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Non-Minimality for a Class of Angles

Tesi di Laurea Magistrale

Candidato
Giulia Vescovo
Matricola: 1041138

Relatore
Prof. Roberto Monti
Università di Padova

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Preface

In this thesis we deal with the problem of regularity of length-minimizing curves in Carnot-Carathéodory spaces.

In the first chapter we study the notion of *sub-Riemannian manifold*, that is a triple $\big(M, \mathcal{D}, g\big)$ where M is a smooth manifold, $\mathcal{D} \subseteq TM$ is a distribution on M and g is a metric on $\mathcal{D}$. Since we deal with a problem of local nature, we identify M with $\mathbb{R}^n$ and $\mathcal{D}$ with a distribution on $\mathbb{R}^n$ generated by a global frame of orthonormal smooth vector fields $X_1, \ldots, X_r$. We introduce the class of *$\mathcal{D}$-horizontal curves*, that are Lipschitz curves γ tangent to the distribution $\mathcal{D}$ at almost every point, i.e. there exists $h \in L^\infty\big([0,1], \mathbb{R}^r\big)$ (whose components are called *controls* of γ) such that

$$\dot{\gamma}(t) = \sum_{i=1}^{r} h_i(t)\, X_i\big(\gamma(t)\big) \qquad \text{for a.e. } t \in [0,1].$$

Thus we can define the *length* of the $\mathcal{D}$-horizontal curve γ as follows:

$$L(\gamma) := \int_0^1 |h(t)|\, dt.$$

We define the *Carnot-Carathéodory distance* between two points x and y in $\mathbb{R}^n$ as the infimum of the lengths of all $\mathcal{D}$-horizontal curves joining x to y. By the Chow-Rashevsky Theorem, if $\mathcal{D}$ is bracket-generating then any couple of points in $\mathbb{R}^n$ can be connected by a $\mathcal{D}$-horizontal curve, hence the Carnot-Carathéodory distance is actually a distance on $\mathbb{R}^n$. A *length-minimizer* joining x to y is a $\mathcal{D}$-horizontal curve that realizes the distance between x and y. In general, length-minimizers do not exist globally, but we will prove their local existence. Moreover, a length-minimizer is in particular an *extremal*, that is a $\mathcal{D}$-horizontal curve which satisfies the first-order necessary conditions of Pontryagin Maximum Principle. Extremals can be either *normal* or *abnormal*: normal extremals are C^∞-smooth, while strictly abnormal extremals (i.e. extremals that are abnormal but not normal) could develop singularities.

The principal open problem in Geometric Control Theory and Calculus of Variations in Carnot-Carathéodory spaces is the regularity of length-minimizers (see [5], Chapter 10, Paragraph 10.1, or [1] and [6]). Originally - by using a wrong argument - Strichartz proved that all length-minimizers are smooth: by applying Pontryagin Maximum Principle, he forgot the case of abnormal extremals. In 1994 Montgomery exhibited the first example of abnormal length-minimizer (see [4]). In 1995 Liu and

Sussmann discovered the class of *regular abnormal extremals*, i.e. abnormal extremals that are always locally length-minimizing (see [9]). On the other hand, all known examples of length-minimizers are smooth. Thus, the open questions are the following:

- Are all length-minimizers C^∞-smooth?
- Are all length-minimizers C^1-smooth?
- Can length-minimizers present angles?

The second chapter contains new results. We prove the non-minimality of angles in a certain class of examples which is not included in the known literature.

Let M be an n-dimensional smooth manifold and let $\mathcal{D} \subseteq TM$ be a bracket-generating distribution of rank r, for some $r = 1, \dots, n$. Suppose that $X_1, \dots, X_r$ constitute a frame of vector fields such that $\mathcal{D} = \operatorname{span}\{X_1, \dots, X_r\}$. For every $\ell \in \mathbb{N}$, let us call $\mathcal{D}_\ell$ (resp. $\mathcal{L}_\ell$) the distribution spanned by the iterated commutators of $X_1, \dots, X_r$ of length *equal to* ℓ (resp. *at most* ℓ), so that $\mathcal{D}_0 = \mathcal{L}_0 = \{0\}$, $\mathcal{D}_1 = \mathcal{L}_1 = \mathcal{D}$ and $\mathcal{L}_\ell = \mathcal{D}_0 + \ldots + \mathcal{D}_\ell$.

In [3] it is proved that if $(M, \mathcal{D}, g)$ is a sub-Riemannian manifold, where $\mathcal{D}$ is *equiregular* (i.e. $\dim(\mathcal{D}_\ell)$ is constant in M) and satisfies

$$[\mathcal{L}_i, \mathcal{L}_j] \subseteq \mathcal{L}_{i+j-1} \quad \text{for every } \ i,j \geq 2 \ \text{ with } \ i+j \geq 5, \tag{1}$$

then a $\mathcal{D}$-horizontal curve on M with a corner-type singularity is not length-minimizing.

Moreover, in [2] it is proved that the same thesis holds if we replace the hypothesis of equiregularity and (1) with the following condition:

$$\mathcal{L}_i(x) \neq \mathcal{L}_{i-1}(x) \implies \mathcal{L}_{i+1}(x) = \mathcal{L}_i(x) \quad \text{for every } \ i \geq 1 \text{ and } x \in M. \tag{2}$$

In this chapter we study a class of non-equiregular distributions that do not satisfy neither (1) nor (2) but in which angles are not length-minimizers. The main result of the thesis is the following:

Theorem 0.1. *Let $\alpha \in \mathbb{N}^+$, $\beta \in \mathbb{N}$ and $\gamma \in \mathbb{N}^+$. Let $\mathcal{D}$ be the distribution in $\mathbb{R}^4$ of 2-planes spanned pointwise by the vector fields*

$$X_1 = \frac{\partial}{\partial x_1}, \qquad X_2 = \frac{\partial}{\partial x_2} + x_1^\alpha x_2^\beta \frac{\partial}{\partial x_3} + x_3^\gamma \frac{\partial}{\partial x_4}. \tag{3}$$

Let $\nu : [-1, 1] \to \mathbb{R}^4$ be the $\mathcal{D}$-horizontal curve defined by

$$\nu(t) = \begin{cases} (0, -t, 0, 0) & \text{if } t \in [-1, 0], \\ (t, 0, 0, 0) & \text{if } t \in [0, 1]. \end{cases} \tag{4}$$

Then ν is not length-minimizing.

The proof of the previous theorem consists of the following steps:

◇ STEP 1: Let $\mathcal{D}$ be the distribution generated by (3). We have that

$$\operatorname{step}\big(\mathcal{D}\big) = \gamma(\alpha+\beta+1)+1,$$

in particular $\mathcal{D}$ is bracket-generating in all of $\mathbb{R}^4$.

◇ STEP 2: For the distribution spanned by (3), the following are equivalent:

(i) the distribution is not equiregular and does not satisfy neither (1) nor (2),

(ii) $(\alpha,\beta)=(1,0)$ and $\gamma\geq 2$.

◇ STEP 3: Hereafter, we shall restrict to the case $(\alpha,\beta)=(1,0)$ and $\gamma\geq 2$. The angle ν defined in (4) is a strictly abnormal extremal.

◇ STEP 4: In order to prove that ν is not length-minimizing, we adapt the shortening technique introduced in [3]: we exhibit a $\mathcal{D}$-horizontal curve joining the points $\nu(-1)=(1,0,0,0)$ and $\nu(1)=(0,1,0,0)$, whose length is strictly smaller than 2, which is the length of ν.

In the case $\gamma\geq 3$ we proceed as follows: first of all, we "cut" the corner ν by considering the $\mathcal{D}$-horizontal curve ν^ε (for some $\varepsilon\in(0,1)$) whose first two components coincide with the polygonal planar curve $(0,1)\to(0,\varepsilon)\to(\varepsilon,0)\to(1,0)$. The curve ν^ε is strictly shorter than ν, but its endpoint has changed into

$$\left(1,0,-\frac{1}{2}\,\varepsilon^2,\frac{(-1)^{\gamma+1}}{2^\gamma}\,\frac{\varepsilon^{2\gamma+1}}{2\gamma+1}\right).$$

In order to correct the endpoint, we have to modify ν^ε: consider the lift μ^ε of the planar curve (depending on suitable positive parameters a, b, c and r) in Figure 1.

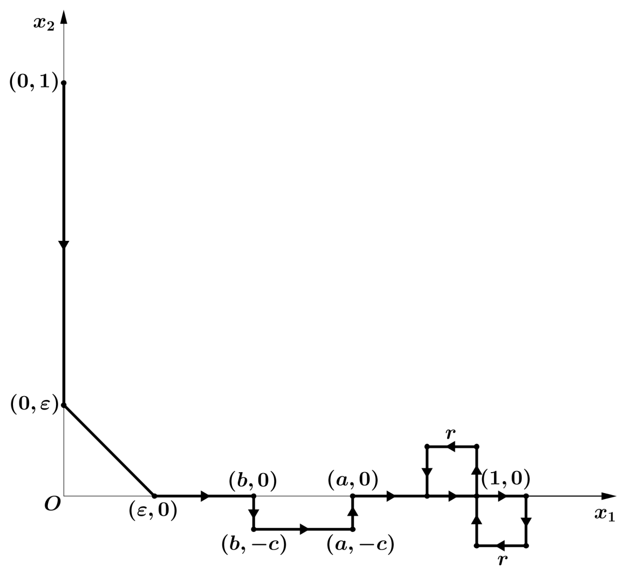

Figure 1: The curve $(\mu_1^\varepsilon,\mu_2^\varepsilon)$.

The endpoint of μ^ε is $(0,1,0,0)$ if and only if

$$c = -\frac{\varepsilon^2}{2(b-a)} \quad \text{and} \quad r = r(\varepsilon),$$

for a uniquely determined function $\varepsilon \mapsto r(\varepsilon)$. For ε sufficiently small, one has that

$$r(\varepsilon) \sim \begin{cases} \varepsilon^{\frac{2\gamma+1}{\gamma+2}} & \text{if } \gamma \text{ is odd,} \\ \\ \varepsilon^{\frac{2\gamma+1}{\gamma+3}} & \text{if } \gamma \text{ is even.} \end{cases} \tag{5}$$

Finally, by means of the previous estimates on $r(\varepsilon)$ we can prove that, for ε sufficiently small, the difference of length ΔL (between ν and μ^ε) is strictly positive, precisely

$$\Delta L = \big(2-\sqrt{2}\,\big)\varepsilon - \frac{\varepsilon^2}{a-b} - 8\,g_\gamma(\varepsilon)\,C_\gamma\,r(\varepsilon) > 0, \tag{6}$$

where g_γ is a function such that $\lim_{\varepsilon\to 0^+} g_\gamma(\varepsilon) = 1$. Therefore, ν is not length-minimizing since μ^ε joins the same points and is strictly shorter.

The case $\gamma = 2$ is more delicate and interesting. The previous argument does not work. The curve μ^ε constructed above is not shorter than ν. Thus we have to find a more refined competitor for ν. To this aim, consider the lift η^ε of the planar

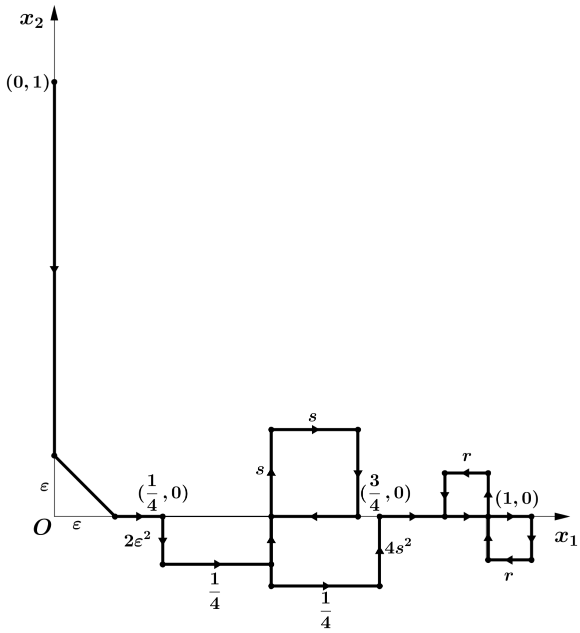

Figure 2: The curve $\big(\eta_1^\varepsilon, \eta_2^\varepsilon\big)$.

curve (depending on suitable positive parameters s and r) in Figure 2. We show that (for ε sufficiently small) there exists a unique positive number

$$s(\varepsilon) \in \left(0, \sqrt[4]{\frac{21}{4}}\,\varepsilon^{\frac{5}{4}}\right)$$

such that, choosing

$$s = s(\varepsilon) \quad \text{and} \quad r = \sqrt[5]{4}\, s(\varepsilon)^{\frac{6}{5}},$$

the endpoint of η^ε is $(0,1,0,0)$ and the difference of length ΔL (between ν and η^ε) is strictly positive, precisely

$$\Delta L = (2-\sqrt{2})\varepsilon - 4\varepsilon^2 - 4\, s(\varepsilon) - 8\, s(\varepsilon)^2 - 8\, r(\varepsilon) > 0,$$

proving that also in this case ν is not a length-minimizer.

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Giulia Vescovo

Contents

Chapter 1

Preliminaries

1.1 Sub-Riemannian manifolds

Our first aim is to introduce the setting we will work in, namely the *sub-Riemannian manifolds*, which is a family of abstract manifolds endowed with a Riemannian metric on a suitable sub-bundle (that we will call *distribution*) of the tangent bundle.

Definition 1.1 (Distribution)**.** Given M an n-dimensional smooth manifold, we define a *distribution* $\mathcal{D}$ *of rank* r on M as follows:

(i) $\mathcal{D}(p)$ is an r-dimensional vector subspace of T_pM for every $p \in M$,

(ii) for every $p \in M$ there exist an open neighborhood U of p in M and smooth vector fields X_1 ,..., X_r on U such that

$$\mathcal{D}(q) = \operatorname{span}_{T_qM}\big\{X_1(q)\,,...,\,X_r(q)\big\} \qquad \text{for every } q \in U\,. \tag{1.1}$$

We will call $X_1, \ldots, X_r$ a *frame of smooth vector fields* of $\mathcal{D}$ and we will denote by $\operatorname{rank}\big(\mathcal{D}\big) := r$ the rank of $\mathcal{D}$. Given a smooth vector field X on M, we write $X \in \operatorname{Sec}\big(\mathcal{D}\big)$ if $X(q) \in \mathcal{D}(q)$ for every $q \in M$.

Let M be an n-dimensional smooth manifold, let $\mathcal{D}$ be a distribution on M of rank r and let $X_1, \ldots, X_r$ be a frame of smooth vector fields of $\mathcal{D}$. For every $\ell \in \mathbb{N}^+$ and $i_1, \ldots, i_\ell \in \{1, \ldots, r\}$, we define

$$X_{i_1 \ldots i_\ell} := [X_{i_1}, [X_{i_2}, \ldots [X_{i_{\ell-1}}, X_{i_\ell}] \ldots]]. \tag{1.2}$$

We say that $X_{i_1 \ldots i_\ell}$ is an *iterated commutator* of $X_1, \ldots, X_r$ of *length* ℓ.
For any point $p \in M$, we define

$$\begin{aligned}
&\mathcal{D}_0(p) := \{0\},\\
&\mathcal{D}_1(p) := \mathcal{D}(p),\\
&\ldots\\
&\mathcal{D}_\ell(p) := \operatorname{span}_{T_pM}\big\{X_{i_1 \ldots i_\ell}(p) \in T_pM \;:\; i_1, \ldots, i_\ell \in \{1, \ldots r\}\big\}.
\end{aligned}$$

Finally, let us define

$$\begin{aligned}
\mathcal{L}_0 &:= \mathcal{D}_0 = \{0\},\\
\mathcal{L}_1 &:= \mathcal{D}_0 + \mathcal{D}_1 = \mathcal{D},\\
&\dots\\
\mathcal{L}_\ell &:= \mathcal{D}_1 + \ldots + \mathcal{D}_\ell.
\end{aligned}$$

One can easily prove that $\mathcal{D}_\ell$ and $\mathcal{L}_\ell$ are distributions on M.
Let us define the function $\operatorname{step}(\mathcal{D},\cdot) : M \to \overline{\mathbb{N}} := \mathbb{N} \cup \{+\infty\}$ as

$$\operatorname{step}(\mathcal{D},p) := \inf\{s \in \mathbb{N} : \mathcal{L}_s(p) = T_pM\} \quad \text{for every p} \in M. \tag{1.3}$$

Definition 1.2 (Bracket-generating distribution)**.** A distribution $\mathcal{D}$ on M is said to be *bracket-generating* (or *completely non-integrable*) if $\operatorname{step}(\mathcal{D},M) \subseteq \mathbb{N}$, i.e.

$$\operatorname{step}(\mathcal{D},p) < +\infty \quad \text{for every } p \in M. \tag{1.4}$$

A bracket-generating distribution $\mathcal{D}$ is *of finite step* if $\operatorname{step}(\mathcal{D},\cdot)$ is bounded, and in this case we define $\operatorname{step}(\mathcal{D}) := \left\|\operatorname{step}(\mathcal{D},\cdot)\right\|_{L^\infty}$, i.e.

$$\operatorname{step}(\mathcal{D}) = \min\{s \in \mathbb{N} : \mathcal{L}_s(p) = T_pM \text{ for every } p \in M\} < +\infty. \tag{1.5}$$

If X is a topological space and $x \in X$, we indicate by $\mathcal{N}_X(x)$ (or briefly $\mathcal{N}(x)$) the set of all open neighborhoods of x in X.

Lemma 1.3. *Let X be a topological space. Let $A \in C(X, M_{m,n}(\mathbb{R}))$, i.e.*

$$A(x) = \big(A_{i,j}(x)\big)_{i,j} \quad \textit{for every } x \in X, \textit{ for suitable } A_{i,j} \in C(X,\mathbb{R}).$$

Let $\operatorname{rk} : M_{m,n}(\mathbb{R}) \to \mathbb{N}$ *be the function that associates to every matrix its rank. Then* $\operatorname{rk} \circ A : X \to \mathbb{R}$ *is lsc (i.e. lower semicontinuous), in other words*

$$\forall x \in X \quad \exists U \in \mathcal{N}(x) : \quad \forall y \in U \quad \operatorname{rk}(A(y)) \geq \operatorname{rk}(A(x)). \tag{1.6}$$

Proof. Fix $x \in X$. Let $k := \operatorname{rk}(A(x))$, hence we can choose $R \subseteq \{1,\ldots,m\}$ and $C \subseteq \{1,\ldots,n\}$ both of cardinality equal to k, such that $\det(M(x)) \neq 0$, where

$$M(y) := \big(A_{i,j}(y)\big)_{i\in R,\, j\in C} \in M_k(\mathbb{R}) \quad \text{for every } y \in X.$$

By continuity of the function $\det : M_k(\mathbb{R}) \to \mathbb{R}$, also $\det\circ M : X \to \mathbb{R}$ is continuous. Thus there exists $U \in \mathcal{N}(x)$ such that $\det(M(y)) \neq 0$ for every $y \in U$. This implies that $\operatorname{rk}(A(y)) \geq k$ for every $y \in U$, hence the thesis. ☐

Proposition 1.4. *Let $\mathcal{D}$ be a distribution on an n-dimensional smooth manifold M. Then* $\operatorname{step}(\mathcal{D},\cdot) : M \to \overline{\mathbb{N}}$ *is usc (i.e. upper semicontinuous), in other words*

$$\forall p \in M \quad \exists U \in \mathcal{N}(p) : \quad \forall q \in U \quad \operatorname{step}(\mathcal{D},q) \leq \operatorname{step}(\mathcal{D},p). \tag{1.7}$$

Proof. We can assume without loss of generality that $M = \mathbb{R}^n$, because of the local nature of the statement. Fix $p \in M$. If $\operatorname{step}\big(\mathcal{D}, p\big) = +\infty$ then (1.7) clearly follows. So assume $s := \operatorname{step}\big(\mathcal{D}, p\big) \in \mathbb{N}$, then there exist $Z_1, \ldots, Z_n \in \operatorname{Sec}(\mathcal{L}_s)$ such that $Z_1(p), \ldots, Z_n(p)$ is a basis of $\mathbb{R}^n$. For every $q \in \mathbb{R}^n$, let us call $M(q) \in M_n\big(\mathbb{R}\big)$ the matrix having $Z_1(q), \ldots, Z_n(q)$ as columns. Thus $\operatorname{rk}\big(M(p)\big) = n$. We deduce, from Lemma 1.3, that $M(q)$ has rank equal to n for every q in some $U \in \mathcal{N}(p)$. In particular, $Z_1(q), \ldots, Z_n(q)$ span all of $\mathbb{R}^n$ for every $q \in U$. Therefore

$$\operatorname{step}\big(\mathcal{D}, q\big) \leq \operatorname{step}\big(\mathcal{D}, p\big),$$

for every $q \in U$, proving (1.7). □

Definition 1.5. Let M be a smooth n-dimensional manifold. A *sub-Riemannian metric* on M is a family $g = g_p(\cdot, \cdot)$ of inner products on each vector space $\mathcal{D}(p)$, such that $g_p(\cdot, \cdot)$ depends smoothly on p. The norm induced by this metric, i.e.

$$\|v\|_p := g_p(v, v)^{1/2}$$

for every $v \in \mathcal{D}(p)$ is called *sub-Riemannian norm*. The triple $(M, \mathcal{D}, g)$ is called *sub-Riemannian manifold*.

1.2 Length-minimizers

Hereafter, we will assume that M coincides with $\mathbb{R}^n$ and that every frame of smooth vector fields is global (i.e. defined in all of $\mathbb{R}^n$), because of the local nature of the problems that we are going to study. We will denote by $\operatorname{Lip}([a, b], \mathbb{R}^n)$ the set of all Lipschitz curves from $[a, b] \subseteq \mathbb{R}$ to $\mathbb{R}^n$.

Definition 1.6 ($\mathcal{D}$-horizontal curve)**.** Let $\mathcal{D}$ be a distribution of rank r on $\mathbb{R}^n$, generated by a frame of smooth (linearly indipendent) vector fields $X_1, \ldots, X_r$.
A curve $\gamma \in \operatorname{Lip}([a, b], \mathbb{R}^n)$ is said to be *$\mathcal{D}$-horizontal* if

$$\dot{\gamma}(t) \in \mathcal{D}(\gamma(t)) \quad \text{for a.e. } t \in [a, b], \tag{1.8}$$

in other words for some $h = (h_1, \ldots, h_r) \in \mathrm{L}^\infty\big([a, b], \mathbb{R}^r\big)$ we have that

$$\dot{\gamma}(t) = \sum_{i=1}^{r} h_i(t)\, X_i\big(\gamma(t)\big) \quad \text{for a.e. } t \in [a, b]. \tag{1.9}$$

We will refer to $h_1, \ldots, h_r$ as *controls* of γ.

Let $(\mathbb{R}^n, \mathcal{D}, g)$ be a sub-Riemannian manifold. The *length* of a $\mathcal{D}$-horizontal curve $\gamma \in \operatorname{Lip}([a, b], \mathbb{R}^n)$ is defined as

$$L(\gamma) := \int_a^b \sqrt{g_{\gamma(t)}\big(\dot{\gamma}(t), \dot{\gamma}(t)\big)}\, dt. \tag{1.10}$$

Definition 1.7 (Carnot-Carathéodory distance)**.** Let $(\mathbb{R}^n, \mathcal{D}, g)$ be a sub-Riemannian manifold, where the distribution $\mathcal{D}$ is bracket-generating. The *Carnot-Carathéodory distance* between two points $x, y \in \mathbb{R}^n$ is given by

$$d(x,y) := \inf\big\{L(\gamma) \bigm| \gamma : [a,b] \to \mathbb{R}^n \text{ is } \mathcal{D}\text{-horizontal}\,, \gamma(0) = x\,, \gamma(1) = y\big\}. \quad (1.11)$$

By the Chow-Rashevsky theorem (see Theorem 2.2, p. 24 in [5]), if the distribution $\mathcal{D}$ is bracket-generating, then any couple of points in $\mathbb{R}^n$ can be connected by a $\mathcal{D}$-horizontal curve. Hence the function $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ is actually a distance on $\mathbb{R}^n$.

Remark 1.8. One can see that the topology induced by the Carnot-Carathéodory distance d coincides with the Euclidean topology of $\mathbb{R}^n$ (see Theorem 2.3 p. 24 in [5], or [7]).

Definition 1.9 (Length-minimizer)**.** Let $(\mathbb{R}^n, \mathcal{D}, g)$ be a sub-Riemannian manifold, where the distribution $\mathcal{D}$ is bracket-generating. A $\mathcal{D}$-horizontal curve $\gamma : [a,b] \to \mathbb{R}^n$ is a *length-minimizer* if it realizes the infimum in (1.11), i.e. $d\big(\gamma(a), \gamma(b)\big) = L(\gamma)$.

In general global length-minimizers do not exist, but the local existence holds true and it is a consequence of Ascoli-Arzelà theorem and of Dunford-Pettis theorem.

Theorem 1.10 (Local existence of length-minimizers in $\mathbb{R}^n$)**.** *Let $(\mathbb{R}^n, \mathcal{D}, g)$ be a sub-Riemannian manifold, where the distribution $\mathcal{D}$ is bracket-generating. Let d be the Carnot-Carathéodory distance defined in* (1.11)*. Let us fix $x \in \mathbb{R}^n$.*
Then there exists $\rho_x > 0$ such that the following property hold: for every

$$y \in B(x, \rho_x) := \left\{z \in \mathbb{R}^n \,\middle|\, d(x,z) < \rho_x\right\},$$

there exists a length-minimizer γ joining x to y.

Proof. By Remark 1.8, we can choose $\rho_x > 0$ such that

$$B(x, \rho_x) := \left\{y \in \mathbb{R}^n \,\middle|\, d(x,y) < \rho_x\right\}$$

is an open bounded subset of $\mathbb{R}^n$. Fix $y \in B(x, \rho_x)$. By definition of d there exists a sequence of $\mathcal{D}$-horizontal curves $\Gamma = \{\gamma^k\}_{k\in\mathbb{N}}$ parametrized on $[a,b]$ joining x to y such that $\lim_{k\to\infty} L(\gamma^k) = d(x,y)$. Hence there exists $\bar{k} \in \mathbb{N}$ such that $[\gamma^k] \subseteq B(x, \rho_x)$ for every $k \geq \bar{k}$, we argue by contradiction: assume that $[\gamma^{k_j}] \nsubseteq B(x, \rho_x)$ for some subsequence $\{\gamma^{k_j}\}_{j\in\mathbb{N}}$, thus $\gamma^{k_j}\left(t^{k_j}\right) \notin B(x, \rho_x)$ for some $t_{k_j} \in [a,b]$. Then

$$L\big(\gamma^{k_j}\big) \geq L\left(\gamma^{k_j}_{|[0,t^{k_j}]}\right) \geq d\left(x, \gamma^k\big(t^{k_j}\big)\right) \geq \rho_x,$$

which gives $d(x,y) = \lim_{j\to\infty} L(\gamma^{k_j}) \geq \rho_x$, contradicting $y \in B(x, \rho_x)$.
Without loss of generality we can assume that for every $k \in \mathbb{N}$ the curve γ^k is parametrized by constant speed and $X_1, \ldots, X_r$ is a frame of orthonormal (with

respect to g) vector fields generating $\mathcal{D}$ such that $\operatorname{rank}\mathcal{D} = r$. Hence for every $k \in \mathbb{N}$,

$$L(\gamma^k) = \int_a^b |h^k(t)|\,dt,$$

where $h^k \in L^\infty\big([a,b],\mathbb{R}^r\big)$ are the controls of γ^k. Then $|h^k(t)| = L(\gamma^k)$, for almost every $t \in [a,b]$ and for every $k \in \mathbb{N}$. Thus $\{\|\dot\gamma^k\|_{L^\infty} \,|\, k \geq \bar{k}\}$ is bounded. Hence the family Γ is equilipschitz, so in particular it is equicontinuous. Moreover, since $[\gamma^k] \subseteq B(x,\rho_x)$ for every $k \geq \bar{k}$, Γ is bounded. By Ascoli-Arzelà theorem we have that Γ is totally bounded. Note that $\overline{\Gamma}$ is a totally bounded closed subset of the complete space $\big(C([a,b],\mathbb{R}^n),\|\cdot\|_\infty\big)$, thus $\overline{\Gamma}$ is also complete. Then, up to a subsequence, $\gamma^k \to \gamma$ uniformly as $k \to \infty$, for some $\gamma \in \mathrm{Lip}\big([a,b],\overline{B(x,\rho_x)}\big)$.
We can choose $M > 0$ such that $|h^k| \leq |\dot\gamma^k| \leq \|\dot\gamma^k\|_\infty \leq M$ a.e. in $[a,b]$ and for every $k \geq \bar{k}$. Fix $\varepsilon > 0$ and $E \subseteq [a,b]$ with $\mathcal{L}^1(E) \leq \frac{\varepsilon}{M}$, then

$$\left|\int_E h^k(t)\,dt\right| \leq M\,\mathcal{L}^1(E) \leq \varepsilon,$$

i.e. the family of controls $\{h^k\}_{k\geq\bar{k}} \subseteq L^1\big([a,b],\mathbb{R}^r\big)$ is uniformly integrable. Then, by Dunford-Pettis theorem, up to a subsequence $h^k \rightharpoonup h$ as $k \to \infty$ for some $h \in L^1\big([a,b],\mathbb{R}^r\big)$. By integrating the equation $\dot\gamma^k(t) = \sum_{i=1}^r h_i^k(t)\,X_i\big(\gamma(t)\big)$ with respect to t, we get

$$\gamma^k(t) - x = \sum_{i=1}^r \int_a^b h_i^k(s)\,X_i\big(\gamma(s)\big)\,ds \qquad \text{for every } t \in [a,b].$$

By letting k go to ∞ we get

$$\gamma(t) - x = \sum_{i=1}^r \int_a^b h_i(s)\,X_i\big(\gamma(s)\big)\,ds \qquad \text{for every } t \in [a,b].$$

By differentiating the above equation with respect to t, we obtain that γ is $\mathcal{D}$-horizontal with controls h:

$$\dot\gamma(t) = \sum_{i=1}^r h_i(t)\,X_i\big(\gamma(t)\big) \qquad \text{for a.e. } t \in [a,b].$$

Note that $\gamma(a) = x$ and $\gamma(b) = y$. Finally, by Fatou Lemma, we have that

$$L(\gamma) = \|h\|_{L^1([a,b],\mathbb{R}^r)} \leq \liminf_{k\to\infty} \|h^k\|_{L^1([a,b],\mathbb{R}^r)} = d(x,y),$$

so that $L(\gamma) = d(x,y)$ as required. □

1.3 Extremal curves

1.3.1 The notion of extremal curve.

Definition 1.11 (Optimal Pair)**.** Let $(\mathbb{R}^n,\mathcal{D},g)$ be a sub-Riemannian manifold, where the distribution $\mathcal{D}$ is bracket-generating. If γ is a length-minimizer with controls h, we say that (γ,h) is an *optimal pair*.

Remark 1.12. If $(\mathbb{R}^n, \mathcal{D}, g)$ is a sub-Riemannian manifold and $X_1, \ldots, X_r$ is a frame of othonormal (with respect to g) vector fields of $\mathcal{D}$, then a $\mathcal{D}$-horizontal curve γ with controls h has length equal to

$$L(\gamma) = \int_0^1 |h(t)|\, dt. \tag{1.12}$$

The 2-*length* of γ is defined as follows:

$$L_2(\gamma) := \left(\int_0^1 |h(t)|^2\, dt \right)^{\frac{1}{2}}. \tag{1.13}$$

The Carnot-Carathéodory distance

$$d(x, y) = \inf\big\{ L(\gamma) \;\big|\; \gamma \text{ is } \mathcal{D}\text{-horizontal}\,, \gamma(0) = x\,, \gamma(1) = y \big\},$$

coincides with the following distance:

$$d_2(x, y) := \inf\big\{ L_2(\gamma) \;\big|\; \gamma \text{ is } \mathcal{D}\text{-horizontal}\,, \gamma(0) = x\,, \gamma(1) = y \big\}. \tag{1.14}$$

Indeed, note that when γ is parametrized by constant speed c, one has that

$$L(\gamma) = |h(t)| = c = L_2(\gamma) \quad \text{for almost every } t \in [0, 1].$$

Now we give the definition of *extremal curve*:

Definition 1.13 (Extremal curve)**.** *Let $(\mathbb{R}^n, \mathcal{D}, g)$ be a sub-Riemannian manifold, where $\mathcal{D}$ is a bracket-generating distribution of rank r, generated by $X_1, \ldots, X_r$. Fix x,$y \in \mathbb{R}^n$. Let γ be a competitor in* (1.11)*. We say that γ is an* extremal curve *if there exist $\xi_0 \in \{0, 1\}$ and a curve $\xi \in \mathrm{Lip}\big([0, 1], \mathbb{R}^n\big)$ such that the following conditions hold:*

(i) *for every $t \in [0, 1]$*

$$\xi_0 + |\xi(t)| \neq 0, \tag{1.15}$$

(ii) *for almost every $t \in [0, 1]$ and for every $i = 1, \ldots, r$*

$$\xi_0\, h_i(t) + \xi(t) \cdot X_i(\gamma(t)) = 0, \tag{1.16}$$

(iii) *for almost every $t \in [0, 1]$*

$$\dot{\xi}(t) + \sum_{i=1}^{r} h_i(t)\, X_i'(\gamma(t))^T\, \xi(t) = 0. \tag{1.17}$$

If γ is an extremal curve and $\xi_0 = 1$ (resp. $\xi_0 = 0$), we say that γ is a normal *(resp.* abnormal*)* extremal*. We say that γ is a* strictly abnormal extremal *if it is abnormal but not normal.*

In the following theorem we see that length-minimizers are extremal curves, so they satisfy some necessary first-oreder conditions. By Remark 1.12, we can suppose γ parametrized by constant speed, thus we can fix $L^2\big([0,1],\mathbb{R}^r\big)$ as space of controls.

Theorem 1.14 (Pontryagin Maximum Principle)**.** *Consider a sub-Riemannian manifold* $(\mathbb{R}^n, \mathcal{D}, g)$*, where* $\mathcal{D}$ *is a distribution of rank* r *with global frame of smooth vector fields* $X_1, \dots, X_r$*. Assume that* $X_1, \dots, X_r$ *are orthonormal with respect to* g*.*
Let $(\gamma, h) \in \mathrm{Lip}\big([0,1],\mathbb{R}^n\big) \times \mathrm{L}^2\big([0,1],\mathbb{R}^r\big)$ *be an optimal pair, with* γ *parametrized by constant speed. Then* γ *is an extremal curve.*

1.3.2 Proof of Pontryagin Maximum Principle

In order to prove Theorem 1.14, we need some preliminary results and definitions. Let $(\mathbb{R}^n, \mathcal{D}, g)$ be a sub-Riemannian manifold, where $\mathcal{D}$ is a distribution of rank r with global frame of smooth vector fields $X_1, \dots, X_r$. Assume that $X_1, \dots, X_r$ are orthonormal with respect to g.

Let $(\gamma, h) \in \mathrm{Lip}\big([0,1],\mathbb{R}^n\big) \times \mathrm{L}^2\big([0,1],\mathbb{R}^r\big)$ be an optimal pair. Suppose that $x_0 \in \mathbb{R}^n$ is the initial point of γ, i.e. $\gamma(0) = x_0$.
For every r-tuple of controls $v \in L^2\big([0,1],\mathbb{R}^r\big)$, consider the unique solution $\gamma^v \in \mathrm{Lip}\big([0,1],\mathbb{R}^n\big)$ of the following Cauchy problem:

$$\begin{cases} \dot{\gamma}^v(t) = \sum_{i=1}^r v_i(t)\, X_i(\gamma^v(t)), \\ \gamma^v(0) = x_0, \end{cases} \tag{1.18}$$

for almost every $t \in [0,1]$. For every $t \in [0,1]$, let $\mathcal{E}_t : L^2\big([0,1],\mathbb{R}^r\big) \to \mathbb{R}^n$ be the map defined by

$$\mathcal{E}_t(v) := \gamma^v(t). \tag{1.19}$$

We say that the map $\mathcal{E} := \mathcal{E}_1$ is the *endpoint map*. Note that $\mathcal{E}(h) = \gamma^h(1) = \gamma(1)$.
For every $x \in \mathbb{R}^n$ consider the unique solution $\gamma_x \in \mathrm{Lip}\big([0,1],\mathbb{R}^n\big)$ of the following Cauchy problem:

$$\begin{cases} \dot{\gamma}_x(t) = \sum_{i=1}^r h_i(t)\, X_i(\gamma_x(t)), \\ \gamma_x(0) = x, \end{cases} \tag{1.20}$$

for almost every $t \in [0,1]$. Note that $\gamma_{x_0} = \gamma$. The family of maps $\big\{\phi_t\big\}_{t\in[0,1]}$, where $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\phi_t(x) := \gamma_x(t), \tag{1.21}$$

is called *optimal flow*. Note that $\phi_t(x_0) = \gamma(t)$ for every $t \in [0,1]$. For every $t \in [0,1]$ we have that the map ϕ_t is a C^1-diffeomorphism.
Let us define the map $\tilde{\mathcal{E}}_t : L^2\big([0,1],\mathbb{R}^r\big) \to \mathbb{R}^n$ as follows:

$$\tilde{\mathcal{E}}_t(v) := \phi_t^{-1}\big(\mathcal{E}_t(v)\big). \tag{1.22}$$

The map $\tilde{\mathcal{E}}_t$ is called the *modified endpoint map* of $\mathcal{E}_t$.

Lemma 1.15. *Let $(\mathbb{R}^n, \mathcal{D}, g)$ be a sub-Riemannian manifold, where $\mathcal{D}$ is a distribution of rank r with global frame of smooth vector fields $X_1, \ldots, X_r$. Assume that $X_1, \ldots, X_r$ are orthonormal with respect to g.*
Let $(\gamma, h) \in \mathrm{Lip}\big([0,1], \mathbb{R}^n\big) \times \mathrm{L}^2\big([0,1], \mathbb{R}^r\big)$ be an optimal pair. Then for every $v \in L^2\big([0,1], \mathbb{R}^r\big)$ the following hold:

$$\frac{\partial}{\partial t}\, \tilde{\mathcal{E}}_t(v) = \phi_t'\big(\tilde{\mathcal{E}}_t(v)\big)^{-1} \sum_{i=1}^{r} (v_i - h_i)\, X_i\big(\mathcal{E}_t(v)\big) \tag{1.23}$$

and

$$D_v\, \tilde{\mathcal{E}}_t(h) = \int_0^1 \phi_t'\big(x_0\big)^{-1} \sum_{i=1}^{r} v_i\, X_i\big(\gamma(t)\big)\, dt, \tag{1.24}$$

where $\mathcal{E}_t$ and $\tilde{\mathcal{E}}_t$ are the maps defined respectively in (1.19) and in (1.22), and the family of maps $\{\phi_t\}_{t\in[0,1]}$ is the optimal flow defined in (1.21).

Proof. First of all, we prove equation (1.23). By differentiating $\mathcal{E}_t(v) = \phi_t\big(\tilde{\mathcal{E}}_t\big)$ with respect to t, we obtain

$$\frac{\partial}{\partial t}\, \mathcal{E}_t(v) = \left(\frac{\partial}{\partial t}\, \phi_t\right) \big(\tilde{\mathcal{E}}_t(v)\big) + \phi_t'\big(\tilde{\mathcal{E}}_t(v)\big) \frac{\partial}{\partial t}\, \tilde{\mathcal{E}}_t(v). \tag{1.25}$$

Note that

$$\left(\frac{\partial}{\partial t}\, \phi_t\right) \big(\tilde{\mathcal{E}}_t(v)\big) = \sum_{i=1}^{r} h_i(t)\, X_i\big(\phi_t\big(\tilde{\mathcal{E}}_t(v)\big)\big) = \sum_{i=1}^{r} h_i(t)\, X_i\big(\mathcal{E}_t(v)\big), \tag{1.26}$$

and that

$$\frac{\partial}{\partial t}\, \mathcal{E}_t(v) = \sum_{i=1}^{r} v_i(t)\, X_i\big(\mathcal{E}_t(v)\big). \tag{1.27}$$

Therefore, by using (1.25), (1.26), (1.27) and since ϕ_t is a C^1-diffeomorphism, we obtain (1.23).
Now we prove the equation (1.24). By integrating (1.23), with $v = h + u$ for some $u \in \mathrm{L}^2\big([0,1], \mathbb{R}^r\big)$, we have that

$$\tilde{\mathcal{E}}(h+u) = \tilde{\mathcal{E}}_1(h+u) = x_0 + \int_0^1 \phi_t'\big(\tilde{\mathcal{E}}_t(h+u)\big)^{-1} \sum_{i=1}^{r} u_i\, X_i\big(\mathcal{E}_t(h+u)\big)\, dt. \tag{1.28}$$

Note that $D_v\, \tilde{\mathcal{E}}_t(h) = \frac{\partial}{\partial s}\, \tilde{\mathcal{E}}_t(h+sv)|_{s=0}$ for every $v \in L^2\big([0,1], \mathbb{R}^r\big)$. By (1.28) we obtain

$$\begin{aligned}\frac{\partial}{\partial s}\, \tilde{\mathcal{E}}_t(h+sv) = &\int_0^1 \frac{\partial}{\partial s} \big(\phi_t'\big(\tilde{\mathcal{E}}_t(h+sv)\big)^{-1}\big) \sum_{i=1}^{r} s\, v_i\, X_i\big(\mathcal{E}_t(h+sv)\big)\, dt \\ &+ \int_0^1 \phi_t'\big(\tilde{\mathcal{E}}_t(h+sv)\big)^{-1} \sum_{i=1}^{r} v_i\, X_i\big(\mathcal{E}_t(h+sv)\big)\, dt.\end{aligned}$$

Hence

$$D_v\,\tilde{\mathcal{E}}_t(h) = \frac{\partial}{\partial s}\,\tilde{\mathcal{E}}_t(h+sv)_{|s=0} = \int_0^1 \phi_t'\big(\tilde{\mathcal{E}}_t(h)\big)^{-1} \sum_{i=1}^r v_i\,X_i\big(\mathcal{E}_t(h)\big)\,dt$$
$$= \int_0^1 \phi_t'\big(x_0\big)^{-1} \sum_{i=1}^r v_i\,X_i\big(\gamma(t)\big)\,dt,$$

getting the thesis. □

The map $\mathcal{F} : L^2\big([0,1],\mathbb{R}^r\big) \to \mathbb{R}^{n+1}$ defined by

$$\mathcal{F}\big(v\big) := \big(\mathcal{L}(v), \mathcal{E}(v)\big), \tag{1.29}$$

where $\mathcal{E}$ is the endpoint map and

$$\mathcal{L}\big(v\big) := \frac{1}{2}\int_0^1 |v(t)|^2\,dt, \tag{1.30}$$

is called the *extended endpoint mapping*. Moreover the map

$$\tilde{\mathcal{F}} : L^2\big([0,1],\mathbb{R}^r\big) \to \mathbb{R}^{n+1}$$

defined by

$$\tilde{\mathcal{F}}\big(v\big) := \big(\mathcal{L}(v), \tilde{\mathcal{E}}(v)\big), \tag{1.31}$$

is called the *modified extended endpoint mapping*.
We denote by $B_n\big(x,r\big)$ the open ball in $\mathbb{R}^n$ of center x and radius r.

Lemma 1.16. *Let $(\mathbb{R}^n, \mathcal{D}, g)$ be a sub-Riemannian manifold, where $\mathcal{D}$ is a distribution of rank r with global frame of smooth vector fields $X_1,\dots,X_r$. Assume that $X_1,\dots,X_r$ are orthonormal with respect to g.*
Let $(\gamma,h) \in \mathrm{Lip}\big([0,1],\mathbb{R}^n\big) \times L^2\big([0,1],\mathbb{R}^r\big)$ be an optimal pair. Then the map

$$\mathcal{F} : L^2\big([0,1],\mathbb{R}^r\big) \to \mathbb{R}^{n+1}$$

is not open at v=h.

Proof. Suppose by contradiction that $\mathcal{F}$ is open at $v = h$. Let U be an open neighborhood of h in $L^2\big([0,1],\mathbb{R}^r\big)$, then $\mathcal{F}(U)$ is an open neighborhood of $\mathcal{F}(h)$ in $\mathbb{R}^{n+1}$. Choose $r > 0$ such that $B_{n+1}\big(\mathcal{F}(h),r\big) \subseteq \mathcal{F}(U)$. So there exists $\varepsilon > 0$ such that

$$\mathcal{F}(h) - \varepsilon\,e_1 \in B_{n+1}\big(\mathcal{F}(h),r\big) \subseteq \mathcal{F}(U),$$

where $e_1 = (1,0,\dots,0) \in \mathbb{R}^{n+1}$. Hence $\big(\mathcal{L}(v)-\varepsilon, \mathcal{E}(v)\big) \in \mathcal{F}(U)$. Then there exists $u \in L^2\big([0,1],\mathbb{R}^r\big)$ such that $\mathcal{F}(u) = \big(\mathcal{L}(v)-\varepsilon, \mathcal{E}(v)\big)$, contradicting the fact that γ is a length-minimizer. □

Lemma 1.17. *Let X, Y be Banach spaces, with* $\dim(Y) = n$. *Let $p \in X$. Assume that $F : X \to Y$ is a differentiable function and that $dF(p) : X \to Y$ is surjective. Then F is open at p.*

Proof. We can assume without loss of generality $p = 0$. Since $dF(0) : X \to Y$ is surjective, we can choose $v_1, \dots, v_n \in X$ such that $dF(v_1), \dots, dF(v_n)$ is a basis of Y. Let $V \leq X$ be the linear subspace of X generated by $v_1, \dots, v_n$. We have that

$$d(F|_V)(0) = dF(0)|_V : V \to Y$$

is a linear isomorphism between V and Y. Thus $G := F|_V$ is open at 0, by the Inverse Function theorem.
Now let $U \subseteq X$ be a neighborhood of 0 in X. Hence $U \cap V$ is a neighborhood of 0 in V. Since G is open at 0, one has that $W := G(U \cap V) = F(U \cap V)$ is a neighborhood of 0 in Y. Thus also $F(U) \supseteq W$ is a neighborhood of 0 in Y. We deduce that F is open at 0. □

We are now ready to prove the above-stated Pontryagin Maximum Principle.

Proof of Theorem 1.14. Let (γ, h) be an optimal pair. By Lemma 1.16 we deduce that $\mathcal{F}$ is not open at $v = h$. Since $\phi_1 : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-diffeomorphism, also $\tilde{\mathcal{F}}$ is not open at $v = h$. Hence, by Lemma 1.17, we have that $d\tilde{\mathcal{F}}(h)$ is not surjective. Thus we can choose $\xi_0 \in \mathbb{R}$ and $\xi(0) \in \mathbb{R}^n$ such that $\xi_0 + |\xi(0)| \neq 0$ and

$$D_v\tilde{\mathcal{F}}(h) \cdot \big(\xi_0, \xi(0)\big) = 0$$

for every $v \in L^2\big([0,1], \mathbb{R}^r\big)$. Explicitly,

$$0 = D_v\tilde{\mathcal{F}}(h) \cdot \big(\xi_0, \xi(0)\big) = \xi_0\, D_v\mathcal{L}(h) + \xi(0)\, D_v\tilde{\mathcal{E}}(h) \tag{1.32}$$

for every $v \in L^2\big([0,1], \mathbb{R}^r\big)$. Note that

$$D_v\mathcal{L}(h) = \frac{\partial}{\partial s}\,\mathcal{L}(h + sv)|_{s=0} = \int_0^1 \sum_{i=1}^r v_i(t)\, h_i(t)\, dt$$

for every $v \in L^2\big([0,1], \mathbb{R}^r\big)$. Then, by (1.24), we can write (1.32) as follows:

$$\int_0^1 \sum_{i=1}^r v_i(t)\, \big\{\xi_0\, h_i(t) + \xi(0) \cdot \phi_t'(x_0)^{-1}\, X_i\big(\gamma(t)\big)\big\}\, dt = 0 \tag{1.33}$$

for every $v \in L^2\big([0,1], \mathbb{R}^r\big)$. Now let $\xi : [0,1] \to \mathbb{R}^n$ be the Lipschitz curve defined by

$$\xi(t) := \big[\phi_t'(x_0)^{-1}\big]^T\, \xi(0). \tag{1.34}$$

Hence ξ satisfies (1.15): indeed if $\xi(t) = 0$ for some $t \in [0,1]$ then $\xi(0) = 0$. By (1.33) and (1.34), we obtain (1.16): for almost every $t \in [0,1]$

$$\xi_0\, h_i(t) + \xi(t) \cdot X_i\big(\gamma(t)\big) = 0.$$

Now we prove (1.17). By differentiating (with respect to t) the following identity

$$\big[\phi_t'(x_0)\big]^T\, \xi(t) = \xi(0), \quad t \in [0,1],$$

we get

$$\left[\phi_t'(x_0)\right]^T \dot{\xi}(t) + \left[\frac{d}{dt}\,\phi_t'(x_0)\right]^T \xi(t) = 0 \quad \text{for a.e. } t \in [0,1]. \tag{1.35}$$

Note that

$$\begin{aligned}\frac{d}{dt}\,\phi_t'(x_0) &= \left(\frac{d}{dt}\,\phi_t(x)\right)'\Big|_{x=x_0} = \left(\sum_{i=1}^r h_i(t)\,X_i(\phi_t(x))\right)'\Big|_{x=x_0}\\ &= \left(\sum_{i=1}^r h_i(t)\,\big(X_i(\phi_t(x))\big)'\right)\Big|_{x=x_0} = \sum_{i=1}^r h_i(t)\,X_i'(\gamma(t))\,\phi_t'(x_0).\end{aligned}$$

Then, by (1.35) we have that

$$\left[\phi_t'(x_0)\right]^T \dot{\xi}(t) + \left[\phi_t'(x_0)\right]^T \sum_{i=1}^r h_i(t)\,X_i'(\gamma(t))^T\,\xi(t) = 0,$$

for almost every $t \in [0,1]$, getting (1.17). □

1.3.3 The open problem of regularity.

We collect here some of the most important known facts about the delicate problem of regularity of length-minimizers:

- We saw in Theorem 1.14 that every length-minimizer is an extremal curve. Thus the focus is moved to study the properties of regularity of extremal curves.
- It is simple to prove (see below) that every normal extremal is smooth, but there are length-minimizers that are strictly abnormal extremals, see [4].
- There are examples of (strictly) abnormal extremals that are not smooth but that are not length-minimizers.
- All known examples of strictly abnormal length-minimizers are smooth.

The problem of regularity of length-minimizers is still open. We now prove that every normal extremal is C^∞ smooth.

Theorem 1.18. *Let $(\mathbb{R}^n, \mathcal{D}, g)$ be a sub-Riemannian manifold, where $\mathcal{D}$ is a distribution of rank r with global frame of smooth vector fields $X_1, \ldots, X_r$. Assume that $X_1, \ldots, X_r$ are orthonormal with respect to g. Let $\gamma : [0,1] \to \mathbb{R}^n$ be a normal extremal with controls h and dual curve ξ. Then $\gamma \in C^\infty\big([0,1], \mathbb{R}^n\big)$.*

Proof. Note that γ and ξ are continuous. Then by (1.16) we deduce that h_i is continuous for every $i = 1, \ldots, r$. Hence (1.17) implies that ξ is C^1 and (1.9) implies that γ is C^1.
Now, by (1.16) we deduce that h_i is continuous for every $i = 1, \ldots, r$. Hence (1.17) implies that ξ is C^2 and (1.9) implies that γ is C^2, and so on. By repeating this argument, we deduce that γ is smooth. □

In the next chapter we will study a class of abnormal extremals presenting corner-type singularities.

Chapter 2

Non-minimality for a class of angles

One of the main open problems in the study of Carnot-Carathéodory spaces is the following: is every length-minimizer smooth or not? To this purpose it was proved, in the papers [3] and [2], that - under suitable assumptions - curves with a corner-type singularity cannot be length-minimizing.

We shall makes use of some definitions: given a sub-Riemannian manifold $(M, \mathcal{D}, g)$, we introduce the following conditions

(A) the distribution $\mathcal{D}$ is *equiregular*, i.e. for every $\ell \in \mathbb{N}$

$$\dim \mathcal{D}_\ell(p) \text{ is independent of the point } p \in M.$$

$(\mathrm{B})_x$ For some $x \in M$, one has that

$$[\mathcal{L}_i, \mathcal{L}_j](x) \subseteq \mathcal{L}_{i+j-1}(x) \quad \text{for every} \quad i,j \geq 2 \text{ with } i+j \geq 5.$$

$(\mathrm{C})_x$ For some $x \in M$, one has that

$$\mathcal{L}_i(x) \neq \mathcal{L}_{i-1}(x) \implies \mathcal{L}_{i+1}(x) = \mathcal{L}_i(x) \quad \text{for every } i \geq 1.$$

The following result was proved in [3]:

Theorem 2.1. *Let $(M, \mathcal{D}, g)$ be a sub-Riemannian manifold satisfying (A) and $(B)_x$ for some $x \in M$. Then any extremal with a corner-type singularity in x is not length-minimizing.*

The following result was proved in [2]:

Theorem 2.2. *Let $(M, \mathcal{D}, g)$ be a sub-Riemannian manifold satisfying $(C)_x$ for some $x \in M$. Then any extremal with a corner-type singularity in x is not length-minimizing.*

The aim of this chapter is to provide an example of an extremal ν in $\mathbb{R}^4$, having an angle in a point $x \in \mathbb{R}^4$, which is not a length-minimizer even if the underlying distribution $\mathcal{D}$ does not satisfy any of the conditions (A), $(\mathrm{B})_x$ and $(\mathrm{C})_x$.
This is the sketch of what we will prove in the following sections:

Section 2.1. Fix $\alpha \in \mathbb{N}^+$, $\beta \in \mathbb{N}$ and $\gamma \in \mathbb{N}^+$. We call $\mathcal{D}$ the 2-dimensional distribution in $\mathbb{R}^4$ generated by

$$\begin{cases} X_1(x) = (1,0,0,0)\,, \\ X_2(x) = \left(0,1,x_1^\alpha x_2^\beta, x_3^\gamma\right)\,, \end{cases} \qquad \text{for every } x \in \mathbb{R}^4.$$

We prove that $\mathcal{D}$ is bracket-generating in all of $\mathbb{R}^4$, with step equal to $\gamma(\alpha+\beta+1)+1$.
Section 2.2. We study when $\mathcal{D}$ satisfies the above conditions (A), $(\mathrm{B})_0$ and $(\mathrm{C})_0$, namely:

- $\mathcal{D}$ satisfies (A) if and only if $(\alpha,\beta,\gamma) = (1,0,1)$.
- $\mathcal{D}$ satisfies $(\mathrm{B})_0$ if and only if $\gamma = 1$.
- $\mathcal{D}$ satisfies $(\mathrm{C})_0$ if and only if $(\alpha,\beta) \neq (1,0)$ and $\gamma \geq 2$.

Hence we restrict our attention to the case $(\alpha,\beta) = (1,0)$ and $\gamma \geq 2$, in such a way that none of (A), $(\mathrm{B})_0$ and $(\mathrm{C})_0$ is satisfied.
Section 2.3. We introduce the $\mathcal{D}$-horizontal curve ν defined as follows:

$$\nu(t) = \begin{cases} (0,-t,0,0) & \text{if } t \in [-1,0], \\ (t,0,0,0) & \text{if } t \in [0,1]. \end{cases}$$

Clearly, ν has an angle in $x = 0$. We prove that ν is a strictly abnormal extremal if and only if $(\alpha,\gamma) \neq (1,1)$.
Section 2.4. We prove that ν is not a length-minimizer when $(\alpha,\beta) = (1,0)$ and $\gamma \geq 3$. We proceed as follows:

- first of all, we "cut" the corner ν with a suitable curve ν^ε (depending on a parameter $0 < \varepsilon < 1$). The length of ν^ε is strictly smaller than the one of ν, but the endpoint of ν^ε is perturbed, since its third and fourth components are non-null.
- In order to correct the third component of the endpoint of ν^ε, we introduce a new curve ζ^ε (depending also on parameters a, b and c), obtained by perturbing ν^ε with a rectangle. For suitable choices of the parameters, ζ^ε is strictly shorter than ν.
- Finally, by concatenating ζ^ε with a suitable circuit, we obtain a curve μ^ε (depending also on parameters r and s) whose third component remains equal to 0 and whose fourth component is sent to 0. For ε sufficiently small, the length of μ^ε remains strictly smaller than the length of ν.

Therefore, for a suitable $\varepsilon > 0$, we have that ν and μ^ε join the same two points and that $L(\mu^\varepsilon) < L(\nu)$, proving that ν is not a length-minimizer.
Section 2.5. We deal with the case $(\alpha, \beta, \gamma) = (1, 0, 2)$, proving that ν is not a length-minimizer also in this case (by exhibiting a suitable curve η^ε obtained with techniques analogous to that of Section 2.4).

2.1 $\mathcal{D}$ is globally bracket-generating

First of all, we prove that the distribution $\mathcal{D}$ introduced above has finite step in $0 \in \mathbb{R}^4$. More precisely, two suitable iterated commutators of X_1 and X_2 of length $\alpha+\beta+1$ and $\gamma(\alpha+\beta+1)+1$, respectively, are multiples of e_3 and e_4, respectively, when evaluated in $x = 0$. We deduce that $\mathcal{D}$ has step smaller than or equal to $\gamma(\alpha+\beta+1)+1$ in 0.

Lemma 2.3. *Let $\alpha \in \mathbb{N}^+$, $\beta \in \mathbb{N}$, $\gamma \in \mathbb{N}^+$. Let $\mathcal{D}$ be the 2-dimensional distribution in $\mathbb{R}^4$ generated by*

$$\begin{cases} X_1(x) = (1, 0, 0, 0)\,, \\ X_2(x) = \big(0, 1, x_1^\alpha x_2^\beta, x_3^\gamma\big)\,, \end{cases} \qquad \text{for every } x \in \mathbb{R}^4. \tag{2.1}$$

Then $\mathcal{D}$ has step smaller than or equal to $\gamma(\alpha+\beta+1)+1$ in $0 \in \mathbb{R}^4$.

Proof. Step 1: We have that $X_1(0) = e_1$ and $X_2(0) = e_2$, so we want to obtain e_3 and e_4. Let us take a vector field V in $\mathbb{R}^4$ of the form $V = (0, 0, p, q)$, where $p \in \mathbb{N}[x_1, x_2]$ and $q \in \big(-\mathbb{N}\big)[x_1, x_2, x_3]$. Then a simple computation yields

$$\begin{cases} [X_1, V] = \Big(0, 0, \frac{\partial p}{\partial x_1}, \frac{\partial q}{\partial x_1}\Big)\,, \\[2ex] [X_2, V] = \Big(0,\ 0,\ \frac{\partial p}{\partial x_2},\ \frac{\partial q}{\partial x_2} + \frac{\partial q}{\partial x_3}\, x_1^\alpha x_2^\beta + \gamma(-p)\, x_3^{\gamma-1}\Big)\,. \end{cases} \tag{2.2}$$

Hence both $[X_1, V]$ and $[X_2, V]$ have the same form of V.
Note that every iterated commutator of X_1 and X_2 of length at least 2 has this form, since

$$[X_1, X_2] = \big(0, 0, \alpha\, x_1^{\alpha-1} x_2^\beta, 0\big). \tag{2.3}$$

Moreover, let us write $L_i(Y) := [X_i, Y]$ for every vector field Y in $\mathbb{R}^4$ and $i = 1, 2$.
Finally, we say that a polynomial $q \in \big(-\mathbb{N}\big)[x_1, x_2, x_3]$ *contains* $a \in \big(-\mathbb{N}\big)[x_1, x_2, x_3]$ if a is an addendum of q and $q - a \in \big(-\mathbb{N}\big)[x_1, x_2, x_3]$.
Step 2: By applying α times the first equation of (2.2) to $V = X_2$, we can easily deduce that $L_1^\alpha(X_2) = (0, 0, \alpha! x_2^\beta, 0)$.
Now, by applying $i = 1, \dots, \beta$ times the second equation of (2.2) to $V = L_1^\alpha(X_2)$, we obtain a vector field whose third component is $\alpha!\beta(\beta-1)\cdots(\beta-i+1)\, x_2^{\beta-i}$ and whose fourth component is a multiple of $x_2^{\beta-i+1}$. Thus the third component of the vector field

$$X_3 := L_2^\beta\big(L_1^\alpha(X_2)\big) \tag{2.4}$$

is $\alpha!\beta!$ and its fourth component is a multiple of x_2, in particular $X_3(0) = \alpha!\beta!\, e_3$.
STEP 3: It only remains to find an iterated commutator of X_1,X_2 having a non-zero fourth component when evaluated at $0 \in \mathbb{R}^4$.
By applying γ times the second equation of (2.2) to $V = [X_1, X_2]$, we get that the fourth component of $L_2^{\gamma}\big([X_1, X_2]\big)$ contains $-\alpha\gamma!\, x_1^{\gamma\alpha-1} x_2^{\gamma\beta}$.
Moreover, by applying (other) $\gamma\beta$ times the second equation of (2.2) to the vector field $V = L_2^{\gamma}\big([X_1, X_2]\big)$, we obtain that the fourth component of $L_2^{\gamma(\beta+1)}\big([X_1, X_2]\big)$ contains the addendum $-\alpha\gamma!(\gamma\beta)!\, x_1^{\gamma\alpha-1}$.
Finally, by applying $\gamma\alpha-1$ times the first equation of (2.2) to $V = L_2^{\gamma(\beta+1)}\big([X_1, X_2]\big)$, we get that the fourth component of

$$X_4 := L_1^{\gamma\alpha-1}\left(L_2^{\gamma(\beta+1)}\big([X_1, X_2]\big)\right) \tag{2.5}$$

contains $m := -\alpha\gamma!(\gamma\beta)!(\gamma\alpha - 1)! \in -\mathbb{N}$.
We infer that the fourth component of $X_4(0)$ is smaller than or equal to m, in particular it is non-zero. So $X_1(0), X_2(0), X_3(0), X_4(0)$ span $\mathbb{R}^4$. Since the commutator X_3 has length $\alpha+\beta+1$ and the commutator X_4 has length $\gamma(\alpha+\beta+1)+1 > \alpha+\beta+1$, we get the thesis. □

Remark 2.4. Actually, the step in $0 \in \mathbb{R}^4$ of the distribution $\mathcal{D}$ of Lemma 2.3 is exactly equal to $\gamma(\alpha + \beta + 1) + 1$. It suffices to show that $e_4 \in \mathcal{L}_s(0)$ implies $s \geq \gamma(\alpha + \beta + 1) + 1$.

We need first to find an iterated commutator of the vector fields X_1 and X_2 whose fourth component is a non-null polynomial, then to commute it again until we get a non-null constant term in the fourth entry.
The iterated commutator with a non-null fourth entry of shortest length is

$$V(x) := [X_2, [X_1, X_2]](x) = \left(0, 0, \alpha\beta\, x_1^{\alpha-1} x_2^{\beta-1}, -\alpha\gamma\, x_1^{\alpha-1} x_2^{\beta} x_3^{\gamma-1}\right).$$

By observing (2.2), we deduce that we need to commute V at least $\gamma - 1$ times with respect to X_2 to get an addendum of its fourth entry having degree 0 in x_3, and the degree in x_1 (respectively in x_2) of this addendum increases of at least $\alpha(\gamma - 1)$ (respectively $\beta(\gamma - 1)$).
Thus we need to commute V at least $\alpha - 1 + \alpha(\gamma - 1)$ times with respect to X_1 and at least $\beta + \beta(\gamma - 1)$ times with respect to X_2, in order to get a non-null constant term on the fourth component.
Therefore, to obtain e_4 we need a commutator of length at least

$$3 + (\gamma - 1) + \big(\alpha - 1 + \alpha(\gamma - 1)\big) + \big(\beta + \beta(\gamma - 1)\big) = \gamma(\alpha + \beta + 1) + 1,$$

as stated above. □

In the following lemma, we will infer from the finiteness of $\mathrm{step}\big(\mathcal{D}, 0\big)$ that $\mathcal{D}$ is bracket-generating in a suitable neighborhood of 0, by using the upper semicontinuity of the function $x \mapsto \mathrm{step}\,\big(\mathcal{D}, x\big)$.

Lemma 2.5. *Let α, β, γ, X_1, X_2 and $\mathcal{D}$ be as in Lemma 2.3. Then $\mathcal{D}$ is bracket-generating in a suitable neighborhood of $0 \in \mathbb{R}^4$.*

Proof. By Lemma 2.3 and Remark 2.4, we have that

$$\operatorname{step}\big(\mathcal{D}, 0\big) = \gamma(\alpha + \beta + 1) + 1 < +\infty.$$

Thus, by Proposition 1.7, there exists $U \in \mathcal{N}(0)$ such that

$$\operatorname{step}\big(\mathcal{D}, x\big) \leq \operatorname{step}\big(\mathcal{D}, 0\big) < +\infty,$$

for every $x \in U$. Hence the distribution $\mathcal{D}$ is bracket-generating in U, as required. □

Actually, $\mathcal{D}$ is bracket-generating in all of $\mathbb{R}^4$. In order to do this, we shall need the following definitions.

Fix $\omega \in (\mathbb{N}^+)^n$. For every $\lambda > 1$, we define the λ-*dilation* $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\delta_\lambda(x) := \big(\lambda^{\omega_1} x_1, \dots, \lambda^{\omega_n} x_n\big) \qquad \text{for every } x \in \mathbb{R}^n. \tag{2.6}$$

Note that δ_λ is a diffeomorphism and that $\delta_\lambda^{-1} = \delta_{\lambda^{-1}}$.
For every $r > 0$, let us define $B_\lambda(r) := \delta_\lambda(B(r))$, where $B(r)$ is the open ball in $\mathbb{R}^n$ of center 0 and radius r. Clearly

$$\bigcup_{\lambda > 1} B_\lambda(r) = \mathbb{R}^n \qquad \text{for every } r > 0. \tag{2.7}$$

Now let X be a smooth vector field in $\mathbb{R}^n$. We call λ-*transform* of X the vector field $\delta_\lambda X : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\delta_\lambda X(x) := \delta_\lambda'\big(\delta_\lambda^{-1}(x)\big)\, X\big(\delta_\lambda^{-1}(x)\big) \qquad \text{for every } x \in \mathbb{R}^n. \tag{2.8}$$

One has that $\delta_\lambda'\big(\delta_\lambda^{-1}(x)\big) = D_\lambda$ for every $x \in \mathbb{R}^n$, where

$$D_\lambda = \begin{pmatrix} \lambda^{\omega_1} & 0 & \cdots & 0 \\ 0 & \lambda^{\omega_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda^{\omega_n} \end{pmatrix}.$$

Hence (2.8) reads as

$$\delta_\lambda X(x) = D_\lambda\, X\big(\delta_{\lambda^{-1}}(x)\big) \qquad \text{for every } x \in \mathbb{R}^n. \tag{2.9}$$

The λ-transform satisfies the following properties:

- if $X_1, \dots, X_k$ (with $1 \leq k \leq n$) are linearly indipendent vector fields in an open $\Omega \subseteq \mathbb{R}^n$, then $\delta_\lambda X_1, \dots, \delta_\lambda X_k$ are linearly indipendent in $\delta_\lambda(\Omega)$,

- for any vector fields X, Y in $\mathbb{R}^n$, we have that

$$\delta_\lambda[X,Y] = [\delta_\lambda X, \delta_\lambda Y]. \tag{2.10}$$

The first statement follows from invertibility of D_λ. For the second one, note that for every $x \in \mathbb{R}^n$ we have

$$\big(\delta_\lambda X\big)'(x) = D_\lambda\, X'\big(\delta_{\lambda^{-1}}(x)\big) D_{\lambda^{-1}} = D_\lambda D_{\lambda^{-1}} X'\big(\delta_{\lambda^{-1}}(x)\big) = X'\big(\delta_{\lambda^{-1}}(x)\big),$$

where we used the facts that $D_\lambda D_{\lambda^{-1}} = \mathbb{I}_n$ and that a diagonal matrix commutes with every other matrix. Hence for every $x \in \mathbb{R}^n$ we get

$$\begin{aligned}[]
[\delta_\lambda X, \delta_\lambda Y](x) &= \big(\delta_\lambda Y\big)'(x)\, \delta_\lambda X(x) - \big(\delta_\lambda X\big)'(x)\, \delta_\lambda Y(x) \\
&= Y'\big(\delta_{\lambda^{-1}}(x)\big) D_\lambda\, X\big(\delta_{\lambda^{-1}}(x)\big) - X'\big(\delta_{\lambda^{-1}}(x)\big) D_\lambda\, Y\big(\delta_{\lambda^{-1}}(x)\big) \\
&= D_\lambda\, [X,Y]\big(\delta_{\lambda^{-1}}(x)\big) = \delta_\lambda[X,Y](x),
\end{aligned}$$

obtaining the second statement above.

By using the above definitions, we can prove that:

Lemma 2.6. *Let α, β, γ, X_1, X_2 and $\mathcal{D}$ be as in Lemma* 2.3*. Then $\mathcal{D}$ is bracket-generating in all of $\mathbb{R}^4$.*

Proof. By Lemma 2.5, the distribution $\mathcal{D}$ is bracket-generating in $B(r)$ for some $r > 0$. Let δ_λ be as in (2.6) with $\omega = \big(1, 1, \alpha+\beta+1, \gamma(\alpha+\beta+1)+1\big)$. Hence for every $x \in B(r)$ we get

$$\begin{aligned}
\delta_\lambda X_1\big(\delta_\lambda(x)\big) &= (\lambda, 0, 0, 0) = \lambda\,(1,0,0,0) = \lambda\, X_1\big(\delta_\lambda(x)\big), \\
\delta_\lambda X_2\big(\delta_\lambda(x)\big) &= \big(0, \lambda, \lambda^{\alpha+\beta+1}\, x_1^\alpha x_2^\beta, \lambda^{\gamma(\alpha+\beta+1)+1}\, x_3^\gamma\big) \\
&= \lambda\, \Big(0, 1, (\lambda\, x_1)^\alpha (\lambda\, x_2)^\beta, \big(\lambda^{\gamma(\alpha+\beta+1)} x_3\big)^\gamma\Big) = \lambda\, X_2\big(\delta_\lambda(x)\big).
\end{aligned}$$

Let X_3 and X_4 be as in (2.4) and (2.5), thus by applying the previous identites and (2.10) we have that

$$\begin{aligned}
\delta_\lambda X_3\big(\delta_\lambda(x)\big) &= \lambda^{\alpha+\beta+1}\, X_3\big(\delta_\lambda(x)\big), \\
\delta_\lambda X_4\big(\delta_\lambda(x)\big) &= \lambda^{\gamma(\alpha+\beta+1)+1}\, X_4\big(\delta_\lambda(x)\big),
\end{aligned}$$

for every $x \in B(r)$. Then one has

$$\begin{aligned}
\delta_\lambda X_1 &= \lambda\, X_1, \\
\delta_\lambda X_2 &= \lambda\, X_2, \\
\delta_\lambda X_3 &= \lambda^{\alpha+\beta+1}\, X_3, \\
\delta_\lambda X_4 &= \lambda^{\gamma(\alpha+\beta+1)+1}\, X_4.
\end{aligned}$$

in all of $B_\lambda(r)$. Now note that

$$X_1, \ldots, X_4 \text{ are linearly indipendent in } B(r)$$

$$\text{if and only if}$$

$$\delta_\lambda X_1, \ldots, \delta_\lambda X_4 \text{ are linearly indipendent in } B_\lambda(r)$$

$$\text{if and only if}$$

$$X_1, \ldots, X_4 \text{ are linearly indipendent in } B_\lambda(r).$$

Therefore we get the thesis by arbitrarity of $\lambda > 1$ and by (2.7). □

We finally collect all of the results seen in this section in the following theorem:

Theorem 2.7. *Let $\alpha \in \mathbb{N}^+$, $\beta \in \mathbb{N}$, $\gamma \in \mathbb{N}^+$. Let $\mathcal{D}$ be the 2-dimensional distribution in $\mathbb{R}^4$ generated by*

$$\begin{cases} X_1(x) = (1, 0, 0, 0)\,, \\ X_2(x) = \big(0, 1, x_1^\alpha x_2^\beta, x_3^\gamma\big)\,, \end{cases} \quad \text{for every } x \in \mathbb{R}^4. \tag{2.11}$$

Then the distribution $\mathcal{D}$ is bracket-generating in all of $\mathbb{R}^4$ and

$$\operatorname{step}\big(\mathcal{D}\big) = \gamma(\alpha + \beta + 1) + 1.$$

Proof. By Lemma 2.6, the distribution $\mathcal{D}$ is bracket-generating in all of $\mathbb{R}^4$. Note that

$$\operatorname{step}\big(\mathcal{D}, x\big) \leq \operatorname{step}\big(\mathcal{D}, 0\big) = \gamma(\alpha + \beta + 1) + 1 \quad \text{for every } x \in \mathbb{R}^4.$$

Thus we deduce that $\operatorname{step}\big(\mathcal{D}\big) = \gamma(\alpha + \beta + 1) + 1$. □

2.2 When $\mathcal{D}$ satisfies (A), $(B)_0$ and $(C)_0$

The aim of this section is to study for which values of α, β and γ the distribution $\mathcal{D}$, defined in 2.11, satisfies the conditions (A), $(B)_0$ and $(C)_0$.

2.2.1 Condition (A)

For the sake of clarity, we repeat the definition of equiregularity of a distribution (in the case $M = \mathbb{R}^n$).

Definition 2.8. Let $(\mathbb{R}^n, \mathcal{D}, g)$ be a sub-Riemannian manifold. Then the distribution $\mathcal{D}$ is said to be *equiregular* if $\dim \mathcal{D}_\ell(p)$ is independent of the point $p \in \mathbb{R}^n$, for every $\ell \in \mathbb{N}$.

One has that:

Theorem 2.9. *Let α, β, γ, X_1, X_2 and $\mathcal{D}$ be as in Theorem 2.7.*
Then the distribution $\mathcal{D}$ is equiregular if and only if $(\alpha, \beta, \gamma) = (1, 0, 1)$.

Proof. Clearly, $\dim \mathcal{D}_0(x) = 0$ and $\dim \mathcal{D}_1(x) = 2$ for every $x \in \mathbb{R}^4$. We have that

$$\begin{aligned} X_{12}(x) &= \big(0, 0, \alpha\, x_1^{\alpha-1} x_2^{\beta}, 0\big), \\ X_{112}(x) &= \big(0, 0, \alpha(\alpha-1)\, x_1^{\alpha-2} x_2^{\beta}, 0\big), \\ X_{212}(x) &= \big(0, 0, \alpha\beta\, x_1^{\alpha-1} x_2^{\beta-1}, -\alpha\gamma\, x_1^{\alpha-1} x_2^{\beta} x_3^{\gamma-1}\big), \end{aligned}$$

for every $x \in \mathbb{R}^4$.
Case 1: Suppose $(\alpha, \beta, \gamma) = (1, 0, 1)$. For every $x \in \mathbb{R}^4$ we have that

$$\begin{aligned} X_{12}(x) &= e_3, \\ X_{112}(x) &= 0, \\ X_{212}(x) &= -e_4. \end{aligned}$$

Then

$$\dim \mathcal{D}_2(x) = \dim \mathcal{D}_3(x) = 1 \quad \text{for every } x \in \mathbb{R}^4.$$

Hence $\mathcal{D}$ is equiregular.
Case 2: Suppose $(\alpha, \beta) = (1, 0)$ and $\gamma \geq 2$.
Since $X_{112}(x) = 0$ and $X_{212}(x) = \big(0, 0, 0, -\gamma\, x_3^{\gamma-1}\big)$ for every $x \in \mathbb{R}^4$, we find that

$$\begin{aligned} X_{112}(0) &= 0, \\ X_{212}(e_3) &= -\gamma\, e_4. \end{aligned}$$

Thus

$$0 = \dim \mathcal{D}_3(0) \neq \dim \mathcal{D}_3(e_3) = 1.$$

Hence $\mathcal{D}$ is not equiregular.
Case 3: Suppose $(\alpha, \gamma) = (1, 1)$ and $\beta \geq 1$.
Since $X_{12}(x) = \big(0, 0, x_2^{\beta}, 0\big)$ for every $x \in \mathbb{R}^4$, we have that

$$\begin{aligned} X_{12}(0) &= 0, \\ X_{12}(e_2) &= e_3. \end{aligned}$$

Thus

$$0 = \dim \mathcal{D}_2(0) \neq \dim \mathcal{D}_2(e_2) = 1.$$

Hence $\mathcal{D}$ is not equiregular.
Case 4: Suppose $\alpha \geq 2$.
Note that $X_{12}(0) = 0$ and $X_{12}(e_1 + e_2) = \alpha\, e_3$. Then

$$0 = \dim \mathcal{D}_2(0) \neq \dim \mathcal{D}_2(e_1 + e_2) = 1.$$

Hence $\mathcal{D}$ is not equiregular. □

2.2.2 Condition $(\mathrm{B})_0$

Now we are interested in studying for which values of α, β and γ our distribution $\mathcal{D}$ satisfies condition $(\mathrm{B})_0$.
We will see that - in the case $\gamma = 1$ - the Lie brackets of two iterated commutators of X_1 and X_2, both having length greater than or equal to 2, is null. By using this fact, we will prove that in this case $(\mathrm{B})_x$ holds for every $x \in \mathbb{R}^4$.

Lemma 2.10. *Let α, β, γ, X_1, X_2 and $\mathcal{D}$ be as in Theorem 2.7. Assume that $\gamma = 1$. Then the distribution $\mathcal{D}$ satisfies $(\mathrm{B})_x$ for every $x \in \mathbb{R}^4$, namely*

$$[\mathcal{L}_i, \mathcal{L}_j](x) \subseteq \mathcal{L}_{i+j-1}(x) \quad \text{for every } i,j \geq 2 \text{ with } i+j \geq 5 \tag{2.12}$$

for every $x \in \mathbb{R}^4$.

Proof. Note that in this case (2.2) reads as

$$\begin{cases} [X_1, V] = \left(0, 0, \frac{\partial p}{\partial x_1}, \frac{\partial q}{\partial x_1}\right), \\ \\ [X_2, V] = \left(0,\ 0,\ \frac{\partial p}{\partial x_2},\ \frac{\partial q}{\partial x_2} + \frac{\partial q}{\partial x_3}\, x_1^{\alpha} x_2^{\beta} - p\right), \end{cases} \tag{2.13}$$

for every vector field $V = (0, 0, p, q)$, with $p,q \in \mathbb{Z}[x_1, x_2]$.
Now observe that, from (2.13), the following facts hold:

- the first and the second entry of every iterated commutator of length greater than or equal to 2 are null,
- the variables x_3 and x_4 do not appear in the iterated commutators of length greater than or equal to 2.

Hence, if V and W are two iterated commutators of X_1 and X_2 of length at least 2 (thus $V = (0, 0, p, q)$, $W = (0, 0, p', q')$ for some p, p', q, $q' \in \mathbb{Z}[x_1, x_2]$), then

$$\begin{aligned} [V, W] = &\left(p\frac{\partial p'}{\partial x_3} - p'\frac{\partial p}{\partial x_3}\right)\frac{\partial}{\partial x_3} + \left(p\frac{\partial q'}{\partial x_3} - p'\frac{\partial q}{\partial x_3}\right)\frac{\partial}{\partial x_3} + \\ &\left(q\frac{\partial p'}{\partial x_4} - q'\frac{\partial p}{\partial x_4}\right)\frac{\partial}{\partial x_4} + \left(q\frac{\partial q'}{\partial x_4} - q'\frac{\partial q}{\partial x_4}\right)\frac{\partial}{\partial x_4} = 0. \end{aligned}$$

Now fix $i,j \geq 2$ with $i+j \geq 5$ and fix $x \in \mathbb{R}^4$. We deduce from the above computation that

$$[\mathcal{L}_i, \mathcal{L}_j](x) = [\mathcal{L}_1, \mathcal{L}_j](x) + [\mathcal{L}_i, \mathcal{L}_1](x). \tag{2.14}$$

Since $[\mathcal{L}_i, \mathcal{L}_1](x) = [\mathcal{L}_1, \mathcal{L}_i](x)$ and by definition of $\mathcal{L}_1(x), \mathcal{L}_2(x), \mathcal{L}_3(x), \dots,$ we find that

$$[\mathcal{L}_1, \mathcal{L}_j](x) + [\mathcal{L}_i, \mathcal{L}_1](x) = \mathcal{L}_{j+1}(x) + \mathcal{L}_{i+1}(x). \tag{2.15}$$

Given that $s \mapsto \mathcal{L}_s(x)$ is a lattice homomorphism between $\mathbb{N}$ (with $\leq$) and the grassmanian of $\mathbb{R}^4$ (with $\subseteq$), we have that

$$\mathcal{L}_{j+1}(x) + \mathcal{L}_{i+1}(x) = \mathcal{L}_{\max\{i,j\}+1}(x). \tag{2.16}$$

Finally, one has that $j \geq 2$ implies $i+1 \leq i+j-1$, and similarly $i \geq 2$ implies $j+1 \leq i+j-1$, thus $\max\{i,j\}+1 \leq i+j-1$ and accordingly

$$\mathcal{L}_{\max\{i,j\}+1}(x) \subseteq \mathcal{L}_{i+j-1}(x). \tag{2.17}$$

Therefore (2.14), (2.15), (2.16) and (2.17) give $[\mathcal{L}_i, \mathcal{L}_j](x) \subseteq \mathcal{L}_{i+j-1}(x)$.
This show that condition $(\mathrm{B})_x$ holds for every $x \in \mathbb{R}^4$. □

Conversely, when $\gamma \geq 2$ condition $(\mathrm{B})_0$ is not satisfied, indeed by commuting X_3 (defined in (2.4), of length $i = \alpha+\beta+1$) with a suitable iterated commutator of length $j = \gamma(\alpha+\beta+1)-(\alpha+\beta)$, we obtain a vector field Z such that $Z(0)$ is a multiple of e_4. Thus the vector e_4 - which does not belong to $\mathcal{L}_{i+j-1}(0) = \mathcal{L}_{\gamma(\alpha+\beta+1)}(0)$ - surely belongs to $[\mathcal{L}_i, \mathcal{L}_j](0)$.

Proposition 2.11. *Let α, β, γ, X_1, X_2 and $\mathcal{D}$ be as in Theorem 2.7. Then condition $(\mathrm{B})_0$ is satisfied if and only if $\gamma = 1$.*

Proof. STEP 1: Suppose $\gamma \geq 2$. As seen in Lemma 2.3, we have that the iterated commutator $X_3 := L_2^{\beta}\big(L_1^{\alpha}(X_2)\big)$ is of the form

$$X_3(x) = \big(0,\, 0,\, \alpha!\beta!,\, x_2 f(x_1,x_2,x_3)\big) \quad \text{for every } x \in \mathbb{R}^4,$$

for some $f \in \big(-\mathbb{N}\big)[x_1,x_2,x_3]$. Then, arguing similarly to what we did in the proof of Lemma 2.3 and using (2.2), we deduce that:

- the fourth component of $L_2\big(X_3\big)$ contains $-\alpha!\beta!\,\gamma\, x_3^{\gamma-1}$,
- the fourth component of $L_2^{\gamma-2}\big(L_2(X_3)\big)$ contains $-\alpha!\beta!\gamma!\, x_1^{\alpha(\gamma-2)} x_2^{\beta(\gamma-2)} x_3$,
- the fourth component of $L_2^{\beta(\gamma-2)}\big(L_2^{\gamma-1}(X_3)\big)$ contains

$$-\alpha!\beta!\gamma!\,(\beta(\gamma-2))!\, x_1^{\alpha(\gamma-2)} x_3,$$

- the fourth component of $V := L_1^{\alpha(\gamma-2)}\big(L_2^{\beta(\gamma-2)+(\gamma-1)}(X_3)\big)$ contains

$$-\alpha!\beta!\gamma!\,(\beta(\gamma-2))!(\alpha(\gamma-2))!\, x_3. \tag{2.18}$$

Since we have commuted $\alpha(\gamma-2)+\beta(\gamma-2)+(\gamma-1) \geq 1$ times the vector field X_3 (with either X_1 or X_2) in order to obtain V, we have that the third component of V vanishes, so $V = (0,0,0,q)$ for some $q \in \big(-\mathbb{N}\big)[x_1,x_2,x_3]$ containing the addendum (2.18). A simple computation gives

$$Z := [X_3, V] = \left(0,0,0,\alpha!\beta!\,\frac{\partial q}{\partial x_3}\right).$$

Hence the fourth component of Z contains

$$-(\alpha!)^2(\beta!)^2\gamma!\,(\beta(\gamma-2))!(\alpha(\gamma-2))!.$$

which implies that $Z(0)$ is a non null multiple of e_4. Moreover the length of X_3 is $i := \alpha+\beta+1$ and the length of V is $j := \gamma(\alpha+\beta+1)-(\alpha+\beta)$ (note that $i \geq 2$ and $j = (\alpha+\beta)(\gamma-1)+\gamma \geq 3$, thus $i+j \geq 5$).
Since $Z \in \mathrm{Sec}\left([\mathcal{L}_i, \mathcal{L}_j]\right)$, one has that $e_4 \in \left[\mathcal{L}_i,\, \mathcal{L}_j\right](0)$, but

$$e_4 \notin \mathcal{L}_{i+j-1}(0) = \mathcal{L}_{\gamma(\alpha+\beta+1)}(0),$$

by Remark 2.4. In other words, $(\mathrm{B})_0$ does not hold when $\gamma \geq 2$.
STEP 2: Suppose $\gamma = 1$. By Lemma 2.10 we have that condition $(\mathrm{B})_0$ is satisfied. □

2.2.3 Condition $(\mathrm{C})_0$

By using Lemma 2.3, it is simple to prove that condition $(\mathrm{C})_0$ is satisfied if and only if $(\alpha,\beta) \neq (1,0)$ and $\gamma \geq 2$:

Proposition 2.12. *Let α,β,γ, X_1, X_2 and $\mathcal{D}$ be as in Theorem 2.7.
Then condition $(\mathrm{C})_0$ is not satisfied if and only if either $(\alpha,\beta) = (1,0)$ or $\gamma = 1$.*

Proof. We deduce from Lemma 2.3 and Remark 2.4 that

$$\begin{aligned}\mathcal{L}_0(0) \neq \mathcal{L}_1(0) = \ldots = \mathcal{L}_{\alpha+\beta}(0) \neq \mathcal{L}_{\alpha+\beta+1}(0)\\ = \ldots = \mathcal{L}_{\gamma(\alpha+\beta+1)}(0) \neq \mathcal{L}_{\gamma(\alpha+\beta+1)+1}(0) = \ldots .\end{aligned}$$

Note that $\mathcal{L}_i(0) \neq \mathcal{L}_{i-1}(0)$ only for

$$\begin{aligned} &i = 1,\\ &i = \alpha+\beta+1,\\ &i = \gamma(\alpha+\beta+1)+1.\end{aligned}$$

Then condition $(\mathrm{C})_0$ does not hold if and only if either $\alpha+\beta = 1$ (i.e. $(\alpha,\beta) = (1,0)$) or $\gamma(\alpha+\beta+1) = \alpha+\beta+1$ (i.e. $\gamma = 1$). □

Hence in this section we proved that:

Corollary 2.13. *Let α, β, γ, X_1, X_2 and $\mathcal{D}$ be as in Theorem 2.7.
Then none of the conditions (A), $(\mathrm{B})_0$ and $(\mathrm{C})_0$ is satisfied if and only if $(\alpha,\beta) = (1,0)$ and $\gamma \geq 2$.*

Proof. It follows from Theorem 2.9, from Proposition 2.11 and from Proposition 2.12. □

2.3 The angle ν is an abnormal extremal

First of all, we give the definition of corner.

Definition 2.14 (Corner)**.** Let $\gamma : [a, b] \to \mathbb{R}^n$ be a Lipschitz curve. We denote by $\dot\gamma_L(t)$ and $\dot\gamma_R(t)$ the *left derivative* and the *right derivative*, respectively, of γ at the time $t \in [a, b]$, whenever they exist. Explicitly,

$$\dot\gamma_L(t) := \lim_{h\to 0^+} \frac{\gamma(t+h) - \gamma(t)}{h}$$
$$\dot\gamma_R(t) := \lim_{h\to 0^-} \frac{\gamma(t+h) - \gamma(t)}{h}.$$

We say that γ has a *corner* (or an *angle*) at the point $x = \gamma(t)$, for some $t \in [a, b]$, if there exist $\dot\gamma_L(t), \dot\gamma_R(t)$ and $\dot\gamma_L(t)$, $\dot\gamma_R(t)$ are linearly independent.

In the next proposition, we introduce an extremal ν having a corner at $0 \in \mathbb{R}^4$ and we study, by using Pontryagin Maximum Principle, for which values of α, β and γ the curve ν is a (strictly) abnormal extremal.

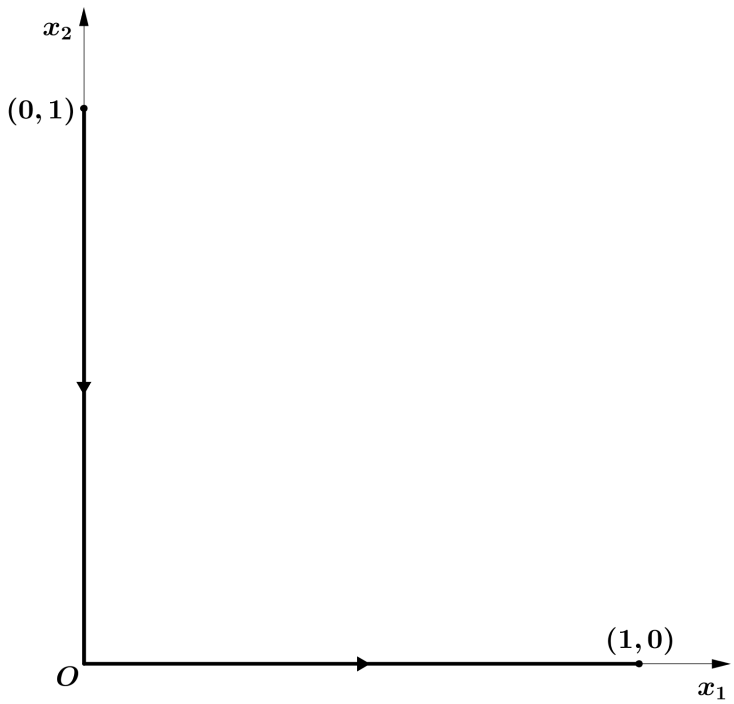

Figure 2.1: The projection of ν on the plane x_1x_2.

Proposition 2.15. *Let α, β, γ, X_1, X_2 and $\mathcal{D}$ be as in Theorem 2.7. Then the $\mathcal{D}$-horizontal curve*

$$\nu(t) = \begin{cases} (0, -t, 0, 0) & \textit{if } t \in [-1, 0], \\ (t, 0, 0, 0) & \textit{if } t \in [0, 1], \end{cases} \tag{2.19}$$

is a strictly abnormal extremal if and only if either $\alpha > 1$ or $\gamma > 1$. Note that ν has a corner at the point $0 \in \mathbb{R}^4$.

Proof. The curve ν is actually $\mathcal{D}$-horizontal, with controls

$$h(t) = \begin{cases} (0,-1) & \text{if } t \in [-1,0], \\ (1,0) & \text{if } t \in (0,1]. \end{cases}$$

In order to prove that the curve ν is an abnormal extremal, we want to find a dual curve ξ which satisfies the following necessary conditions (of Pontryagin Maximum Principle):

$$\xi(t) \cdot X_i\big(\nu(t)\big) = 0 \quad \text{for every } t \in [-1,1] \text{ and } i = 1,2, \tag{2.20}$$

$$\xi(t) \neq 0 \quad \text{for every } t \in [-1,1], \tag{2.21}$$

$$\dot{\xi}(t) = -\Big(h_1(t)\, X_1'\big(\nu(t)\big) + h_2(t)\, X_2'\big(\nu(t)\big)\Big)^T \xi(t) \quad \text{for a.e. } t \in [-1,1]. \tag{2.22}$$

In this case, (2.20) and (2.22) read as

$$\xi_1(t) = \xi_2(t) = 0 \quad \text{for every } t \in [-1,1] \tag{2.23}$$

and

$$\begin{cases} \dot{\xi}(t) = X_2'\big(\nu(t)\big)^T \xi(t) & \text{for a.e. } t \in [-1,0], \\ \dot{\xi}(t) = 0 & \text{for a.e. } t \in (0,1], \end{cases} \tag{2.24}$$

respectively. We have that

$$X_2'\big(\nu(t)\big)^T = \begin{pmatrix} 0 & 0 & \alpha\,\nu_1(t)^{\alpha-1}\nu_2(t)^{\beta} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma\nu_3(t)^{\gamma-1} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence:
Case 1: Suppose $\alpha > 1$, $\gamma = 1$. We get $\xi(t) = \big(0,0,\mu(t \wedge 0) + \lambda, \mu\big)$ for every $t \in [-1,1]$, for any λ, $\mu \in \mathbb{R}$ with $(\lambda,\mu) \neq (0,0)$.
Case 2: Suppose $\alpha = 1$, $\gamma > 1$. We get $\xi(t) = (0,0,0,\mu)$ for every $t \in [-1,1]$, for any $\mu \neq 0$.
Case 3: Suppose $\alpha, \gamma > 1$. We get that $\xi(t) = (0,0,\lambda,\mu)$ for every $t \in [-1,1]$, for any λ, $\mu \in \mathbb{R}$ with $(\lambda,\mu) \neq (0,0)$.
Case 4: Suppose $\alpha = \gamma = 1$. We obtain that $\xi(t) = (0,0,0,0)$ for every $t \in [-1,1]$. Thus in this case the curve ν is not an abnormal extremal, by (2.21).
Finally, ν is not a normal extremal, because it is not smooth in $t = 0$, hence the thesis. □

2.4 The curve ν is not a length-minimizer

As a consequence of Section 2.3 and Section 2.2, we are interested in studying $\mathcal{D}$ in the case $(\alpha,\beta) = (1,0)$ and $\gamma \geq 2$. Explicitly, $\mathcal{D} = \operatorname{span}\{X_1, X_2\}$ where

$$\begin{cases} X_1(x) = (1,0,0,0), \\ X_2(x) = (0,1,x_1,x_3^{\gamma}), \end{cases} \tag{2.25}$$

for every $x \in \mathbb{R}^4$.

By Remark 2.4 we have that $\mathrm{step}(\mathcal{D}) = 2\gamma + 1$. By Proposition 2.15 we have that the angle ν (defined in (2.19)) is a strictly abnormal extremal. The aim of this section is to find a $\mathcal{D}$-horizontal curve joining the points $\nu(-1) = (0,1,0,0)$ and $\nu(1) = (1,0,0,0)$ with length strictly smaller than that of ν.

2.4.1 The "cut" ν^ε

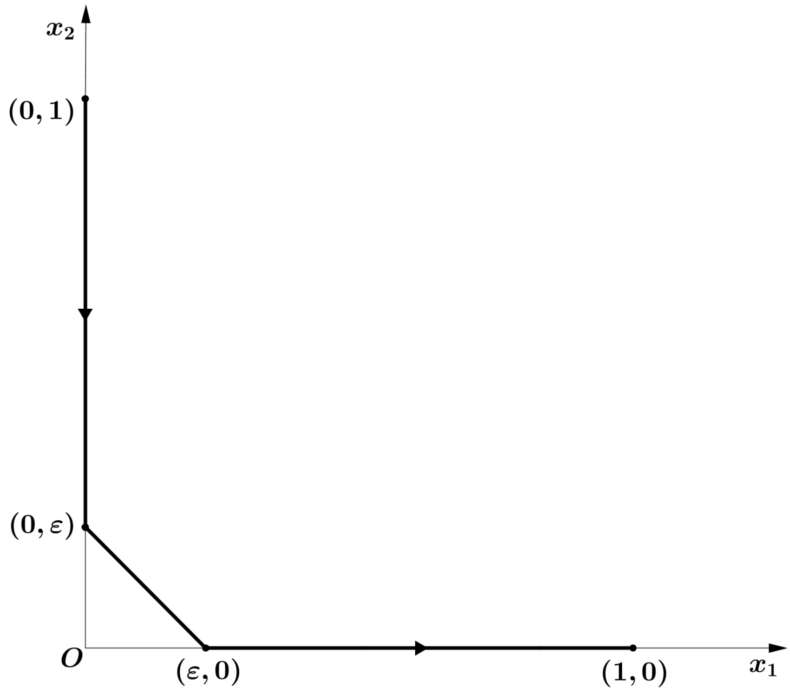

Figure 2.2: The curve $(\nu_1^\varepsilon, \nu_2^\varepsilon)$.

The first step is to construct a $\mathcal{D}$-horizontal curve starting from the same initial point of ν, whose length is strictly smaller than that of ν. The problem is that this new curve and ν have a different endpoint, hence we will need further corrections.

Fix $0 < \varepsilon < 1$. Consider the polygonal planar curve $(\nu_1^\varepsilon, \nu_2^\varepsilon) : [-1,1] \to \mathbb{R}^2$ obtained by the concatenation of the segment joining $(0,1)$ to $(0,\varepsilon)$, the segment joining $(0,\varepsilon)$ to $(\varepsilon, 0)$ and finally the segment joining $(\varepsilon,0)$ to $(1,0)$.
A parametrization of $(\nu_1^\varepsilon, \nu_2^\varepsilon)$ can be chosen as follows:

$$(\nu_1^\varepsilon, \nu_2^\varepsilon)(t) := \begin{cases} (0, -t) & \text{if } t \in [-1, -\varepsilon], \\ \left(\frac{t+\varepsilon}{2}, \frac{\varepsilon - t}{2}\right) & \text{if } t \in [-\varepsilon, \varepsilon], \\ (t, 0) & \text{if } t \in [\varepsilon, 1]. \end{cases}$$

We want to find the *lift* $\nu^\varepsilon = (\nu_1^\varepsilon, \nu_2^\varepsilon, \nu_3^\varepsilon, \nu_4^\varepsilon) : [-1,1] \to \mathbb{R}^4$ of $(\nu_1^\varepsilon, \nu_2^\varepsilon)$ starting from the point $(0,1,0,0)$, i.e.

(i) ν^ε is $\mathcal{D}$-horizontal, thus

$$\begin{aligned}\dot{\nu^\varepsilon}(t) &= h_1(t)\, X_1\big(\nu^\varepsilon(t)\big) + h_2(t)\, X_2\big(\nu^\varepsilon(t)\big) \\ &= h_1(t)\,\big(1,0,0,0\big) + h_2(t)\,\big(0,\, 1,\, \nu_1^\varepsilon(t),\, \nu_3^\varepsilon(t)^\gamma\big)\end{aligned}$$

for almost every $t \in [-1,1]$ and for suitable controls $h \in \mathrm{L}^\infty\big([-1,1], \mathbb{R}^2\big)$,

(ii) $\nu^\varepsilon(-1) = (0,1,0,0)$.

We deduce from (i) that

$$h = \big(h_1(t), h_2(t)\big) = \big(\dot{\nu_1^\varepsilon}(t),\, \dot{\nu_2^\varepsilon}(t)\big) \quad \text{for almost every } t \in [-1,1]. \tag{2.26}$$

◇ Case $t \in [-1, -\varepsilon]$.
By (i) and (2.26) we have that $\dot{\nu_3^\varepsilon}(t) = 0$ and $\dot{\nu_4^\varepsilon}(t) = -\nu_3^\varepsilon(t)^\gamma$ for almost every $t \in [-1,-\varepsilon]$. By (ii) it follows that

$$\nu_3^\varepsilon(t) = \nu_3^\varepsilon(-1) = 0 \quad \text{for every } t \in [-1,-\varepsilon],$$

hence

$$\nu_4^\varepsilon(t) = \nu_4^\varepsilon(-1) = 0 \quad \text{for every } t \in [-1,-\varepsilon].$$

◇ Case $t \in [-\varepsilon, \varepsilon]$.
We have that $\dot{\nu_3^\varepsilon}(t) = -\frac{1}{2}\,\nu_1^\varepsilon(t) = -\frac{1}{4}\,(t+\varepsilon)$ and $\dot{\nu_4^\varepsilon}(t) = -\frac{1}{2}\,\nu_3^\varepsilon(t)^\gamma$ for almost every $t \in [-\varepsilon, \varepsilon]$. Therefore for every $t \in [-\varepsilon, \varepsilon]$

$$\nu_3^\varepsilon(t) = \nu_3^\varepsilon(-\varepsilon) - \frac{1}{4}\int_{-\varepsilon}^{t} \big(y+\varepsilon\big)\,dy = -\frac{1}{8}(t+\varepsilon)^2$$

and

$$\dot{\nu_4^\varepsilon}(t) = -\frac{1}{2}\,\nu_3^\varepsilon(t)^\gamma = \frac{(-1)^{\gamma+1}}{2^{3\gamma+1}}\,(t+\varepsilon)^{2\gamma}.$$

Then for every $t \in [-\varepsilon, \varepsilon]$

$$\nu_4^\varepsilon(t) = \nu_4^\varepsilon(-\varepsilon) + \frac{(-1)^{\gamma+1}}{2^{3\gamma+1}}\int_{-\varepsilon}^{t} \big(s+\varepsilon\big)^{2\gamma}\,ds = \frac{(-1)^{\gamma+1}}{2^{3\gamma+1}}\,\frac{(t+\varepsilon)^{2\gamma+1}}{2\gamma+1}.$$

◇ Case $t \in [\varepsilon, 1]$.
We find that $\dot{\nu_3^\varepsilon}(t) = 0$ and $\dot{\nu_4^\varepsilon}(t) = 0$ for almost every $t \in [\varepsilon, 1]$. Therefore

$$\nu_3^\varepsilon(t) = \nu_3^\varepsilon(\varepsilon) = -\frac{1}{2}\,\varepsilon^2 \quad \text{for every } \ t \in [\varepsilon, 1]$$

and

$$\nu_4^\varepsilon(t) = \nu_4^\varepsilon(\varepsilon) = \frac{(-1)^{\gamma+1}}{2^\gamma}\,\frac{\varepsilon^{2\gamma+1}}{2\gamma+1} \quad \text{for every } \ t \in [\varepsilon, 1].$$

Then the lift of $(\nu_1^\varepsilon, \nu_2^\varepsilon)$ is

$$\nu^\varepsilon(t) = \begin{cases} (0, -t, 0, 0) & \text{if } t \in [-1, -\varepsilon], \\ \left(\frac{t+\varepsilon}{2}, \frac{\varepsilon-t}{2}, -\frac{1}{8}(t+\varepsilon)^2, \frac{(-1)^{\gamma+1}}{2^{3\gamma+1}} \frac{(t+\varepsilon)^{2\gamma+1}}{2\gamma+1}\right) & \text{if } t \in [-\varepsilon, \varepsilon], \\ \left(t, 0, -\frac{1}{2}\varepsilon^2, \frac{(-1)^{\gamma+1}}{2^\gamma} \frac{\varepsilon^{2\gamma+1}}{2\gamma+1}\right) & \text{if } t \in [\varepsilon, 1]. \end{cases}$$

Note that the endpoint of ν^ε is not $\nu(1) = (1, 0, 0, 0)$, indeed

$$\nu_3^\varepsilon(1) = -\frac{1}{2}\varepsilon^2 \tag{2.27}$$

and

$$\nu_4^\varepsilon(1) = \frac{(-1)^{\gamma+1}}{2^\gamma} \frac{\varepsilon^{2\gamma+1}}{2\gamma+1}. \tag{2.28}$$

Moreover, the length of ν is $L(\nu) = 2$, while the length of ν^ε is $L(\nu^\varepsilon) = 2(1-\varepsilon)+\sqrt{2}\varepsilon$. Hence ν^ε is strictly shorter than ν, precisely

$$L(\nu) - L(\nu^\varepsilon) = \left(2 - \sqrt{2}\right)\varepsilon > 0. \tag{2.29}$$

2.4.2 The first perturbation ζ^ε of ν^ε

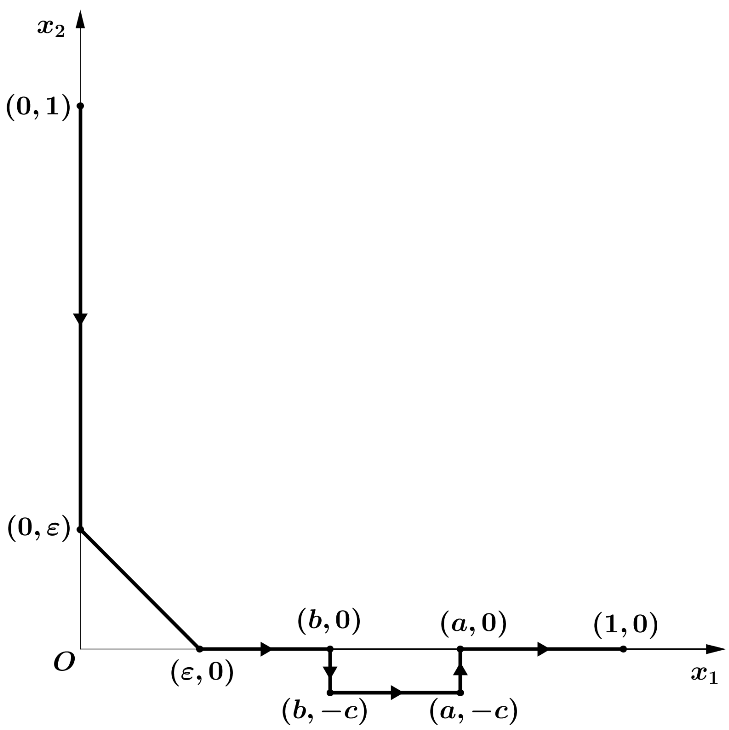

Figure 2.3: The curve $(\zeta_1^\varepsilon, \zeta_2^\varepsilon)$.

In order to correct the third component of the endpoint of ν^ε, we now construct a curve ζ^ε (obtained by modifying ν^ε with a rectangle), which depends on some parameters of position.

Take $0 < \varepsilon < b < a < 1$ and suppose $c < \frac{a-b}{2}$. Consider the polygonal planar curve $(\zeta_1^\varepsilon, \zeta_2^\varepsilon)$ joining the following points:

$$(0,1) \to (0,\varepsilon) \to (\varepsilon, 0) \to (b,0) \to (b,-c) \to (a,-c) \to (a,0) \to (1,0).$$

Then $(\zeta_1^\varepsilon, \zeta_2^\varepsilon)$ can be parametrized as follows:

$$(\zeta_1^\varepsilon, \zeta_2^\varepsilon)(t) := \begin{cases} (0,-t) & \text{if } t \in [-1,-\varepsilon], \\ \left(\frac{t+\varepsilon}{2}, \frac{\varepsilon - t}{2}\right) & \text{if } t \in [-\varepsilon, \varepsilon], \\ (t,0) & \text{if } t \in [\varepsilon, b], \\ (b, b-t) & \text{if } t \in [b, b+c], \\ \left(\frac{a-b}{a-b-2c}t - \frac{c(a+b)}{a-b-2c}, \ -c\right) & \text{if } t \in [b+c, a-c], \\ (a, t-a) & \text{if } t \in [a-c, a], \\ (t,0) & \text{if } t \in [a,1]. \end{cases}$$

We now calculate the lift $\zeta^\varepsilon = \left(\zeta_1^\varepsilon, \zeta_2^\varepsilon, \zeta_3^\varepsilon, \zeta_4^\varepsilon\right) : [-1,1] \to \mathbb{R}^4$ of $(\zeta_1^\varepsilon, \zeta_2^\varepsilon)$ starting from the point $(0,1,0,0)$. Since ζ^ε must be $\mathcal{D}$-horizontal, we impose that

$$\dot{\zeta}_3^\varepsilon = \dot{\zeta}_2^\varepsilon \, \zeta_1^\varepsilon \tag{2.30}$$

and that

$$\dot{\zeta}_4^\varepsilon = \dot{\zeta}_2^\varepsilon \, \zeta_3^\gamma \tag{2.31}$$

almost everywhere in $[-1,1]$.

$\diamond$ Case $t \in [-1, b]$.
Clearly $\zeta^\varepsilon(t) = \nu^\varepsilon(t)$ for every $t \in [-1,b]$.
$\diamond$ Case $t \in [b, b+c]$.
By (2.30) we have $\dot{\zeta}_3^\varepsilon(t) = -b$ for almost every $t \in [b, b+c]$. Then for every $t \in [b, b+c]$

$$\zeta_3^\varepsilon(t) = \zeta_3^\varepsilon(b) - \int_b^t b \, dy = -\frac{\varepsilon^2}{2} - b(t-b).$$

By (2.31) we find that $\dot{\zeta}_4^\varepsilon(t) = (-1)^{\gamma+1}\left(b(t-b) + \frac{\varepsilon^2}{2}\right)^\gamma$ for almost every $t \in [b, b+c]$. Hence for every $t \in [b, b+c]$

$$\begin{aligned} \zeta_4^\varepsilon(t) &= \frac{(-1)^{\gamma+1}}{2^\gamma} \frac{\varepsilon^{2\gamma+1}}{2\gamma+1} + (-1)^{\gamma+1} \int_b^t \left(b(y-b) + \frac{\varepsilon^2}{2}\right)^\gamma dy \\ &= (-1)^{\gamma+1} \frac{\varepsilon^{2\gamma+1}}{2^\gamma} \left(\frac{1}{2\gamma+1} - \frac{\varepsilon}{2b(\gamma+1)}\right) + \frac{(-1)^{\gamma+1}}{b(\gamma+1)} \left(b(t-b) + \frac{\varepsilon^2}{2}\right)^{\gamma+1}. \end{aligned}$$

⋄ Case $t \in [b+c, a-c]$.
By (2.30) and (2.31) we deduce that for every $t \in [b+c, a-c]$

$$\zeta_3^\varepsilon(t) = \zeta_3^\varepsilon(b+c) = -bc - \frac{\varepsilon^2}{2}$$

and that

$$\begin{aligned}\zeta_4^\varepsilon(t) &= \zeta_4^\varepsilon(b+c)\\ &= (-1)^{\gamma+1}\frac{\varepsilon^{2\gamma+1}}{2^\gamma}\left(\frac{1}{2\gamma+1} - \frac{\varepsilon}{2b(\gamma+1)}\right) + \frac{(-1)^{\gamma+1}}{b(\gamma+1)}\left(bc + \frac{\varepsilon^2}{2}\right)^{\gamma+1}.\end{aligned}$$

⋄ Case $t \in [a-c, a]$.
By (2.30) we have $\dot{\zeta}_3^\varepsilon(t) = a$ for almost every $t \in [a-c, a]$. Then

$$\zeta_3^\varepsilon(t) = -bc - \frac{\varepsilon^2}{2} + a(t-a+c)$$

for every $t \in [a-c, a]$. Thus, by (2.31), we find that $\dot{\zeta}_4^\varepsilon(t) = \left(-bc - \frac{\varepsilon^2}{2} + a(t-a-c)\right)^\gamma$ for almost every $t \in [a-c, a]$. Hence for every $t \in [a-c, a]$

$$\begin{aligned}\zeta_4^\varepsilon(t) =&(-1)^{\gamma+1}\frac{\varepsilon^{2\gamma+1}}{2^\gamma}\left(\frac{1}{2\gamma+1} - \frac{\varepsilon}{2b(\gamma+1)}\right) + \frac{(-1)^{\gamma+1}}{b(\gamma+1)}\left(bc + \frac{\varepsilon^2}{2}\right)^{\gamma+1}\\ &+ \frac{1}{a(\gamma+1)}\left(\left(a(t-a+c) - bc - \frac{\varepsilon^2}{2}\right)^{\gamma+1} - \left(-bc - \frac{\varepsilon^2}{2}\right)^{\gamma+1}\right).\end{aligned}$$

⋄ Case $t \in [a, 1]$.
By (2.30) and (2.31) we deduce that for every $t \in [a, 1]$

$$\zeta_3^\varepsilon(t) = \zeta_3^\varepsilon(a) = -bc - \frac{\varepsilon^2}{2} + ac$$

and

$$\begin{aligned}\zeta_4^\varepsilon(t) =&(-1)^{\gamma+1}\frac{\varepsilon^{2\gamma+1}}{2^\gamma}\left(\frac{1}{2\gamma+1} - \frac{\varepsilon}{2b(\gamma+1)}\right) + \frac{(-1)^{\gamma+1}}{b(\gamma+1)}\left(bc + \frac{\varepsilon^2}{2}\right)^{\gamma+1}\\ &+ \frac{1}{a(\gamma+1)}\left(\left(ac - bc - \frac{\varepsilon^2}{2}\right)^{\gamma+1} - \left(-bc - \frac{\varepsilon^2}{2}\right)^{\gamma+1}\right).\end{aligned}$$

Thus we deduce that

$$\zeta_3^\varepsilon(1) = -bc - \frac{\varepsilon^2}{2} + ac.$$

Hence, if we choose

$$c = \frac{\varepsilon^2}{2(a-b)} \tag{2.32}$$

we have that $\zeta_3^\varepsilon(1) = 0$. Hereafter, we will always fix $c = \frac{\varepsilon^2}{2(a-b)}$. By (2.32) one has that

$$\lambda(\varepsilon) := \zeta_4^\varepsilon(1) = \frac{(-1)^{\gamma+1}}{2^\gamma(2\gamma+1)}\,\varepsilon^{2\gamma+1}\left(1 + \varepsilon\,\frac{2\gamma+1}{2b(\gamma+1)}\left(\frac{a^\gamma}{(a-b)^\gamma} - 1\right)\right). \tag{2.33}$$

Remark 2.16. We give a geometric interpretation of the third entry of every $\mathcal{D}$-horizontal curve, which is consistent with what we observe in the previous section. Namely, the third component furnishes information about the area enclosed by the first two entries in $\mathbb{R}^2$. In order to give a geometric meaning to the third component of a $\mathcal{D}$-horizontal curve, we shall make use of the well-known *Green's Formula*:

Theorem 2.17. *Let $\mu : [a,b] \to \mathbb{R}^2$ be a continuous, piecewise C^1 circuit in $\mathbb{R}^2$. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field of class C^1. Let us call D the region in $\mathbb{R}^2$ enclosed by the curve μ. Then*

$$\int_\mu F \cdot n_\mu \, d\mu = \int_D \mathrm{div} F(x,y)\, dx\, dy, \tag{2.34}$$

where:

- $\int_\mu F \cdot n_\mu \, d\mu := \int_a^b F(\mu(t)) \cdot n_\mu\big(\mu(t)\big)\, |\dot\mu(t)|\, dt$ *is the path integral of F along μ,*
- n_μ *is the unit external normal to D, explicitly*

$$n_\mu\big(\mu(t)\big) := \frac{\big(\dot\mu_2(t), -\dot\mu_1(t)\big)}{\sqrt{\dot\mu_1(t)^2 + \dot\mu_2(t)^2}} \tag{2.35}$$

for every $t \in [a,b]$ such that $\dot\mu(t)$ exists non-null and arbitrarily defined elsewhere.

Let $\mu : [t_1, t_2] \to \mathbb{R}^4$ be a $\mathcal{D}$-horizontal curve such that $\big(\mu_1(t_1), \mu_2(t_1), \mu_3(t_1)\big) = (0,1,0)$ and $\big(\mu_1(t_2), \mu_2(t_2)\big) = (1,0)$. Let $\sigma : [0,1] \to \mathbb{R}^2$ be the segment joining $(1,0)$ to $(0,1)$, defined by

$$\sigma(t) := (1-t, t) \quad \text{for every } t \in [0,1].$$

Hence the curve obtained by concatenating (μ_1, μ_2) and σ is a circuit in $\mathbb{R}^2$. Let us call D the region enclosed by such curve. Thus for every $F \in C^1\big(\mathbb{R}^2, \mathbb{R}^2\big)$ one has

$$\begin{aligned}\int_D \mathrm{div} F(x,y)\, dx\, dy = &\int_{t_1}^{t_2} F\big(\mu_1(t), \mu_2(t)\big) \cdot n_{(\mu_1,\mu_2)}\big(\mu_1(t), \mu_2(t)\big)\, \big|(\dot\mu_1, \dot\mu_2)(t)\big|\, dt \\ &+ \int_0^1 F(\sigma(t)) \cdot n_\sigma(\sigma(t))\, |\dot\sigma(t)|\, dt.\end{aligned}$$

Note that

$$n_\sigma\big(\sigma(t)\big) = \frac{(1,1)}{\sqrt{2}} \quad \text{for every } t \in (0,1).$$

Now take $F(x,y) := (x, 0)$ for every $(x,y) \in \mathbb{R}^2$. Thus $\mathrm{div} F = 1$ and accordingly

$$\mathcal{A}(D) := \int_D dx\, dy = \int_{t_1}^{t_2} \mu_1(t)\dot\mu_2(t)\, dt + \int_0^1 (1-t)\, dt.$$

Since μ is a $\mathcal{D}$-horizontal curve, one has that $\dot\mu_3 = \mu_1 \dot\mu_2$ a.e. in $[a,b]$. Therefore $\mu_3(b) = \int_{t_1}^{t_2} \dot\mu_3(t)\, dt = \int_{t_1}^{t_2} \mu_1(t)\dot\mu_2(t)\, dt$, which gives

$$\mathcal{A}(D) = \mu_3(t_2) + \frac{1}{2}.$$

This equality can be interpreted in the following way: the third component of μ at the final time t_2 is equal to 0 if and only if the (signed) area enclosed by (μ_1, μ_2) (concatenated with the segment joining $(0,1)$ to $(1,0)$) is equal to $\frac{1}{2}$. Notice that this is coherent with what we saw in the previous section for the curves ζ^ε and μ^ε. For example, note that - by (2.32) - the area of the rectangle of vertices $(b,0)$,$(b,-c)$,$(a,-c)$,$(a,0)$ coincides with the area of the triangle of vertices $(0,\varepsilon)$,$(0,0)$,$(\varepsilon,0)$.

2.4.3 The second perturbation μ^ε of ν^ε

We now concatenate ζ^ε with a suitable circuit, obtaining a new curve μ^ε, in such a way that the fourth component of the endpoint becomes equal to 0, but leaving unchanged its third component.

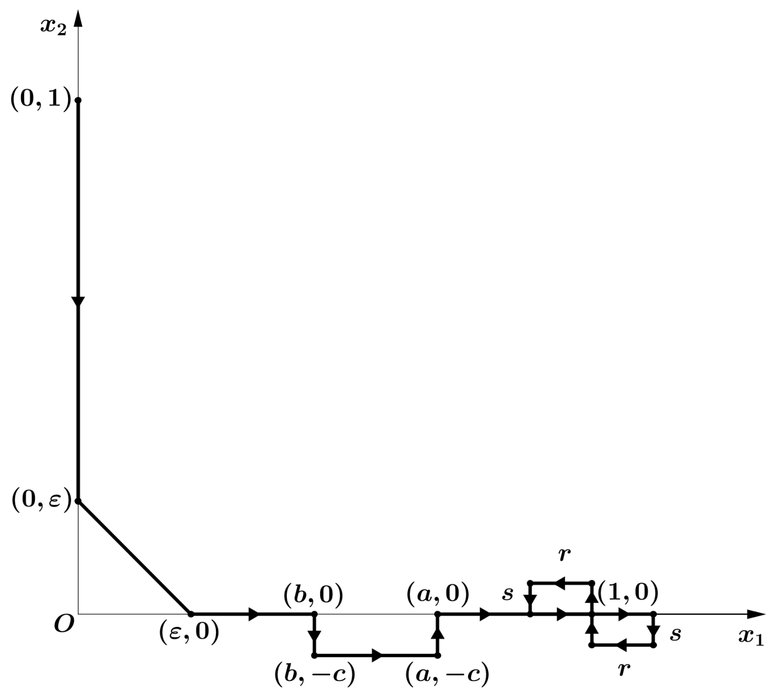

Figure 2.4: The curve $(\mu_1^\varepsilon, \mu_2^\varepsilon)$.

Fix two parameters r,$s > 0$. To get $(\mu_1^\varepsilon, \mu_2^\varepsilon)$, we add to $(\zeta_1^\varepsilon, \zeta_2^\varepsilon)$ the polygonal planar curve joining the following points:

$$(1,0) \to (1+r,0) \to (1+r,-s) \to (1,-s) \to (1,s) \to$$
$$\to (1-r,s) \to (1-r,0) \to (1,0).$$

We parametrize $(\mu_1^\varepsilon, \mu_2^\varepsilon)$ on the interval $[-1,\, 1+4r+4s]$ as follows:

$$(\mu_1^\varepsilon, \mu_2^\varepsilon)(t) := \begin{cases} \big(\zeta_1^\varepsilon(t), \zeta_2^\varepsilon(t)\big) & \text{if } t \in [-1,1], \\ (t,0) & \text{if } t \in [1,1+r], \\ \big(1+r, r-t+1\big) & \text{if } t \in [1+r, 1+r+s], \\ \big(-t+2+s+2r, -s\big) & \text{if } t \in [1+r+s, 1+2r+s], \\ \big(1, t-1-2r-2s\big) & \text{if } t \in [1+2r+s, 1+2r+3s], \\ \big(-t+2+2r+3s, s\big) & \text{if } t \in [1+2r+3s, 1+3r+3s], \\ \big(1-r, -t+1+3r+4s\big) & \text{if } t \in [1+3r+3s, 1+3r+4s], \\ \big(t-4r-4s, 0\big) & \text{if } t \in [1+3r+4s, 1+4r+4s]. \end{cases}$$

We now want to compute the lift $\mu^\varepsilon = \big(\mu_1^\varepsilon, \mu_2^\varepsilon, \mu_3^\varepsilon, \mu_4^\varepsilon\big) : [-1, 1+4r+4s] \to \mathbb{R}^4$ of the curve $(\mu_1^\varepsilon, \mu_2^\varepsilon)$. Since μ^ε must be $\mathcal{D}$-horizontal, we impose that

$$\dot\mu_3^\varepsilon = \dot\mu_2^\varepsilon\, \mu_1^\varepsilon \tag{2.36}$$

and

$$\dot\mu_4^\varepsilon = \dot\mu_2^\varepsilon\, \big(\mu_3^\varepsilon\big)^\gamma \tag{2.37}$$

almost everywhere in $[-1, 1+4r+4s]$.

◇ Case $t \in [-1,1]$.
Clearly $\mu^\varepsilon(t) = \zeta^\varepsilon(t)$ for every $t \in [-1,1]$.
◇ Case $t \in [1, 1+r]$.
By (2.36) and (2.37) we have that for every $t \in [1,1+r]$

$$\begin{aligned} \mu_3^\varepsilon(t) &= \mu_3^\varepsilon(1) = 0, \\ \mu_4^\varepsilon(t) &= \mu_4^\varepsilon(1) = \lambda(\varepsilon). \end{aligned}$$

◇ Case $t \in [1+r, 1+r+s]$.
By (2.36) we see that $\dot\mu_3^\varepsilon(t) = -(1+r)$ for almost every $t \in [1+r, 1+r+s]$. Thus

$$\mu_3^\varepsilon(t) = -(1+r)(t-r-1) \qquad \text{for every } t \in [1+r, 1+r+s].$$

Therefore, by (2.37) we deduce that $\dot\mu_4^\varepsilon(t) = (-1)^{\gamma+1}\,(1+r)^\gamma (t-r-1)^\gamma$ for almost every $t \in [1+r, 1+r+s]$. Thus

$$\mu_4^\varepsilon(t) = \lambda(\varepsilon) + \frac{(-1)^{\gamma+1}}{\gamma+1}\,(1+r)^\gamma (t-r-1)^{\gamma+1} \qquad \text{for every } t \in [1+r, 1+r+s].$$

◇ Case $t \in [1+r+s, 1+2r+s]$.
By (2.36) and (2.37) we deduce that for every $t \in [1+r+s, 1+2r+s]$

$$\mu_3^\varepsilon(t) = \mu_3^\varepsilon(1+r+s) = -s\,(1+r),$$
$$\mu_4^\varepsilon(t) = \mu_4^\varepsilon(1+r+s) = \lambda(\varepsilon) + \frac{(-1)^{\gamma+1}}{\gamma+1}\,(1+r)^\gamma s^{\gamma+1}.$$

◇ Case $t \in [1+2r+s, 1+2r+3s]$.
By (2.36) we have that $\dot\mu_3^\varepsilon(t) = 1$ for almost every $t \in [1+2r+s, 1+2r+3s]$. Then

$$\mu_3^\varepsilon(t) = -s(1+r) + (t-1-2r-s) \qquad \text{for every } t \in [1+2r+s, 1+2r+3s].$$

By (2.37) we have that $\dot\mu_4^\varepsilon(t) = \big((t-1-2r-s) - s(1+r)\big)^\gamma$ for almost every $t \in [1+2r+s, 1+2r+3s]$. Then for every $t \in [1+2r+s, 1+2r+3s]$

$$\begin{aligned}\mu_4^\varepsilon(t) =& \lambda(\varepsilon) + \frac{(-1)^{\gamma+1}}{\gamma+1}\,(1+r)^\gamma s^{\gamma+1} \\ &+ \frac{1}{(\gamma+1)}\Big(\big(t-1-2r-s-s(1+r)\big)^{\gamma+1} - (-s(1+r))^{\gamma+1}\Big).\end{aligned}$$

◇ Case $t \in [1+2r+3s, 1+3r+3s]$.
By (2.36) and (2.37) we deduce that for every $t \in [1+2r+3s, 1+3r+3s]$

$$\begin{aligned}\mu_3^\varepsilon(t) &= \mu_3^\varepsilon(1+2r+3s) = s\,(1-r),\\ \mu_4^\varepsilon(t) &= \mu_4(1+2r+3s) = \lambda(\varepsilon) + \frac{(-1)^{\gamma+1}}{\gamma+1}\,(1+r)^\gamma s^{\gamma+1} r \\ &\quad + \frac{1}{(\gamma+1)}\,s^{\gamma+1}(1-r)^{\gamma+1}.\end{aligned}$$

◇ Case $t \in [1+3r+3s, 1+3r+4s]$.
By (2.36) we have that $\dot\mu_3^\varepsilon(t) = (r-1)$ for almost every $t \in [1+3r+3s, 1+3r+4s]$. Then

$$\mu_3^\varepsilon(t) = (1-r)(-t+1+3r+4s) \qquad \text{for every } t \in [1+3r+3s, 1+3r+4s].$$

By (2.37) we have that $\dot\mu_4^\varepsilon(t) = -(1-r)^\gamma(-t+1+3r+4s)^\gamma$ for almost every $t \in [1+3r+3s, 1+3r+4s]$. Thus for every $t \in [1+3r+3s, 1+3r+4s]$

$$\begin{aligned}\mu_4^\varepsilon(t) =& \lambda(\varepsilon) + \frac{(-1)^{\gamma+1}}{\gamma+1}\,(1+r)^\gamma s^{\gamma+1} r \\ &+ \frac{1}{(\gamma+1)}\,s^{\gamma+1}(1-r)^{\gamma+1} + \frac{(1-r)^\gamma}{\gamma+1}\,\big((-t+1+3r+4s)^{\gamma+1} - s^{\gamma+1}\big).\end{aligned}$$

◇ Case $t \in [1+3r+4s, 1+4r+4s]$.
By (2.36) and (2.37) we deduce that for every $t \in [1+3r+4s, 1+4r+4s]$

$$\begin{aligned}\mu_3^\varepsilon(t) =& \mu_3^\varepsilon(1+3r+4s) = 0,\\ \mu_4^\varepsilon(t) =& \lambda(\varepsilon) + \frac{(-1)^{\gamma+1}}{\gamma+1}\,(1+r)^\gamma s^{\gamma+1} r + \frac{1}{(\gamma+1)}\,s^{\gamma+1}(1-r)^{\gamma+1} \\ &- \frac{(1-r)^\gamma}{\gamma+1}\,s^{\gamma+1}.\end{aligned}$$

Therefore, we conclude that

$$\mu_3^\varepsilon(1+4r+4s) = 0,$$
$$\mu_4^\varepsilon(1+4r+4s) = \lambda(\varepsilon) - \frac{(-1)^{\gamma+1}}{(\gamma+1)}\, r\, s^{\gamma+1}\big((r+1)^\gamma - (r-1)^\gamma\big).$$

Note that $\mu_4^\varepsilon(1+4r+4s) = 0$ if and only if

$$\lambda(\varepsilon) = \frac{(-1)^{\gamma+1}}{(\gamma+1)}\, r\, s^{\gamma+1}\big((r+1)^\gamma - (r-1)^\gamma\big). \tag{2.38}$$

Moreover, the length of μ^ε is

$$L(\mu^\varepsilon) = 2(1-\varepsilon) + \varepsilon\sqrt{2} + 2c + 4r + 4s.$$

2.4.4 Final considerations

Now let us fix $r = s$ in the definition of μ^ε. Recall that $c < \frac{a-b}{2}$ and $c = \frac{\varepsilon^2}{2(a-b)}$, thus $\varepsilon < a-b$. Hence - given that also $\varepsilon < b$ - we shall take $0 < \varepsilon < \bar{\varepsilon} := \min\{b, a-b\}$.

Let us define

$$\Psi(\varepsilon) := \frac{\varepsilon^{2\gamma+1}}{2^\gamma(2\gamma+1)}\big(1 + \varepsilon\, k\big) \qquad \text{for every } \varepsilon \in (0, \bar{\varepsilon}), \tag{2.39}$$

where $k := \frac{2\gamma+1}{2b(\gamma+1)}\frac{a^\gamma - (a-b)^\gamma}{(a-b)^\gamma} > 0$, and

$$\Phi(r) := \frac{r^{\gamma+2}}{(\gamma+1)}\big((r+1)^\gamma - (r-1)^\gamma\big) \qquad \text{for every } r > 0. \tag{2.40}$$

Observe that

$$\Phi(r) = \frac{r^{\gamma+2}}{(\gamma+1)}\sum_{i=0}^{\gamma}\binom{\gamma}{i} r^i\big(1 - (-1)^{\gamma-i}\big) \qquad \text{for every } r > 0. \tag{2.41}$$

We deduce from (2.38) that, given $\varepsilon \in (0, \bar{\varepsilon})$, one has that

$$\mu_4^\varepsilon\big(1 + 8r\big) = 0 \qquad \text{if and only if} \qquad \Psi(\varepsilon) = \Phi(r).$$

Hence our task is to find, for $\varepsilon \in (0, \bar{\varepsilon})$ sufficiently small, a suitable $r > 0$ (if it exists) such that both $\Psi(\varepsilon) = \Phi(r)$ and $L(\mu^\varepsilon) < L(\nu)$.
Note that $\Psi(\varepsilon) > 0$ for every $\varepsilon \in (0, \bar{\varepsilon})$ and that $\Phi :\ (0, +\infty) \to (0, +\infty)$ is a polynomial function with positive coefficients such that $\Phi(0) = 0$, in particular Φ is strictly increasing and continuous. Given that $\lim_{r\to+\infty}\Phi(r) = +\infty$ and $\lim_{r\to 0^+}\Phi(r) = 0$, for every $\varepsilon \in (0, \bar{\varepsilon})$ there exists a unique $r(\varepsilon) > 0$ such that

$$\Psi(\varepsilon) = \Phi\big(r(\varepsilon)\big). \tag{2.42}$$

By (2.42) we deduce that $\lim_{\varepsilon\to 0^+}\Phi\big(r(\varepsilon)\big)=\lim_{\varepsilon\to 0^+}\Psi(\varepsilon)=0$, hence necessarily

$$\lim_{\varepsilon\to 0^+} r(\varepsilon)=0. \tag{2.43}$$

We now study what happens to the length of μ^ε when we choose $r=r(\varepsilon)$. We distinguish three cases:

⋄ Case $\gamma=2$.
We have that

$$\Phi(r)=\frac{4}{3}\,r^5,$$
$$\Psi(\varepsilon)=\frac{\varepsilon^5}{20}\,(1+\varepsilon\,k).$$

In this case, we can explicitly compute the function $r:\,(0,\bar\varepsilon)\to(0,+\infty)$. Namely

$$r(\varepsilon)=\sqrt[5]{\frac{3(1+\varepsilon\,k)}{80}}\,\varepsilon.$$

Note that μ^ε is strictly shorter than ν if and only if

$$L(\nu)-L(\mu^\varepsilon)=\big(2-\sqrt{2}\,\big)\varepsilon-\frac{\varepsilon^2}{a-b}-8\,\sqrt[5]{\frac{3(1+\varepsilon\,k)}{80}}\,\varepsilon>0. \tag{2.44}$$

Since $k>0$, we have that

$$\big(2-\sqrt{2}\,\big)\varepsilon-\frac{\varepsilon^2}{a-b}-8\,\sqrt[5]{\frac{3(1+\varepsilon\,k)}{80}}\,\varepsilon<\left(2-\sqrt{2}-8\,\sqrt[5]{\frac{3}{80}}\right)\varepsilon-\frac{\varepsilon^2}{a-b}.$$

But the right hand side is positive if and only if $\varepsilon<(a-b)\left(2-\sqrt{2}-8\,\sqrt[5]{\frac{3}{80}}\right)<0$, which is impossible because $\varepsilon\in(0,\bar\varepsilon)$. Hence (2.44) is not satisfied in the case $\gamma=2$.

⋄ Case $\gamma\geq 3$ odd.
We have that

$$\Phi(r)=\frac{2}{(\gamma+1)}\sum_{i=0}^{\frac{(\gamma-1)}{2}}\binom{\gamma}{2i}\,r^{\gamma+2+2i}=\frac{2\,r^{\gamma+2}}{\gamma+1}\big(1+r^2\,p(r)\big)$$

for a suitable polynomial $p\in\mathbb{N}[r]$. Equation (2.42) yields

$$r(\varepsilon)=g_\gamma(\varepsilon)\,C_\gamma\,\varepsilon^{\frac{2\gamma+1}{\gamma+2}}, \tag{2.45}$$

where

$$C_\gamma:=\sqrt[\gamma+2]{\frac{\gamma+1}{2^{\gamma+1}(2\gamma+1)}}$$

and

$$g_\gamma(\varepsilon):=\sqrt[\gamma+2]{\frac{1+\varepsilon\,k}{1+r(\varepsilon)^2\,p(r(\varepsilon))}}.$$

In order to prove that ν is not a length minimizer, we have to verify that

$$L(\mu^\varepsilon) < L(\nu),$$

which is equivalent to

$$\big(2-\sqrt{2}\,\big)\varepsilon - \frac{\varepsilon^2}{a-b} - 8\, g_\gamma(\varepsilon)\, C_\gamma\, \varepsilon^{\frac{2\gamma+1}{\gamma+2}} > 0. \tag{2.46}$$

By collecting ε, we have that (2.46) is equivalent to

$$\big(2-\sqrt{2}\,\big) - \frac{\varepsilon}{a-b} - 8\, g_\gamma(\varepsilon)\, C_\gamma\, \varepsilon^{\frac{\gamma-1}{\gamma+2}} > 0. \tag{2.47}$$

Note that, since $\gamma > 1$, one has

$$\frac{\gamma-1}{\gamma+2} > 0,$$

and note that $\lim_{\varepsilon\to 0^+} g_\gamma(\varepsilon) = 1$. Hence the limit of the left hand side of (2.47) as $\varepsilon \to 0^+$ is $2-\sqrt{2} > 0$, proving that there exists $\varepsilon_\gamma \in (0,\bar\varepsilon]$ such that (2.47) is satisfied for every $\varepsilon \in (0,\varepsilon_\gamma)$.

$\diamond$ Case $\gamma \geq 4$ even.
We have that

$$\Phi(r) = \frac{2}{(\gamma+1)} \sum_{i=0}^{\frac{(\gamma-2)}{2}} \binom{\gamma}{2i+1} r^{\gamma+2+(2i+1)} = \frac{2\, r^{\gamma+3}}{\gamma+1}\big(1 + r^2\, q(r)\big)$$

for a suitable polynomial $q \in \mathbb{N}[r]$. Equation (2.42) yields

$$r(\varepsilon) = g_\gamma(\varepsilon)\, C_\gamma\, \varepsilon^{\frac{2\gamma+1}{\gamma+3}}, \tag{2.48}$$

where

$$C_\gamma := \sqrt[\gamma+2]{\frac{\gamma+1}{2^{\gamma+1}\gamma\,(2\gamma+1)}}$$

and

$$g_\gamma(\varepsilon) := \sqrt[\gamma+2]{\frac{1+\varepsilon\, k}{1+r(\varepsilon)^2\, q(r(\varepsilon))}}.$$

In order to prove that ν is not a length minimizer, we have to verify that

$$L(\mu^\varepsilon) < L(\nu),$$

which is equivalent to

$$\big(2-\sqrt{2}\,\big)\varepsilon - \frac{\varepsilon^2}{a-b} - 8\, g_\gamma(\varepsilon)\, C_\gamma\, \varepsilon^{\frac{2\gamma+1}{\gamma+3}} > 0. \tag{2.49}$$

By collecting ε, we have that (2.49) is equivalent to

$$\left(2-\sqrt{2}\right)-\frac{\varepsilon}{a-b}-8\,g_\gamma(\varepsilon)\,C_\gamma\,\varepsilon^{\frac{\gamma-2}{\gamma+3}}>0. \tag{2.50}$$

Note that, since $\gamma>2$, one has

$$\frac{\gamma-2}{\gamma+3}>0,$$

and note that $\lim_{\varepsilon\to0^+} g_\gamma(\varepsilon)=1$. Hence the limit of the left hand side of (2.50) as $\varepsilon\to0^+$ is $2-\sqrt{2}>0$, proving that there exists $\varepsilon_\gamma\in(0,\bar\varepsilon]$ such that (2.50) is satisfied for every $\varepsilon\in(0,\varepsilon_\gamma)$.

We finally summarize what we did in this section: we proved that

Theorem 2.18. *Let $\gamma\in\mathbb{N}^+$ with $\gamma\geq3$. Let $\mathcal{D}$ be the distribution in $\mathbb{R}^4$ generated by*

$$\begin{cases} X_1(x)=(1,0,0,0),\\ X_2(x)=\left(0,1,x_1,x_3^\gamma\right),\end{cases}$$

for every $x\in\mathbb{R}^4$. Let $\nu:[-1,1]\to\mathbb{R}^4$ be the strictly abnormal extremal for $\mathcal{D}$ defined by

$$\nu(t)=\begin{cases}(0,-t,0,0) & \text{if } t\in[-1,0],\\ (t,0,0,0) & \text{if } t\in[0,1],\end{cases} \tag{2.51}$$

so that ν has a corner at 0. Then ν is not length-minimizing.

2.5 Case $(\alpha,\beta,\gamma)=(1,0,2)$

In this section we shall make use of the following definitions:

- Given $\sigma^1\in\mathrm{Lip}\big([a,b],\mathbb{R}^n\big)$ and $\sigma^2\in\mathrm{Lip}\big([c,d],\mathbb{R}^n\big)$ such that $\sigma^1(b)=\sigma^2(c)$, the *concatenation* $\sigma^1*\sigma^2\in\mathrm{Lip}\big([a,b+d-c],\mathbb{R}^n\big)$ between σ^1 and σ^2 is defined by

  $$(\sigma^1*\sigma^2)(t):=\begin{cases}\sigma^1(t) & \text{if } t\in[a,b],\\ \sigma^2(t-b+c) & \text{if } t\in[b,b+d-c].\end{cases}$$

  If $\mathcal{D}$ is a distribution on $\mathbb{R}^n$, generated by a global frame of orthonormal vector fields, and σ^1, σ^2 are $\mathcal{D}$-horizontal, then also $\sigma^1*\sigma^2$ is $\mathcal{D}$-horizontal and satisfies $L(\sigma^1*\sigma^2)=L(\sigma^1)+L(\sigma^2)$.
  Since $*$ is associative, for any Lipschitz curves σ^1,σ^2 and σ^3, we will write $\sigma^1*\sigma^2*\sigma^3$ instead of $(\sigma^1*\sigma^2)*\sigma^3$.

- Given $\sigma\in\mathrm{Lip}\big([a,b],\mathbb{R}^n\big)$, we will denote by

  $$\mathcal{E}(\sigma):=\sigma(b) \tag{2.52}$$

  its endpoint.

- With abuse of notation, for every $x, y \in \mathbb{R}^n$ we will denote by $[x, y]$ the following parametrization of the segment joining x to y:

$$[x, y](t) := (1-t)x + ty \quad \text{for every } t \in [0, 1].$$

In Section 2.4 we showed that the $\mathcal{D}$-horizontal angle ν (where ν is defined in (2.19) and $\mathcal{D}$ is defined in Theorem 2.18) is not a length-minimizer for $\gamma \geq 3$. However, the same technique doesn't show that ν is not a length-minimizer in the case $\gamma = 2$, since there was a problem in the balance of length (see Subsection 2.4.4). The aim of this section is to provide an example of a curve η^ε, which shows that ν is not a length-minimizer also in the case $\gamma = 2$.
In order to do this, let us fix $\varepsilon < \frac{1}{4}$. We want to concatenate the following planar curves:

- We follow the "cut" ν^ε (defined in Subsection 2.4.1) from $(0, 1)$ to $\left(\frac{1}{4}, 0\right)$, i.e. we consider

$$\nu^\varepsilon|_{\left[-1, \frac{1}{4}\right]}.$$

- The rectangle R^1 joining the following points:

$$\left(\frac{1}{4}, 0\right) \to \left(\frac{1}{4}, -2\varepsilon^2\right) \to \left(\frac{1}{2}, -2\varepsilon^2\right) \to \left(\frac{1}{2}, 0\right).$$

- The square Q^1 (of suitable side $s > 0$) joining the following points:

$$\left(\frac{1}{2}, 0\right) \to \left(\frac{1}{2}, s\right) \to \left(\frac{1}{2} + s, s\right) \to \left(\frac{1}{2} + s, 0\right) \to \left(\frac{1}{2}, 0\right).$$

- The rectangle R^2 joining the following points:

$$\left(\frac{1}{2}, 0\right) \to \left(\frac{1}{2}, -4s^2\right) \to \left(\frac{3}{4}, -4s^2\right) \to \left(\frac{3}{4}, 0\right).$$

- The segment $S := \left[\left(\frac{3}{4}, 0\right), (1, 0)\right]$.

- The curve Q^2 (of suitable parameter $r > 0$) joining the following points:

$$\begin{aligned} &(1, 0) \to (1+r, 0) \to (1+r, -r) \to \\ &(1, -r) \to (1, r) \to (1-r, r) \to (1-r, 0) \to (1, 0). \end{aligned}$$

We parametrize the above-mentioned curves as follows:

$$R^1(t) := \begin{cases} \left(\frac{1}{4}, -t\right) & \text{if } t \in \left[0, 2\varepsilon^2\right], \\ \left(t - 2\varepsilon^2 + \frac{1}{4}, -2\varepsilon^2\right) & \text{if } t \in \left[2\varepsilon^2, 2\varepsilon^2 + \frac{1}{4}\right], \\ \left(\frac{1}{2}, t - 4\varepsilon^2 - \frac{1}{4}\right) & \text{if } t \in \left[2\varepsilon^2 + \frac{1}{4}, 4\varepsilon^2 + \frac{1}{4}\right]. \end{cases}$$

$$Q^1(t) := \begin{cases} \left(\frac{1}{2}, t\right) & \text{if } t \in [0, s], \\ \left(t + \frac{1}{2} - s, s\right) & \text{if } t \in [s, 2s], \\ \left(\frac{1}{2} + s, -t + 3s\right) & \text{if } t \in [2s, 3s], \\ \left(-t + 4s + \frac{1}{2}, 0\right) & \text{if } t \in [3s, 4s]. \end{cases}$$

$$R^2(t) := \begin{cases} \left(\frac{1}{2}, -t\right) & \text{if } t \in \left[0, 4s^2\right], \\ \left(t - 4s^2 + \frac{1}{2}, -4s^2\right) & \text{if } t \in \left[4s^2, 4s^2 + \frac{1}{4}\right], \\ \left(\frac{3}{4}, t - 8s^2 - \frac{1}{4}\right) & \text{if } t \in \left[4s^2 + \frac{1}{4}, 8s^2 + \frac{1}{4}\right]. \end{cases}$$

$$Q^2(t) := \begin{cases} (t+1, 0) & \text{if } t \in [0, r], \\ (r+1, r-t) & \text{if } t \in [r, 2\,r], \\ (-t + 3\,r + 1, -r) & \text{if } t \in [2\,r, 3\,r], \\ (1, t - 4\,r) & \text{if } t \in [3\,r, 5\,r], \\ (-t + 5\,r + 1, r) & \text{if } t \in [5\,r, 6\,r], \\ (1 - r, -t + 7\,r) & \text{if } t \in [6\,r, 7\,r], \\ (t + 1 - 8\,r, 0) & \text{if } t \in [7\,r, 8\,r]. \end{cases}$$

We want to choose $s, r > 0$ in such a way that, having called η^ε the lift of the concatenated curve

$$\nu^\varepsilon|_{[-1,\frac{1}{4}]} * R^1 * Q^1 * R^2 * S * Q^2,$$

one has that $\mathcal{E}(\eta^\varepsilon) = (1, 0, 0, 0)$. We proceed in the following way:

$\diamond$ Step 1: Let us call $\pi^1 := \nu^\varepsilon|_{[-1,\frac{1}{4}]} * R^1$ and $\bar{\pi}^1$ its lift to $\mathbb{R}^4$. Then

$$\mathcal{E}(\bar{\pi}^1) = \left(\frac{1}{2}, 0, 0, -\frac{\varepsilon^5}{20} - \frac{\varepsilon^6}{2}\right).$$

$\diamond$ Step 2: Let us call $\pi^2 := \pi^1 * Q^1$ and $\bar{\pi}^2$ its lift to $\mathbb{R}^4$. We want to find $s > 0$ in such a way that the fourth component of

$$\mathcal{E}(\bar{\pi}^2) = \left(\frac{1}{2}, 0, -s^2, -\frac{\varepsilon^5}{20} - \frac{\varepsilon^6}{2} + \frac{1-2s}{6}s^4\right)$$

is equal to 0. Note that $\mathcal{E}(\bar{\pi}^2)_4 = 0$ if and only if

$$f(s) := -3\,\varepsilon^5 - 30\,\varepsilon^6 + 10\,s^4 - 20\,s^5 = 0. \tag{2.53}$$

By differentiating f with respect to s, we obtain that

$$\frac{d}{ds}\,f(s) = 40\,s^3 - 100\,s^4 > 0$$

when $s \in \left(0, \frac{2}{5}\right)$, hence f is strictly increasing in $\left[0, \frac{2}{5}\right]$.
Note that $f(0) = -(3\,\varepsilon^5 + 30\,\varepsilon^6) < 0$. Moreover, if $\varepsilon < \frac{2}{5}\sqrt[5]{\frac{5}{33}}$ then

$$f\left(\frac{2}{5}\right) = -3\,\varepsilon^5 - 30\,\varepsilon^6 + \frac{2^5}{5^4} > 0.$$

Indeed, since $\varepsilon < \frac{2}{5}\sqrt[5]{\frac{5}{33}} < 1$, we have that

$$3\,\varepsilon^5 + 30\,\varepsilon^6 = 3\,\varepsilon^5(1 + 10\,\varepsilon) < 33\,\varepsilon^5 < 33\left(\frac{2}{5}\sqrt[5]{\frac{5}{33}}\right) = \frac{2^5}{5^4}.$$

Therefore, since $\frac{1}{4} < \frac{2}{5}\sqrt[5]{\frac{5}{33}}$, for every $\varepsilon \in \left(0, \frac{1}{4}\right)$ there exists an unique positive solution $s(\varepsilon) \in \left(0, \frac{2}{5}\right)$ of (2.53). Now we want to estimate $s(\varepsilon)$: note that $f\big(s(\varepsilon)\big) = 0$ if and only if

$$s(\varepsilon)^4 = \frac{3\varepsilon^5}{10}\,\frac{1 + 10\varepsilon}{1 - 2s(\varepsilon)}.$$

Since $\varepsilon \in \left(0, \frac{1}{4}\right)$ and $s(\varepsilon) \in \left(0, \frac{2}{5}\right)$, we deduce that

$$\frac{1 + 10\varepsilon}{1 - 2s(\varepsilon)} < \frac{1 + 10\frac{1}{4}}{1 - 2\frac{2}{5}} = \frac{35}{2},$$

hence $s(\varepsilon)^4 < \frac{21}{4}\varepsilon^5$. Thus

$$0 < s(\varepsilon) < \sqrt[4]{\frac{21}{4}}\,\varepsilon^{\frac{5}{4}}. \tag{2.54}$$

⋄ Step 3: Let us call $\pi^3 := \pi^2 * R^2$ and $\bar{\pi}^3$ its lift to $\mathbb{R}^4$ (having chosen $s = s(\varepsilon)$ in the definition of Q^1). Then

$$\mathcal{E}(\bar{\pi}^3) = \left(\frac{3}{4}, 0, 0, -\frac{16}{3}\,s(\varepsilon)^6\right).$$

⋄ Step 4: Let us call $\pi^4 := \pi^3 * S * Q^2$ and $\bar{\pi}^4$ its lift to $\mathbb{R}^4$. Then

$$\mathcal{E}(\bar{\pi}^4) = \left(1, 0, 0, -\frac{16}{3}\,s(\varepsilon)^6 + \frac{4}{3}\,r^5\right).$$

Now let us choose $r = r(\varepsilon) := \sqrt[5]{4}\,s(\varepsilon)^{\frac{6}{5}}$. Therefore $\mathcal{E}(\bar{\pi}^4)_4 = 0$.

Now let us call $\eta^\varepsilon := \bar{\pi}^4$ with the choice $s = s(\varepsilon)$ and $r = r(\varepsilon)$. We have that this curve joins $e_2 = (0,1,0,0)$ to $e_1 = (1,0,0,0)$. Thus it is a competitor for $d(e_2, e_1)$. It only remains to show that, for a suitable $\varepsilon \in \left(0, \frac{1}{4}\right)$, one has that $L(\eta^\varepsilon) < L(\nu) = 2$. Notice that $L(\eta^\varepsilon) < 2$ if and only if

$$(2-\sqrt{2})\varepsilon - 4\varepsilon^2 - 4\,s(\varepsilon) - 8\,s(\varepsilon)^2 - 8\,r(\varepsilon) > 0. \tag{2.55}$$

From (2.54) we deduce that

$$(2-\sqrt{2})\varepsilon - 4\varepsilon^2 - 4\,s(\varepsilon) - 8\,s(\varepsilon)^2 - 8\,r(\varepsilon) > g(\varepsilon),$$

where

$$g(\varepsilon) := (2-\sqrt{2})\,\varepsilon - 4\varepsilon^2 - 4\sqrt[4]{\frac{21}{4}}\,\varepsilon^{\frac{5}{4}} - 8\sqrt[2]{\frac{21}{4}}\,\varepsilon^{\frac{5}{2}} - 8\sqrt[5]{4}\left(\frac{21}{4}\right)^{\frac{3}{10}}\varepsilon^{\frac{3}{2}}.$$

Note that there exists $\bar{\varepsilon} \in \left(0, \frac{1}{4}\right)$ such that $g(\varepsilon) > 0$ for every $\varepsilon \in (0, \bar{\varepsilon})$. This shows that ν is not a length-minimizer.

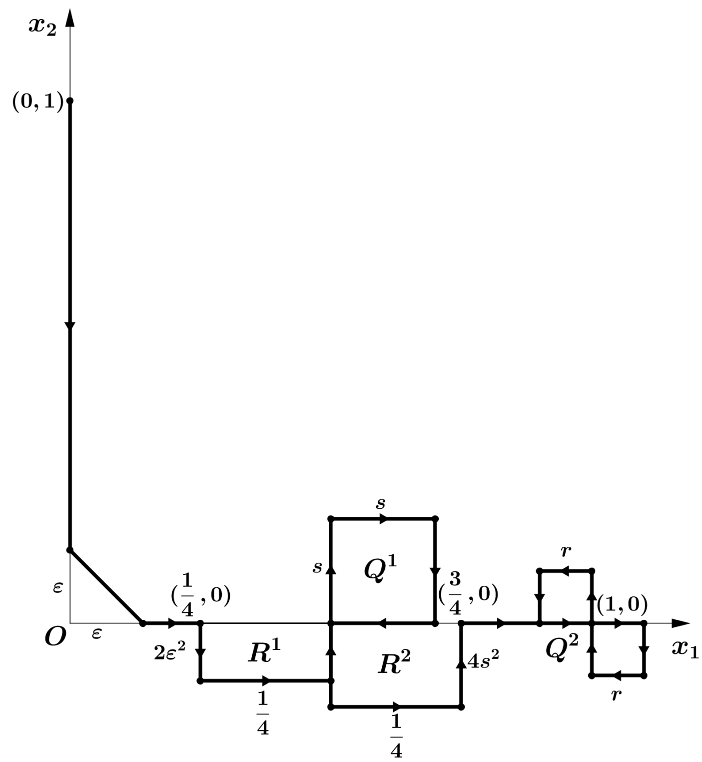

Figure 2.5: The curve $\left(\eta_1^\varepsilon, \eta_2^\varepsilon\right)$.

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