

$$\begin{aligned} g(r)^{\frac{n-1}{n}} &= \mathcal{L}^n(B(x,r) \cap E)^{\frac{n-1}{n}} \le C||\partial(B(x,r) \cap E)||(\mathbb{R}^n) \\ &\le 3C\mathcal{H}^{n-1}(\partial B(x,r) \cap E) \le 3Cg'(r) \end{aligned} \tag{4.5}$$

where C is the Isoperimetric constant. Hence

$$\frac{r}{3Cn} \le \int_0^r \left(g(r)^{\frac{1}{n}}\right)' dr = g(r)^{\frac{1}{n}}$$

and so

$$g(r) \le \frac{1}{3nC} r^n.$$

At this point, we have that for $0 < r \ge r_0$

$$\liminf_{r \searrow 0} \frac{\mathcal{L}^n(E \cap B(x,r))}{r^n} \le \frac{1}{3nC}$$

and this proves (i).
Now we observe that for all $\varphi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$ we have

$$\int_E \mathrm{div}\varphi \, dx + \int_{\mathbb{R}^n \setminus E} \mathrm{div}\varphi \, dx = \int_{\mathbb{R}^n} \mathrm{div}\varphi \, dx = 0.$$

Therefore

$$||\partial E|| = ||\partial(\mathbb{R}^n \setminus E)|| \qquad \text{and} \qquad n_E = -n_{\mathbb{R}^n\setminus E}$$

and (ii) follows from (i). In (4.3) we have found that for all $0 < r < r_0$

$$||\partial E||(B(x,r)) \le 2\mathcal{H}^{n-1}(E \cap \partial B(x,r)) \le Cr^{n-1}$$

which implies

$$\limsup_{r \searrow 0} \frac{||\partial E||(B(x,r))}{r^{n-1}} \le \text{const.} < \infty.$$

This together with (4.1) gives (iii). $\square$

4.3 The reduced boundary is $n-1$ rectifiable

Let E be a bounded set of $\mathbb{R}^n$ with locally finite perimeter. We consider the distributional derivative $D1_E = n_E \cdot ||D1_E||$: this is a vector valued measure such that for all $\varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$

$$< \varphi, Df > := \int_{\mathbb{R}^n} f \mathrm{div}\varphi dx.$$

We know that the measure $\mu = ||D1_E||$ is a Radon measure. Our aim is to prove that this measure (and the reduced boundary $\partial^\star E$) is $n-1$ rectifiable.

We chose a point $x \in \partial^\star E$ and we define the homothetic expansion of E in x of ratio r as

$$E_{x,r} := \frac{E-x}{r}.$$

The following theorem proves that this homothetic expansion tends to an appropriate half space in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Definition 64. For each $x \in \partial^\star E$, define the hyperplane

$$H(x) \equiv \{n_E(x) \cdot (y-x) = 0\}$$

and the half-spaces

$$H^+(x) \equiv \{n_E(x) \cdot (y-x) \geq 0\},$$
$$H^-(x) \equiv \{n_E(x) \cdot (y-x) \leq 0\}.$$

Theorem 65 (Homotetic Expansion Theorem)**.** *Let $x \in \partial^\star E$. Then*

$$1_{E_{x,r}} \longrightarrow 1_{H^-(x)} \ in L^1_{loc}(\mathbb{R}^n)$$

as $r \to 0$.

Proof: Without loss of generality we can assume

$$\begin{cases} x = 0, \\ n_E(0) = e_n = (0, \dots, 0, 1), \\ H(0) = \{y \in \mathbb{R}^n | y_n = 0\}, \\ H^+(0) = \{y \in \mathbb{R}^n | y_n \geq 0\}, \\ H^-(0) = \{y \in \mathbb{R}^n | y_n \leq 0\}. \end{cases}$$

Let r_k be any sequence which tends to 0: it suffices to prove that there exists a subsequence $\{s_j\}_{j=1}^\infty$ of $\{r_k\}_{k=1}^\infty$ such that

$$1_{E_{s_j}} \overset{j\to\infty}{\longrightarrow} 1_{H^-(0)} \ \mathrm{in} L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

Now we fix $L > 0$ and pose

$$D_r \equiv E_r \cap B(0, L), \qquad g_r(y) = \frac{y}{r}.$$

For any $\varphi \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$, $|\varphi| \leq 1$, we have

$$\begin{aligned} \int_{D_r} \mathrm{div}\varphi \, dz &= \frac{1}{r^{n-1}} \int_{E \cap B(0,rL)} \mathrm{div}(\varphi \circ g_r) \, dy \\ &= \frac{1}{r^{n-1}} \int (\varphi \circ g_r) \cdot n_{E \cap B(0,rL)} \, d||\partial(E \cap B(0, rL))|| \\ &\leq \frac{||\partial(E \cap B(0, rL))||(\mathbb{R}^n)}{r^{n-1}} \\ &\leq C < \infty, \end{aligned}$$

where $r \in (0,1]$ and in the last line we have used Lemma 63-(iii).
But so for $r \in (0,1]$ we have also

$$||\partial D_r||(\mathbb{R}^n) \leq C < \infty.$$

Noticing that

$$||1_{D_r}||_{L^1(\mathbb{R}^n)} \;=\; \mathcal{L}^n(D_r) \;\leq\; \mathcal{L}^n(B(0,L)) < \infty$$

we have

$$||1_{D_r}||_{\mathrm{BV}(\mathbb{R}^n)} \;=\; ||1_{D_r}||_{L^1(\mathbb{R}^n)} + ||\partial D_r||(\mathbb{R}^n) \;\leq C < \infty.$$

Using the Compactness Theorem (Theorem 27), we find that there exists a subsequence $\{s_j\}_{j=1}^{\infty} \subset \{r_k\}_{k=1}^{\infty}$ and a function $f \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$1_{E_{s_j}} \longrightarrow f \;\text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

Possibly passing to another subsequence of $\{s_j\}_{j=1}^{\infty}$ we can assume that $1_{E_{s_j}}$ converges to f $\mathcal{L}^n-$a.e. too: therefore $f(x) \in \{0,1\}$ for $\mathcal{L}^n-$a.e. x and there exists a subset $F \subset \mathbb{R}^n$ with locally finite perimeter such that

$$f = 1_F \;\; \mathcal{L}^n - \text{a.e.}$$

and so if $\varphi \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$,

$$\int_F \mathrm{div}\varphi \, dy \;=\; \int_{\mathbb{R}^n} \varphi \cdot n_F \, d||\partial F||$$

where n_F is a $||\partial F||-$measurable function with $|n_F| = 1$ $||\partial F||-$a.e.

At this point our goal is to prove that $F = H^-(0)$.

First step: $n_F = e_n$ $||\partial F||-$a.e.
Proof: Let $\varphi \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$, we have $\forall\, j$

$$\int_{\mathbb{R}^n} \varphi \cdot n_{E_j} \, d||\partial E_j|| \;=\; \int_{E_j} \mathrm{div}\varphi \, dy.$$

Since $1_{E_j} \to 1_F$ in L^1_{loc}, we have for all $\varphi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$

$$\int_{E_j} \mathrm{div}\varphi \, dy \;=\; \int_F \mathrm{div}\varphi \, dy$$

and so

$$\int_{\mathbb{R}^n} \varphi \cdot n_{E_j} \, d||\partial E_j|| \overset{j\to\infty}{\longrightarrow} \int_{\mathbb{R}^n} \varphi \cdot n_F \, d||\partial F||.$$

Therefore

$$n_{E_j}||\partial E_j|| \overset{*}{\rightharpoonup} n_F||\partial F||$$

in the weak* convergence of Radon measures.
Then we choose $L > 0$ such that $||\partial F||(\partial B(0,L))$, and we have

$$\int_{B(0,L)} n_{E_j} \, d||\partial E_j|| \;\to\; \int_{B(0,L)} n_F \, d||\partial F||.$$

Next we note that $(\star)$

- $||\partial E_j||(B(0,L)) \;=\; \left|\left|\partial\left(\dfrac{1}{s_j}E\right)\right|\right|\left(\dfrac{1}{s_j}B(0,s_jL)\right) \;=\; \dfrac{1}{s_j^{n-1}}\,||\partial E||\Big(B(0,s_jL)\Big);$

- $\displaystyle\int_{B(0,L)} n_{E_j}\, d||\partial E_j|| \;=\; \frac{1}{s_j^{n-1}} \int_{B(0,s_jL)} n_E\, d||\partial E||.$

So it follows that

$$\lim_{j\to\infty} \fint_{B(0,s_jL)} n_{E_j}\, d||\partial E_j|| \;=\; \lim_{j\to\infty} \frac{\displaystyle\int_{B(0,s_jL)} n_E\, d||\partial E||}{||\partial E||\big(B(0,s_jL)\big)} \overset{0\in\partial^\star E}{=} n_E(0) \;=\; e_n.$$

At this point we use the fact that $||\partial F||(\partial B(0,L)) = 0$ and the Lower Semicontinuity Theorem to find

$$||\partial F||(B(0,L)) \le \liminf_{j\to\infty} ||\partial E_j||(B(0,L)),$$

and, possibily passing to a subsequence, we can consider the limit instead of the lim inf:

$$||\partial F||(B(0,L)) \le \lim_{j\to\infty} ||\partial E_j||(B(0,L)); \tag{4.6}$$

moreover

$$\begin{aligned} \lim_{j\to\infty} ||\partial E_j||(B(0,L)) \;&=\; \lim_{j\to\infty} \int_{B(0,L)} e_n \cdot n_{E_j}\, d||\partial E_j|| \\ &\overset{\text{weak conv.}}{=} \int_{B(0,L)} e_n \cdot n_F\, d||\partial F|| \\ &\overset{\text{Cauchy-Schwartz ineq.}}{\le} ||\partial F||(B(0,L)). \end{aligned} \tag{4.7}$$

But then we find that the inequalities in (4.6) and (4.7) are actually equalities and, in particular

$$e_n \cdot n_F = 1 \qquad ||\partial F|| - \text{a.e.}$$

and, recalling that $|n_F| = 1$ $||\partial F||-$a.e., this is equivalent to

$$n_F = e_n \qquad ||\partial F|| - \text{a.e.}$$

Second step: F is a half space, i.e. $F \;=\; \big\{y \in \mathbb{R}^n : y_n \le \gamma\big\}$ for some $\gamma \in \mathbb{R}$.
Proof: In the proof of the first step we have seen that for all $\varphi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$

$$\int_F \mathrm{div}\varphi\, dz \;=\; \int_{\mathbb{R}^n} \varphi \cdot e_n\, d||\partial F||.$$

Then we fix $\epsilon > 0$ and we consider the mollified function $f_\epsilon \;=\; 1_F * \eta_\epsilon$ with $\eta_\epsilon(x) := \dfrac{1}{\epsilon^n}\,\eta\left(\dfrac{x}{\epsilon}\right)$ Now $f_\epsilon \in C^\infty(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} f_\epsilon\, \mathrm{div}\varphi\, dy \;=\; \int_F \mathrm{div}\varphi_\epsilon\, dy \;=\; \int_{\mathbb{R}^n} \eta_\epsilon * (\varphi \cdot e_n)\, d||\partial F||.$$

On the other hand, integrating by parts we find:

$$\int_{\mathbb{R}^n} f_\epsilon \mathrm{div}\varphi\, dz \;=\; -\int_{\mathbb{R}^n} Df_\epsilon \cdot \varphi\, dz.$$

a) If $\varphi \in C^1_c(< e_n >^\perp, \mathbb{R}^n)$, then $\varphi_n = 0$ and $\varphi \cdot n_E \equiv 0$, and so

$$-\int_{\mathbb{R}^n} Df_\epsilon \cdot \varphi\, dy \;=\; \int_{\mathbb{R}^n} \varphi_\epsilon \cdot n_E\, d||\partial E|| \;=0$$

by the Fundamental Theorem of Calculus of Variations we have

$$Df_\epsilon = 0 \qquad \text{a.e. in} \quad < e_n >^\perp,$$

i.e. in each hyperplane perpendicular to e_n (each of them is connected) f_ϵ is constant.

b) If $\varphi \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$ with $\varphi \cdot n_E \neq 0$ ($\varphi_n \neq 0$): then there exists a function $a : \mathbb{R}^n \longrightarrow \mathbb{R}_{>0}$ such that
$$\varphi(x) = a(x) \cdot n_E.$$
Therefore
$$\begin{aligned} -\int_{\mathbb{R}^n} Df_\epsilon \cdot a(x) n_E \, dx &= \int_{\mathbb{R}^n} f_\epsilon \operatorname{div}(a(x) n_E) \, dx \\ &= \int_{\mathbb{R}^n} f_\epsilon \cdot \nabla a(x) n_E \, dx = \int_E \underbrace{\nabla a(x) n_E}_{=\operatorname{div}\varphi_\epsilon} dx \\ &= \int_{\mathbb{R}^n} \underbrace{a(x)}_{\varphi_\epsilon \cdot n_E} d||\partial E|| \geq 0 \end{aligned}$$

Therefore we get the conclusion:
$$\frac{\partial f_\epsilon}{\partial x_i} \equiv 0 \qquad \text{for} \quad i = 1, \ldots, n-1, \qquad \text{and} \qquad \frac{\partial f_\epsilon}{\partial x_n} \leq 0$$
(1_F is equal to 1 or 0 in each hyperplane perpendicular to e_n, while it's decreasing in the e_n-direction). We recall that $f_\epsilon \to 1_F$ $\mathcal{L}^n-$a.e. if ϵ tends to zero and so there exists a $\gamma \in \mathbb{R}$ such that, up to a set of $\mathcal{L}^n-$measure zero,
$$F = \{y \in \mathbb{R}^n : y_n \leq \gamma\}.$$

Third step: $F = H^-(0)$, i.e. $\gamma = 0$.
Proof: We know that $x \in \partial^\star E$ and we suppose that $\gamma > 0$. Next we take the sequence defined above $E_j = \dfrac{1}{s_j} E$, $1_{E_j} \longrightarrow 1_E$ in $L^1_{\text{loc}}(\mathbb{R}^n)$.
Then
$$\begin{aligned} \alpha(n)\gamma^n = \mathcal{L}^n(B(0,\gamma) \cap E) &= \lim_{j\to\infty} \mathcal{L}^n(B(0,\gamma) \cap E_j) \\ &= \lim_{j\to\infty} \frac{\mathcal{L}^n(B(0,\gamma s_j) \cap E)}{s_j^n} \end{aligned} \tag{4.8}$$
which gives a contradiction to Lemma 63-(i). If we suppose $\gamma < 0$, the same argument leads to a contradiction to Lemma 63-(ii). □

But since $1_{E_{x,r}} \xrightarrow{L^1_{\text{loc}}} 1_{H^-(x)}$, it is true that $||D1_{E_{x,r}}|| \longrightarrow ||D1_{H^-(x)}||$ and $D1_{E_{x,r}} \rightharpoonup D1_{H^-(x)} = \mathcal{H}^{n-1} \llcorner \underbrace{\partial H^-(x)}_{=H(x) \in \mathbb{G}(n-1,\mathbb{R}^n)}$ in the weakly* topology of $C_c(\mathbb{R}^n, \mathbb{R}^n)^*$.

Indeed the following lemma is true:

Lemma 66. *If $\{f_j\}_j$, $f \in L^1_{loc}$ such that $f_j \longrightarrow f$ in L^1, then $Df_j \overset{*}{\rightharpoonup} Df$.*

Proof: Choose $\varphi \in C^\infty_c(\mathbb{R}^n, \mathbb{R}^n)$
$$< \varphi, Df_j > = \int f_j \operatorname{div}\varphi \xrightarrow{L^1} \int f \operatorname{div}\varphi = < \varphi, Df > . \quad \square$$

Therefore the tangent measure is flat almost everywhere: if we pose $\mu = ||D1_E||$, $\operatorname{Tan}^{n-1}(\mu, x) = \{\mathcal{H}^{n-1} \llcorner W, \text{ with } W \in \mathbb{G}(n-1, \mathbb{R}^n)\}$ a.e. $x \in \partial^\star E$ which is equivalent to say that $\partial^\star E$ is $n-1$ rectifiable, by Theorem 42.

In addiction as we have seen in Remark 61 $||D1_E||(\mathbb{R}^n \setminus \partial^\star E) = 0$ and we observe that

$||D1_E|| << \mathcal{H}^{n-1}$; then, by the Isoperimetric Inequality (Gagliardo-Nirenberg-Sobolev, Theorem 58), we get

$$||D1_E|| = C\mathcal{H}^{n-1}\llcorner\partial^\star E$$

where $C \in \mathbb{R}^+$. In order to conclude that $C = 1$ we have to use an argument of density and, in particular, the following:

Proposition 67. *Let* $x \in \partial^\star E$

$$\lim_{r\to 0} \frac{||\partial E||(B(x,r))}{\alpha(n-1)r^{n-1}} = 1.$$

Proof: By $(\star)$,

$$\frac{||\partial E||(B(x,r))}{r^{n-1}} = ||\partial E_r||(B(x,1)).$$

Now $||\partial H^-(x)||\partial(B(x,1)) = \mathcal{H}^{n-1}(\partial B(x,1)) \cap H(x)) = 0$ and, by the Homotetic Expanction Theorem, we find

$$\begin{aligned}\lim_{r\to 0} \frac{||\partial E||(B(x,r))}{r^{n-1}} &= ||\partial H^-(x)||(B(x,1)) = \mathcal{H}^{n-1}(B(x,1)\cap H(x)) \\ &= \alpha(n-1). \quad \square\end{aligned}$$

Remark 68. (on Lemma 66) $f_j \longrightarrow f$ in L^1 $\not\Rightarrow$ $||Df_j|| \overset{*}{\rightharpoonup} ||Df||$, where we recall that for all $U \subset \mathbb{R}^n$

$$||Df||(U) = \sup\left\{ \int \text{div}\varphi f \ : \ \text{supp}\varphi \subseteq U \text{ and } ||\varphi||_\infty \le 1\right\}$$

Example 69. In $\mathbb{R}^2$ we consider the function $f_r = \dfrac{1}{r}\, 1_{B_r}$. Then we have:

- $||f_r||_{L^1} = \pi r \underset{r\to 0^+}{\longrightarrow} 0,$
- $Df_r = \dfrac{1}{r} \cdot \dfrac{x}{|x|}\, \mathcal{H}^1\llcorner\partial B(x,r),$
- $Df_r \rightharpoonup 0$ weakly*,

but $||Df_r|| \rightharpoonup 2\pi\delta_0$. Indeed by Structure Theorem for BV_{loc} Functions (Theorem 55)

$$Df_r = \sigma\mu_r$$

where $\sigma : U \to \mathbb{R}^n$, $|\sigma| = 1$: in our case $\sigma(x) = \dfrac{x}{|x|}$ and $Df_r = \dfrac{1}{r} \cdot \dfrac{x}{|x|}\, \mathcal{H}^1\llcorner\partial B(x,r)$, so

$$\mu_r = ||Df_r|| = \frac{1}{r}\mathcal{H}^1\llcorner\partial B(x,r).$$

We observe that μ_r has no more orientation and that $(\mu_r)_{r>0}$ is a sequence of measures with the same mass.
Moreover

$$||Df_r|| \rightharpoonup 2\pi\delta_0$$

since we have $||Df_r||(\mathbb{R}^2) = 2\pi$ and for all $\varphi \in C^1_c(U)$

$$\lim_{r\to 0} \int_U \varphi(x)\, \frac{1}{r}\, d\mathcal{H}^1\llcorner\partial B(x,r) = 2\pi\varphi(0).$$

However if $0 \in \partial^\star E$ and $f_r = 1_{\frac{E}{r}}$, we know that $||D1_E|| = n \cdot ||D1_E||$ and that

- $x \in \partial^\star E$ if and only if $||D1_E||\big(B(x,r)\big) > 0,\ \forall\, r > 0,$

- $\lim_{r\to 0} \fint_{B(x,r)} n(y)||D1_E||(y) = n^*(x)$ exists and $||n^*(x)|| = 1.$

This is equivalent to say that there is not loss of mass and it's true that if $1_{\frac{E}{r}} \underset{L^1}{\longrightarrow} 1_{H^-(0)}$, then $||D1_{\frac{E}{r}}|| \rightharpoonup ||D1_{H^-(0)}||$.

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