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MASTER THESIS

Singular extremals of constrained variational problems

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Introduction

As an introductory example let us consider the following minimum problem: Let $I = (a, b)$, $a, b \in \mathbb{R}$ and

$$\min\{\, \mathcal{F}(u) \,|\, u \in H^1(I);\ u(a) = 0,\ u(b) = \beta \text{ and } \mathcal{G}(u) = 1\} \tag{1}$$

where $H^1(I)$ denotes the Sobolev-space consisting of L^2 functions with weak derivatives in L^2 and

$$\mathcal{F}(u) := \frac{1}{2}\int_I u'(x)^2 dx \qquad \text{and} \qquad \mathcal{G}(u) := \frac{1}{2}\int_I u(x)^2 dx.$$

We assume that there exists a minimizer of (1). In fact, the existence follows by the theory which we will develop in Chapter 1.

We say that u is a regular extremal if the first variation of $\mathcal{G}$ does not vanish for all $\psi \in C_c^\infty(I)$, meaning there exists some $\psi \in C_c^\infty(I)$ such that

$$\int_I u(x)\psi(x)dx \neq 0.$$

In this case, by the Lagrange multiplier Theorem (this will be introduced in Chapter 2, Thm 5) there exists some $\lambda \in \mathbb{R}$ such that

$$\frac{d}{d\epsilon}\left(\mathcal{F}(u + \epsilon\varphi) + \lambda\mathcal{G}(u + \epsilon\varphi)\right)\big|_{\epsilon=0} = 0.$$

Together with the boundary conditions we get the following ordinary differential equation

$$\begin{cases} u''(x) = \lambda u(x) \\ u(a) = 0 \text{ and } u(b) = \beta. \end{cases} \tag{2}$$

For a given λ the solution of this problem is unique and is given by

If $\lambda > 0:$ $u(x) = \gamma_1 e^{\sqrt{\lambda}x} + \gamma_2 e^{\sqrt{\lambda}x}$ where γ_1 and γ_2 are constants determined by the boundary conditions.

If $\lambda < 0:$ $u(x) = \gamma_1 \sin(\sqrt{-\lambda}x) + \gamma_2 \cos(\sqrt{-\lambda}x)$ where γ_1 and γ_2 are constants determined by the boundary conditions.

If $\lambda = 0:$ $u(x) = \frac{\beta}{b-a}(x - a).$

So for all possible λ the solution of (2) is smooth. We will actually see that a regular minimizer of a general minimization problem is always smooth.

But what happens if u is not regular (meaning singular)? In this case we can not apply the Lagrange multiplier theorem and it is not clear how to deduce an Euler Lagrange equation. Is a singular minimizer also smooth? Is it of class C^1? In our case it is easy to see that a singular minimizer is smooth, since the singularity of u together with $\mathcal{G}(u) = 0$ and $\beta = 0$ implies that $u(x) \equiv 0$ on I and $u \equiv 0$ also minimizes $\mathcal{F}$. Can we find examples of singular minimizers that are not smooth?

In this thesis we will study the regularity of minimizers for constrained problems. In the regular case we will prove that a minimizer inherits the regularity of the Lagrangian of $\mathcal{F}$ and $\mathcal{G}$. This will be the first part. We will then focus on singular minimizers in Chapter 3 and Chapter 4

Chapter 1

Existence of constrained minimizers

We are interested in the following minimization problem: Let $I = (a, b) \subseteq \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$ and

$$\min \{\mathcal{F}(u) \,|\, u \in AC(I),\, u(a) = \alpha,\, u(b) = \beta \quad \text{and} \quad \mathcal{G}(u) = 0\}\,, \tag{1.1}$$

where $AC(I)$ denotes the space of absolutely continuous functions which we will define rigorously in the next section. We fix the following functional setting:

Let $I = (a, b)$ be a bounded interval and $n \geq 1$. Let $G \in C^1(I\times\mathbb{R};\mathbb{R}^n)$ and $F \in C^1(I\times\mathbb{R}\times\mathbb{R};\mathbb{R})$. We define the functionals

$$\mathcal{F} : AC(I) \to \mathbb{R} \quad \text{and} \quad \mathcal{G} : AC(I) \to \mathbb{R}^n$$

as

$$\mathcal{F}(u) := \int_a^b F(x, u(x), u'(x))dx \quad \text{and} \quad \mathcal{G}(u) := \int_a^b G(x, u(x))dx,$$

respectively. The aim of this chapter is to prove existence for the constrained variational problem (1.1). More precisely, we will prove the following theorem:

Theorem 1 (Tonneli's existence theorem)**.** *Suppose $F \in C^1(I \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$ is such that:*

1. *$F(x, z, \xi)$ is convex in ξ, meaning that the map $\xi \mapsto F(x, z, \xi)$ is convex for all fixed $(x, z) \in \overline{I} \times \mathbb{R}$;*

2. *$F(x, z, \xi)$ has quadratic growth: there exist positive constants c_0, c_1 such that for all $\xi \in \mathbb{R}$*
$$c_0|\xi|^2 \leq F(x, z, \xi) \leq c_1(1 + |\xi|^2) \qquad \textit{for all } (x, z) \in \overline{I} \times \mathbb{R} \textit{ fixed}; \tag{1.2}$$

3. *There exists $u \in \mathcal{C}(\alpha, \beta)$ with $\mathcal{G}(u) = 0$.*

Then there exists a minimizer of $\mathcal{F}$ under the constraint $\mathcal{G}(u) = 0$ in the class

$$\mathcal{C}(\alpha, \beta) := \left\{ u \in AC(I) \,|\, u(a) = \alpha,\, u(b) = \beta \textit{ and } u' \in L^2(I)\right\},$$

where α, $\beta \in \mathbb{R}$ are fixed.

Tonelli's Theorem is presented in [1, Chapter 1-4] but without the constraint $\mathcal{G}(u) = 0$. We shall therefore state and prove the theory adapted to our setting.

Remark 1. We choose $p = 2$ since $H^1(I)$ is a Hilbert space and so the characterization of weak convergence is slightly easier.

1.1 Background

In this thesis we will work with absolutely continuous functions, therefore we give a short overview about the necessary background.

Definition 1 (Absolutely continuous functions)**.** A function $u : (a, b) \to \mathbb{R}$ is said to be *absolutely continuous* if for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sum_{i=1}^{N} (\beta_i - \alpha_i) < \delta \quad \text{implies} \quad \sum_{i=1}^{N} |u(\beta_i) - u(\alpha_i)| < \epsilon$$

whenever $(\alpha_1, \beta_1), ..., (\alpha_N, \beta_N)$ are disjoint line segments in (a, b). The class of absolutely continuous functions is denoted by $AC(a, b)$.

On the real line we have the following characterisation:

Theorem 2. *For all $I \subseteq \mathbb{R}$ we have*

$$AC(I) = H^{1,1}(I)$$

where $H^{1,1}(I)$ denotes the Sobolev space whose elements are the L^1 functions with weak derivative in L^1.

More precisely, every $u \in AC(I)$ has an almost everywhere classical derivative u' which belongs to $L^1(I)$ and viewed as an element of L^1, u' is the weak derivative of u. Conversely, every $u \in H^{1,1}(I)$ is an absolutely continuous function, modulo a modification on a set of measure zero.

Finally, $u \in AC(I)$ if and only if u is almost everywhere differentiable in a classical sense, u' belongs to $L^1(I)$ and the fundamental theorem of calculus holds true, i.e.: for all x, $y \in I$ we have

$$u(x) - u(y) = \int_y^x u'(s)ds.$$

A proof is given in [1, Thm 2.17].

Remark 2 (Uniform continuity). For an absolutely continuous function $u \in AC(I)$ and for all x, $y \in I$ we have by Hölder's inequality that

$$|u(x) - u(y)| \leq \int_y^x |u'(s)|\, ds \leq \|u'\|_{L^2(I)} |x - y|^{1/2},$$

meaning that u is $\frac{1}{2}$-Hölder equicontinuous so in particular uniformly continuous.

Remark 3. Condition (1.2) assures that the functional $\mathcal{F}$ is well-defined for $u \in \mathcal{C}(\alpha, \beta)$. In fact, since u is absolutely continuous u' exists almost everywhere in I and by definition of $\mathcal{C}(\alpha, \beta)$ it belongs to L^2. Since $\xi \mapsto F(x, z, \xi)$ is of class C^1 the composition $F(x, u(x), u'(x))$ is measurable. The upper and lower bounds in (1.2) then guarantee integrability, since I is bounded. Moreover, $x \mapsto G(x, u(x))$ is continuous and so $\mathcal{G}(u) = 0$ is well-defined.

Remark 4. Since $H^{1,1}(I)$ is not reflexive also $AC(I)$ is not reflexive. We shall therefore work in the Sobolev space $H^1(I) := H^{1,2}(I)$ containing of functions $u \in L^2(I)$ with weak derivative $u' \in L^2(I)$, which is a Hilbert space. Let us also notice that we have $H^1(I) = \left\{ u \in AC(I) \,|\, u' \in L^2(I) \right\}$.

1.2 Tonneli's semicontinuity theorem

To prove Theorem 1 we need a result from the direct methods of Calculus of Variation, called Tonelli's semicontinuity theorem, which provides lower semicontinuity of $\mathcal{F}(u)$ under the assumption that F is convex in ξ.

Definition 2 (Weak convergence in $H^1(I)$)**.** Let $(u_k)_{k\in\mathbb{N}} \subseteq H^1(I)$. We say that u_k *converges weakly* to u in $H^1(I)$ if u_k and u_k' converge weakly to u and u' respectively, where weak convergence in L^2 is characterized as follows:

u_k converges weakly to u in $L^2(I)$ if for all $\psi \in L^2(I)$ we have

$$\lim_{k\to\infty} \int_I u_k(x)\psi(x)dx = \int_I u(x)\psi(x)dx.$$

If u_k converges weakly to u in $H^1(I)$ we write $u_k \rightharpoonup u$.

Definition 3 (Sequential lower semicontinuity)**.** We say that the functional $\mathcal{F}$ is (weakly) *lower sequentially semicontinuous* (weakly-lsc), if

$$\mathcal{F}(u) \leq \liminf_{k\to\infty} \mathcal{F}(u_k)$$

for all sequences $(u_k)_{k\in\mathbb{N}} \subseteq H^1(I)$ converging weakly to $u \in H^1(I)$.

Theorem 3 (Tonelli's semicontinuity theorem)**.** *Let $F \in C^1(I \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$ be such that:*

1. *either $F(x, z, \xi) \geq 0$ or there exists some $f \in L^1(I)$ such that $F(x, z, \xi) \geq f(x)$ for all $(x, z, \xi) \in I \times \mathbb{R} \times \mathbb{R}$;,*
2. *$F(x, z, \xi)$ is convex in $\xi \in \mathbb{R}$ for all $(x, z) \in I \times \mathbb{R}$.*

Then $\mathcal{F}$ is weakly-lsc in $H^1(I)$, meaning if $(u_k)_{k\in\mathbb{N}} \subseteq H^1(I)$ converges weakly to some $u \in H^1(I)$ then we have

$$\mathcal{F}(u) \leq \liminf_{k\to\infty} \mathcal{F}(u_k).$$

The proof relies on these two standart results from measure theory, which we state for the sake of completeness.

Theorem (Egorov)**.** *Let $f_k : I \to \mathbb{R}$, $k \in \mathbb{N}$ be a sequence of measurable functions such that*

$$f_k(x) \to f(x) \text{ a.e. in } I \qquad \text{and} \qquad |f(x)| < \infty \text{ for a.e. } x \in I.$$

Then for all $\epsilon > 0$ there exists a compact set $K \subseteq I$ with $|I \setminus K| < \epsilon$ and f_k converges uniformly to f on K.

Proof. See [3, Thm. 2.33] □

Theorem (Lusin)**.** *Let $f : [a, b] \to \mathbb{R}$ be measurable. Then for all $\epsilon > 0$ there exists a compact set $K \subseteq I$ such that $|I \setminus K| < \epsilon$ and $f : K \to \mathbb{R}$ is continuous.*

Proof. See [3, Thm. 7.10] □

Proof of Theorem 1.2. Let $(u_k)_{k\in\mathbb{N}} \in H^1(I)$ be such that u_k converges weakly to $u \in H^1(I)$. This implies, after possibly passing to a subsequence, that u_k converges strongly to u in $L^2(I)$ by the Arzelà-Ascoli compactness Theorem. So we can assume, again after possibly passing to a subsequence, that $u_k(x)$ converges to $u(x)$ for a.e. $x \in I$. Therefore by Egorov's theorem there exists some $K \subseteq I$ compact such that

$$u_k(x) \to u(x) \qquad \text{uniformly on } K. \tag{1.3}$$

Moreover by Lusin's theorem, since u and u' are measurable, we can also assume that $u : K \to \mathbb{R}$ and $u' : K \to \mathbb{R}$ are continuous.

Now F is of class C^1 and since $u \in H^1(I)$ the composition $F(x, u, u')$ is in $L^1(I)$. Therefore by Lebesgue's absolute continuity theorem (see [3, Cor. 3.6] for a proof) we have

$$\begin{aligned}\int_I F(x,u,u')dx &= \int_K F(x,u,u')dx + \int_{I\setminus K} F(u,x,u')dx \\ &< \int_K F(x,u,u')dx + \epsilon.\end{aligned}$$

Since F is convex in ξ and F is C^1 we have

$$F(x,z,\xi_1) \geq F(x,z,\xi_2) + \frac{\partial F}{\partial \xi}(x,z,\xi_2)(\xi_1 - \xi_2) \qquad \text{for all} \xi_1, \xi_2 \in \mathbb{R}. \tag{1.4}$$

We obtain

$$\begin{aligned}\mathcal{F}(u_k) &\geq \int_K F(x,u_k,u_k')dx \\ &\overset{(1.4)}{\geq} \int_K \frac{\partial F}{\partial \xi}(x,u_k,u')\,(u_k'-u')\,dx + \int_K F(x,u_k,u')dx \\ &= \int_K F(x,u_k,u')dx + \int_K \frac{\partial F}{\partial \xi}(x,u,u')\,(u_k'-u')\,dx \\ &+ \int_K \left(\frac{\partial F}{\partial \xi}(x,u_k,u') - \frac{\partial F}{\partial \xi}(x,u,u')\right)(u_k'-u')\,dx.\end{aligned}$$

Now since K is compact, u, u' are bounded in K. Since $\partial_\xi F$ is continuous we have that $\partial_\xi F(x,u,u')$ is bounded in K as well. In particular $\partial_\xi F(\cdot, u(\cdot), u'(\cdot)) \in L^2(I)$ and therefore

$$\int_K \frac{\partial F}{\partial \xi}(x,u,u')\,(u_k'-u')\,dx \to 0, \qquad \text{for } k \to \infty,$$

since u_k' converges weakly to u.

Also $(\partial_\xi F(x,u_k,u') - \partial_\xi F(x,u,u')) \in L^2(K)$ and $(u_k' - u') \in L^2(K)$ so by Hölder we have

$$\begin{aligned}\int_K \left(\frac{\partial F}{\partial \xi}(x,u_k,u') - \frac{\partial F}{\partial \xi}(x,u,u')\right)(u_k'-u')\,dx &\leq \|\partial_\xi F(x,u_k,u') - \partial_\xi F(x,u,u')\|_{L^2(K)} \underbrace{\|u_k'-u'\|_{L^2(K)}}_{\leq C \text{ for some } C>0} \\ &\leq C\,\|\partial_\xi F(x,u_k,u') - \partial_\xi F(x,u,u')\|_{L^2(K)}.\end{aligned}$$

As mentioned in Remark 2, u_k and u are uniformly continuous. Since $\partial_\xi F$ is continuous and K is compact also $\partial_\xi F$ is uniformly continuous. So for all $\epsilon > 0$ there exists some $\delta = \delta(\epsilon) > 0$ such that for all $k \in \mathbb{N}$

$$|\partial_\xi F((x, u_k, u') - \partial_\xi F(y, u_k, u')| \, , |\partial_\xi F(x, u, u') - \partial F(y, u, u')| < \frac{\epsilon}{2}$$

whenever $|x - y| < \delta$ so for all $k \in \mathbb{N}$ we have

$$|\partial_\xi F((x, u_k, u') - \partial_\xi F(x, u, u') - \partial_\xi F(y, u_k, u') + \partial F(y, u, u')| < \epsilon$$

whenever $|x - y < \delta$ meaning that $(\partial_\xi F(x, u_k, u') - \partial_\xi F(x, u, u')_{k\in\mathbb{N}}$ is equicontinuous. It is also bounded on K and by Arzelà-Ascoli we can pass to a subsequence that converges uniformly to 0 on K.

Therefore we have

$$\int_K \left(\frac{\partial F}{\partial \xi}(x, u_k, u') - \frac{\partial F}{\partial \xi}(x, u, u') \right) (u_k' - u') \, dx \to 0 \qquad \text{for } k \to \infty.$$

Finally we can conclude that for all $\epsilon > 0$ we have

$$\begin{aligned} \liminf_{k\to\infty} \mathcal{F}(u_k) &\geq \liminf_{k\to\infty} \int_K F(x, u_k, u')dx \\ &\geq \int_K F(x, u, u')dx \geq \int_I F(x, u, u')dx - \epsilon, \end{aligned}$$

where we used the lower bound on F and Fatou's lemma to exchange the limes inferior and integration. Since $\epsilon > 0$ can be chosen as small as we want, the result follows. $\square$

1.3 Existence

We are now ready to prove Theorem 1.

Proof of Thm 1. By the quadratic growth (1.2) the functional $\mathcal{F}$ is bounded from below by 0. Let $(u_k)_{k\in\mathbb{N}}$ be a minimizing sequence of $\mathcal{F}$ in the class $\mathcal{C}(\alpha, \beta)$ with $\mathcal{G}(u_k) = 0$, i.e. $\lim_{k\to\infty} \mathcal{F}(u_k) = \inf_{v\in\mathcal{C}(\alpha,\beta)} \{\mathcal{F}(v)\}$ and $\mathcal{G}(v) = 0$. Such a v does exists by the third assumption of the Theorem. If $\mathcal{F}(u) = \infty$ then $\mathcal{F} \equiv +\infty$ on $\mathcal{C}(\alpha, \beta)$. Therefore we may assume without loss of generality that $\mathcal{F}(u) < \infty$.

Our goal is to show that the sequence u_k is bounded in $H^1(I)$. Since $H^1(I)$ is reflexive, we then can extract a subsequence which converges weakly in $H^1(I)$ and by Tonelli's lower semicontinuity theorem we see that u is a candidate for a minimizer of (1.1).

The bound for u_k' follows by the quadratic growth of F and by the fact that u_k is a minimizing sequence. Now since $u_k \in AC(I)$ we can bound u_k by u_k' since we have

$$u(x) = \alpha + \int_a^x u_k'(s)ds$$

and therefore

$$\|u_k\|_{L^2(I)} \leq C \|u_k'\|_{L^2(I)}$$

for some C depending on α and I. Therefore u_k is bounded in $H^1(I)$ and so by Theorem 1.2 we have

$$\mathcal{F}(u) \leq \liminf_{k\to\infty} \mathcal{F}(u_k),$$

meaning that u is a candidate minimizer of problem (1.1).

It remains to check that $u(a) = \alpha$, $u(b) = \beta$ and $\mathcal{G}(u) = 0$. The boundary conditions follow by the fact that $\overline{I} = [a, b]$ is compact and so u_k converges uniformly to u on $\overline{I}$ which implies $u(a) = \alpha$ and $u(b) = \beta$. The constraint follows since u_k and G are continuous so $x \mapsto G(x, u_k(x))$ is bounded and by dominated convergence we have

$$\mathcal{G}(u) = \lim_{k\to\infty} \mathcal{G}(u_k) = 0.$$

$\square$

Remark 5. Let us mention that Theorem 1 holds also for a general choice of $p > 1$. Assumption (1.2) has to be changed to

$$c_0|\xi|^p \leq F(x, z, \xi) \leq c_1(1 + |\xi|^p) \qquad \text{for all } (x, z) \in \overline{I} \times \mathbb{R} \text{ fixed.}$$

In fact, it even holds for $p = 1$. But since L^1 and therefore $H^{1,1}$ is not reflexive, it requires a weak compactness criterion. A detailed analysis can be found in Chapter 2 of [1].

Chapter 2

Regular minimizers

In this chapter we will prove regularity for $\mathcal{G}$-regular minimizers of Problem (1.1). The main result will be the following:

Theorem 4. *Let $I = (a,b)$ be a bounded interval in $\mathbb{R}$ and let $F : \overline{I} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be of class C^1 with $F(x,z,\cdot) \in C^2(\mathbb{R})$ and $G : \overline{I} \times \mathbb{R} \to \mathbb{R}^n$ be of class C^1 satisfying the following conditions:*

1. *There exist $c \in \mathbb{R}$ such that for all $(x,z,\xi) \in \overline{I} \times \mathbb{R} \times \mathbb{R}$ we have:*
$$F(x,z,\xi) \leq c(1+|\xi|^2). \tag{2.1}$$

2. *There exists $c_3 \in \mathbb{R}$ such that for all $(x,z,\xi) \in \overline{I} \times \mathbb{R} \times \mathbb{R}$ we have*
$$\left|\frac{\partial F}{\partial z}(x,z,\xi)\right| + \left|\frac{\partial F}{\partial \xi}(x,z,\xi)\right| \leq c_3(1+|\xi|). \tag{2.2}$$

3. *There exists some $\delta > 0$ such that for all $(x,z,\xi) \in \overline{I} \times \mathbb{R} \times \mathbb{R}$*
$$\frac{\partial^2 F}{\partial \xi^2}(x,z,\xi) > \delta. \tag{2.3}$$

If $u \in AC(I)$ is a $\mathcal{G}$-regular minimizer Problem (1.1) then $u \in C^1(\overline{I})$. Moreover, if F and G are of class C^k for $2 \leq k \leq \infty$ then $u \in C^k(\overline{I})$.

Remark 6. If u is a minimizer we don't need additional assumptions on G in order for $\mathcal{G}(u)$ to be well-defined. Since u is a minimizer we have $\mathcal{F}(u) \leq C < \infty$ and $\|u\|_{H^1(I)} \leq C$ by (2.1) for some $C > 0$. So if G is at least continuous on $\overline{I} \times \mathbb{R}$ then also $G(x,u(x))$ is bounded and $\mathcal{G}(u)$ is therefore well defined.

For this whole chapter we assume that we have at least $F \in C^1(\overline{I} \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$ with $F(x,z,\cdot) \in C^2(\mathbb{R})$ and $G \in C^1(\overline{I} \times \mathbb{R}; \mathbb{R}^n)$.

2.1 Lagrange Multipliers for $\mathcal{G}$-regular extremal

In this subsection we introduce and prove the Lagrange Multiplier Theorem for Problem (1.1). We will then use this result to prove that a $\mathcal{G}$-regular extremal always inherits the regularity of F and G.

Definition 4 (Singular extremal)**.** We say $u \in AC(I)$ is a $\mathcal{G}$-singular extremal if for all $\psi_1, ..., \psi_n \in C_c^\infty(I;\mathbb{R})$ we have

$$\det J_\tau \Psi(0) = 0,$$

where $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is given by $\Psi(\tau) := \mathcal{G}\left(u + \sum_{i=1}^n \tau_i \psi_i\right)$ for $\tau \in \mathbb{R}^n$ and J_τ denotes the Jacobian matrix of Ψ with respect to τ.

We say that $u \in AC(I)$ is $\mathcal{G}$-regular if it is not $\mathcal{G}$-singular.

Remark 7 (Regular extremal). If $u \in AC(I)$ is a $\mathcal{G}$-regular extremal then there exist $\psi_1, ..., \psi_n \in C_c^\infty(I;\mathbb{R})$ such that $\det(J_\tau \Psi(0)) \neq 0$ Now for $i, j \in \{1, ..., n\}$ we have

$$\begin{aligned}(J_\tau \Psi)_{i,j}(0) = \frac{\partial \Psi_i}{\partial \tau_j}(0) &= \frac{\partial}{\partial \tau_j}\ \mathcal{G}_i\left(u + \sum_{k=1}^n \tau_k \psi_k\right)\Bigg|_{\tau=0} \\ &= \frac{\partial}{\partial \tau_j} \int_I G_i\left(x, u + \sum_{k=1}^n \tau_k \psi_k\right) dx\Bigg|_{\tau=0}.\end{aligned}$$

Since all derivatives are continuous we can exchange differentiation and integration by Leibniz integral rule [4, Chapter 8] and we get

$$(J_\tau \Psi)_{i,j}(0) = \int_I \frac{\partial G_i}{\partial z}(x,u)\psi_j dx. \tag{2.4}$$

We will use the observation from this Remark for the proof of the following Theorem.

Theorem 5 (Lagrange multiplier Theorem)**.** *Let $u \in AC(I)$ be a regular minimizer of our problem (1.1). Then there exist $\lambda_1, ..., \lambda_n \in \mathbb{R}$ such that*

$$\frac{d}{d\epsilon}\left(\int_I F(x, u+\epsilon\varphi, u'+\epsilon\varphi')dx\ + \sum_{i=1}^n \lambda_i \int_I G_i\left(x, u+\epsilon\varphi\right)dx\right)\Bigg|_{\epsilon=0} = 0 \tag{2.5}$$

for all $\varphi \in C_c^\infty(I)$. Moreover if $u \in C^2(I)$ it satisfies:

$$\frac{\partial F}{\partial z}(x,u,u') - \frac{d}{dx}\frac{\partial F}{\partial \xi}(x,u,u') - \sum_{i=1}^n \lambda_i \frac{\partial G_i}{\partial z}(x,u) = 0, \qquad \forall x \in I. \tag{2.6}$$

Remark 8. We call (2.5) the weak Euler-Lagrange equation for (1.1) and (2.6) the (strong) Euler-Lagrange equation for (1.1).

Proof. Let $\mathcal{Q} := \{(\epsilon,\tau) \in \mathbb{R}\times\mathbb{R}^n \mid \quad |\epsilon| < \epsilon_0, |\tau_i| < \tau_0 \quad \forall i \in \{1,...,n\}\}$ where $0 < \epsilon_0, \tau_0 << 1$ are fixed. For $\varphi \in C_c^\infty(I)$ and $\psi_1, ..., \psi_n \in C_c^\infty(I)$ we define

$$\Phi : \mathcal{Q} \to \mathbb{R} \quad \text{and} \quad \Psi : \mathcal{Q} \to \mathbb{R}^n$$

as

$$\Phi(\epsilon,\tau) := \mathcal{F}\left(u + \epsilon\varphi + \sum_{i=1}^{n} \tau_i\psi_i\right) \quad \text{and} \quad \Psi(\epsilon,\tau) := \mathcal{G}\left(u + \epsilon\varphi + \sum_{i=1}^{n} \tau_i\psi_i\right)$$

respectively.

By assumption we have $\Psi(0,0) = 0$ and by Remark 7, since u is a regular minimizer, there exist $\psi_1, ..., \psi_n \in C_c^\infty(I)$ such that $\det(J_\tau\Psi(0,0)) \neq 0$. This means the matrix $J_\tau\Psi$ in(2.4) is invertible at the point $(0,0)$ and since $\Psi \in C^1(Q;\mathbb{R}^n)$ we can apply the implicit function theorem (see [5, Thm. 9.28, p. 224] to obtain some $\tau \in C^1((-\epsilon_0,\epsilon_0);\mathbb{R}^n)$ such that

$$\Psi(\epsilon,\tau(\epsilon)) \equiv 0, \qquad \text{for } |\epsilon| < \epsilon_0.$$

This means

$$0 = \frac{d}{d\epsilon}\Psi(\epsilon,\tau(\epsilon))\Big|_{\epsilon=0} = \int_I D_zG(x,u)\left(\varphi + \sum_{k=1}^{n} \tau_k'(0)\psi_k\right)dx$$

and so

$$-\int_I D_zG(x,u)\varphi dx = \sum_{k=1}^{n} \tau_k'(0)\int_I D_zG(x,u)\psi_k dx.$$

Notice that

$$\sum_{k=1}^{n} \tau_k'(0)\int_I D_zG(x,u)\psi_k dx = J_\tau\Psi(0,0)\cdot\tau'(0)$$

where $\underbrace{J_\tau\Psi(0,0)}_{\in\mathbb{R}^n\times\mathbb{R}^n}\cdot\underbrace{\tau'(0)}_{\in\mathbb{R}^n} \in \mathbb{R}^n$. Using that $J_\tau\Psi(0,0)$ is invertible we get

$$\tau'(0) = -[J_\tau\Psi(0,0)]^{-1}\cdot\int_I D_zG(x,u)\varphi dx, \tag{2.7}$$

which component-wise reads

$$\tau_i'(0) = -\sum_{k=1}^{n} M_{ik}\int_I \frac{\partial G_k}{\partial z}(y,u)\varphi(y)dy, \qquad i \in \{1,...,n\},$$

where we let $M := [J_\tau\Psi(0,0)]^{-1} \in \mathbb{R}^{n\times n}$. The matrix M is independent of φ.

Let us now compute the derivative of $\Phi(\epsilon,\tau(\epsilon))$ with respect to ϵ. For this we let $\varphi \in C_c^\infty(I)$ be arbitrary. We have:

$$\begin{aligned}
0 &= \frac{d}{d\epsilon}\Phi(\epsilon,\tau(\epsilon))\Big|_{\epsilon=0} \\
&= \int_I \frac{d}{d\epsilon}\left(F(x, u + \epsilon\varphi + \sum_{i=1}^{n}\tau_i(\epsilon)\psi_i, u' + \epsilon\varphi' + \sum_{i=1}^{n}\tau_i(\epsilon)\psi_i')\right)\Big|_{\epsilon=0} dx \\
&= \int_I \left\{\frac{\partial F}{\partial z}(x,u,u')\left(\varphi + \sum_{i=1}^{n}\tau_i'(0)\psi_i\right) + \frac{\partial F}{\partial \xi}(x,u,u')\left(\varphi' + \sum_{i=1}^{n}\tau_i'(0)\psi_i'\right)\right\}dx \\
&=: I + II + III + IV.
\end{aligned}$$

For term II we use (2.7):

$$\begin{aligned}
II &:= \int_I \frac{\partial F}{\partial z}(x,u,u')\sum_{i=1}^{n}\tau_i'(0)\psi_i(x)dx \\
&= -\int_I \frac{\partial F}{\partial z}(x,u,u')\sum_{i=1}^{n}\left(\sum_{k=1}^{n} M_{ik}\int_I \frac{\partial G_k}{\partial z}(y,u)\varphi(y)dy\right)\psi_i(x)dx \\
&= -\sum_{i,k=1}^{n} M_{ik}\int_I \frac{\partial F}{\partial z}(x,u,u')\psi_i(x)\int_I \frac{\partial G_k}{\partial z}(y,u)\varphi(y)dydx \\
&= -\sum_{i,k=1}^{n} M_{ik}\int_I \frac{\partial G_k}{\partial z}(y,u)\varphi(y)\int_I \frac{\partial F}{\partial z}(x,u,u')\psi_i(x)dxdy
\end{aligned}$$

where we used Fubini-Tonelli in the last line. For IV we use a similar computation to get

$$\begin{aligned}
IV &:= \int_I \frac{\partial F}{\partial \xi}(x,u,u')\sum_{i=1}^{n}\tau_i'(0)\psi_i'(x)dx \\
&= -\sum_{i,k=1}^{n} M_{ik}\int_I \frac{\partial G_k}{\partial z}(y,u)\varphi(y)\int_I \frac{\partial F}{\partial \xi}(x,u,u')\psi_i(x)dxdy.
\end{aligned}$$

Now we add everything together again and get, for all $\varphi\in C_c^\infty(I)$

$$\int_I \Bigg\{\left(\frac{\partial F}{\partial z}(x,u,u') - \sum_{i,k=1}^{n} M_{ik}\frac{\partial G_k}{\partial z}(x,u)\int_I\left[\frac{\partial F}{\partial z}(y,u,u')\psi_i(y) - \frac{\partial F}{\partial \xi}(y,u,u')\psi_i'(y)\right]dy\right)\varphi(x) + \frac{\partial F}{\partial \xi}(x,u,u')\varphi'(x)\Bigg\}dx = 0.$$

We define the Lagrange multipliers as

$$\lambda_k := -\sum_{i=1}^{n} M_{ik}\int_I\left[\frac{\partial F}{\partial z}(y,u,u')\psi_i(y) - \frac{\partial F}{\partial \xi}(y,u,u')\psi_i'(y)\right]dy, \qquad \forall k\in\{1,...,n\}.$$

We notice that λ_k does not depend on φ for any $k\in\{1,...,n\}$. In terms of our funtions Φ and Ψ we have just shown

$$\partial_\epsilon\Phi(0,0) - \underbrace{\partial_\tau\Phi(0,0)^T\left[J_\tau\Psi(0,0)\right]^{-1}}_{=:\lambda\in\mathbb{R}^n}\partial_\epsilon\Psi(0,0) = 0$$

where $\lambda\in\mathbb{R}^n$ corresponds to the Lagrange multipliers. This shows (2.5).
If we have $u\in C^2(I)$ we can perform an integration by parts for the term involving $\varphi'(x)$. We then get for all $\varphi\in C_c^\infty(I)$

$$\int_I\left\{\frac{\partial F}{\partial z}(x,u,u') - \frac{d}{dx}\frac{\partial F}{\partial \xi}(x,u,u') + \sum_{k=1}^{n}\lambda_k\frac{\partial G_k}{\partial z}(x,u)\right\}\varphi(x)dx = 0.$$

This shows (2.6) since the equation holds for all $\varphi\in C_c^\infty(I)$. □

2.2 Regularity of $\mathcal{G}$-regular extremal

In this section we prove that a $\mathcal{G}$-regular minimizer of Problem (1.1) inherits the regularity of F and G. For this we define $H : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$$H(x, z, \xi) := F(x, z, \xi) - \sum_{k=1}^{n} \lambda_k G_k(x, z).$$

This is the Lagrangian of the constrained problem, where $\lambda_1, ..., \lambda_n \in \mathbb{R}$ are the multipliers from Theorem 5. We define the functional $\mathcal{H} : AC(I) \to \mathbb{R}$ as

$$\mathcal{H}(u) := \int_I H(x, u(x), u'(x))dx.$$

Remark 9. Condition (2.1) assures that the functional $\mathcal{H}$ is well defined for $u \in AC(I)$. In fact since u is absolutely continuous u' exists almost everywhere in I and it belongs to L^1. Since $\xi \mapsto H(x, z, \xi)$ is of class C^2, so in particular continuous, the composition $H(x, u(x), u'(x))$ is measurable. The bound in (2.1) then guarantee integrability, since I is bounded.

We divide the proof into two steps. We first prove that a $\mathcal{G}$-regular minimizer $u \in AC(I)$ is actually in $C^1(I)$. We then continue by proving that a $C^1(I)$ minimizer is actually $C^k(I)$ if F and G are in C^k.

Proposition 1. *Under the same assumptions as in Theorem 4, if $u \in AC(I)$ is a $\mathcal{G}$-regular minimizer of Problem (1.1) then $u \in C^1(I)$.*

Proof. We claim that there exists some $c \in \mathbb{R}$ such that

$$\frac{\partial H}{\partial \xi}(x, u, u') = c + \int_a^x \frac{\partial H}{\partial z}(s, u, u')ds \qquad \text{for a.e. } x \in I. \tag{2.8}$$

First we observe that the mappings $x \mapsto \frac{\partial H}{\partial z}(x, u(x), u'(x))$ and $x \mapsto \frac{\partial H}{\partial \xi}(x, u(x), u'(x))$ are measurable since H is of class C^1 and u' is integrable. By (2.2) they are in $L^1(I)$. So the weak Euler-Lagrange equation (2.5) reads

$$\int_I \left(\frac{\partial H}{\partial z}(x, u, u')\varphi(x) + \frac{\partial H}{\partial \xi}(x, u, u')\varphi'(x) \right) dx = 0, \qquad \varphi \in C_c^\infty(I). \tag{2.9}$$

We do an integration by parts in the first term:

$$\begin{aligned}\int_I \frac{\partial H}{\partial z}(x, u, u')\varphi(x)dx &= \underbrace{\left(\int_a^x \frac{\partial H}{\partial z}(s, u, u')ds \right) \varphi(x)\Big|_{x=a}^{x=b}}_{=0} - \int_I \left(\int_a^x \frac{\partial H}{\partial u}(s, u, u')ds \right) \varphi'(x)dx \\ &= - \int_I \left(\int_a^x \frac{\partial H}{\partial z}(s, u, u')ds \right) \varphi'(x)dx,\end{aligned}$$

where we used that φ has compact support. So (2.9) becomes

$$\int_I \left(\frac{\partial H}{\partial \xi}(x, u, u') - \int_a^x \frac{\partial H}{\partial z}(s, u, u')ds \right) \varphi'(x)dx = 0, \qquad \varphi \in C_c^\infty(I).$$

By the Lemma of du Bois-Raymond (see [6, Lemma 2, p.10] there exists a constant $c \in \mathbb{R}$ such that

$$\frac{\partial H}{\partial \xi}(x, u, u') = c + \int_a^x \frac{\partial H}{\partial z}(s, u, u')ds \qquad \text{for a.e. } x \in I. \tag{2.10}$$

We notice that the function

$$\pi(x) := c + \int_a^x \frac{\partial H}{\partial z}(s, u, u')ds \tag{2.11}$$

is absolutely continuous.

Let us now define the mapping $\Gamma : \overline{I} \times \mathbb{R} \times \mathbb{R} \to \overline{I} \times \mathbb{R} \times \mathbb{R}$ as

$$\Gamma(x, z, \xi) := (x, z, \partial_\xi H(x, z, \xi)).$$

Let moreover $\Omega := \left\{\Gamma(x, z, \xi) \,|\, (x, z, \xi) \in \overline{I} \times \mathbb{R} \times \mathbb{R}\right\}$ be the image of Γ. Since the mapping $\xi \mapsto \partial_\xi H(x, z, \xi)$ is continuously differentiable and $\partial_\xi^2 H(x, z, \xi) \neq 0$ for all $(x, z, \xi) \in \overline{I} \times \mathbb{R} \times \mathbb{R}$ by (2.3) by the implicit function theorem the inverse map $\partial_\xi H^{-1}$ exists and is C^1. Therefore $\Gamma : \overline{I} \times \mathbb{R} \times \mathbb{R} \to \Omega$ is a C^1 diffeomorphism.

We define

$$\sigma(x) := (x, u(x), u'(x)) \qquad \text{and} \qquad e(x) := (x, u(x), \pi(x)).$$

The function σ is defined a.e. in I while e is actually defined for all $x \in \overline{I}$, since $\pi(x)$ and $u(x)$ are absolutely continuous. Since $\pi(x) = \partial_\xi H(x, u, u')$, identity (3.2) reads

$$\Gamma(\sigma(x)) = e(x), \qquad \text{a.e. in } \overline{I}. \tag{2.12}$$

We now need to check that $\Gamma^{-1}(e(x))$ is well-defined, meaning we need to check that $e(x) \in \Omega$ for all $x \in \overline{I}$. This follows from (2.3), which implies that $\partial_\xi H(x, u, \mathbb{R}) = \mathbb{R}$. I.e. $\Omega = \overline{I} \times \mathbb{R} \times \mathbb{R}$, and so $e(x) \in \Omega$ and $\Gamma^{-1}(e(x))$ is well defined and continuous for all $x \in \overline{I}$.

Therefore

$$\Gamma^{-1}(e(x)) =: (x, u(x), v(x))$$

for some $v(x) \in C(I)$. But then (2.12) implies

$$(x, u(x), u'(x)) = (x, u(x), v(x)), \qquad \text{a.e. in } \overline{I},$$

so $u'(x) = v(x)$ a.e. in $\overline{I}$. This proves Proposition 1 since we have

$$u(x) = u(a) + \int_a^x u'(s)ds = u(a) + \int_a^x v(s)ds$$

and therefore $u \in C^1(\overline{I})$. □

We are now ready to prove Theorem 4.

Proof of Thm. 4. Let $u \in AC(I)$ be a $\mathcal{G}$-regular minimizer of Problem (1.1). We argue by induction that we have $u \in C^k(\overline{I})$ for all $k \in \mathbb{N}$ whenever F and G are of class C^k.

By Proposition 1 we have $u \in C^1(\overline{I})$. Let us show that we have

$$u \in C^2(\overline{I}),$$

which will be our base step. To this aim let $P : \overline{I} \times \mathbb{R} \to \mathbb{R}$ be given by

$$P(x,\xi) := \frac{\partial H}{\partial \xi}(x, u(x), \xi) - \pi(x),$$

where $\pi(x)$ is defined as in (2.11). Since $u \in C^1(\overline{I})$ we have that $(x,\xi) \mapsto \partial_\xi H(x, u(x), \xi)$ is of class C^1 and also

$$\pi(x) = c + \int_a^x \underbrace{\frac{\partial H}{\partial z}(s, u, u')ds}_{\in C^(\overline{I})}$$

is of class C^1. So we have $P \in C^1(\overline{I} \times \mathbb{R}; \mathbb{R})$. Moreover we have $P(x, u') = 0$ and

$$\frac{\partial P}{\partial \xi}(x,\xi) = \frac{\partial^2 F}{\partial \xi^2}(x, u(x), \xi) > 0, \qquad (x,\xi) \in \overline{I} \times \mathbb{R},$$

by assumption (2.3). In particular we have $\partial_\xi P(x, u') > 0$. By the implicit function theorem there exists a neighbourhood $U_x \subseteq \overline{I}$ and a function $v : U_x \to \mathbb{R}$, which is of class C^1, such that $v(x) = u'(x)$ for all $x \in U_x$. This means $u \in C^2(U_x)$. But $x \in \overline{I}$ can be chosen arbitrary and we conclude $u \in C^2(\overline{I})$.

Let now $k \in \mathbb{N}$, $k \geq 2$ be arbitrary and let $u \in C^k(\overline{I})$. We show that $u \in C^{k+1}(\overline{I})$. With the same arguments as above we see that $P \in C^k(\overline{I} \times \mathbb{R})$ and the implicit function theorem yields the existence of some $v \in C^k(U_x)$ such that $v(x) = u'(x)$ in some neighbourhood $U_x \subseteq \overline{I}$ of x. Again this argument holds for all $x \in \overline{I}$ so we have $u \in C^{k+1}(\overline{I})$. This concludes the induction step.

So we just have shown

$$u \in C^k(\overline{I}), \qquad k \in \mathbb{N},$$

which is precisely what we wanted.

$\square$

Chapter 3

Non minimality for a class of singular extremal

In this chapter we turn to the study of a class of singular extremal that have a cusp of arbitrary type. We will show that they are not minimizers for the energy.

The minimization problem we consider is the following:

$$\min_{u\in H^1(I)} \{\, \mathcal{F}(u) \,|\, u(\pm 1) = 1 \text{ and } \mathcal{G}(u) = V_{h,k} \,\} \tag{3.1}$$

where $I := [-1, 1]$ and

$$\mathcal{F}(u) := \frac{1}{2}\int_I |u'(x)|^2 dx \qquad \text{and} \qquad \mathcal{G}(u) := \int_I \left(x^{2h}u(x) - \frac{u(x)^{2k+1}}{2k+1} \right) dx$$

where $k, h \in \mathbb{N}_{\geq 1}$ are fixed and $V_{h,k} := \frac{4k^2}{(2k+1)(2kh+k+h)}$.

Remark 10. The Lagrangian $F = F(\xi) = \frac{1}{2}|\xi|^2$ and the constraint $G(x, z) = x^{2h}z - \frac{z^{2k+1}}{2k+1}$ are smooth. $\mathcal{F}$ is well defined for $u \in H^1(I)$ and also $\mathcal{G}$ is well-defined since u is continuous.

Remark 11. The value of $V_{h,k}$ may seem a bit arbitrary. We will see later that $V_{h,k}$ is the exact value of $\mathcal{G}(u)$ for a singular extremal u.

Our aim is to prove that for this choice of F and G a singular extremal is not a minimizer of (3.1).

Theorem 6. *For all $h, k \in \mathbb{N}$ with $h \neq k$ there exists no $\mathcal{G}$-singular minimizer $u \in H^1(I)$ of (3.1).*

But what happens if $h = k$? It turns out that this case is more difficult.

Open question 1. *If $h = k$, can a $\mathcal{G}$-singular extremal be a minimizer of Problem (3.1)?*

We will focus on this question in the next Chapter. Before we turn to the proof we notice that there exists a minimizer $u \in H^1(I)$ of (3.1), because the conditions of Theorem 1 of Chapter 1 hold:

Obviously F has quadratic growth and is convex. So we know that (3.1) has a solution. Let us first study the case of when u is regular.

3.1 Regular extremals

If u is a regular minimizer we can use the theory developed in Chapter 2. First let us check that u is smooth so we can compute the Euler Lagrange equation for u. We need to check that the conditions of Theorem 4 hold.

By assumption $u \in H^1(I)$, $F \in C^\infty(\mathbb{R})$ and $G \in C^\infty(I \times \mathbb{R}; \mathbb{R})$. F has quadratic growth and $\partial_\xi F = \xi \leq (1 + \xi)$. Finally we have $\partial_\xi^2 F = 1 > 0$ so all conditions of Theorem 4 are fulfilled and therefore $u \in C^\infty(I)$ if u is a regular minimizer of (3.1). In particular, $u \in C^2(I)$ and we can compute the Euler Lagrange equation: There exists some $\lambda \in \mathbb{R}$ such that

$$\underbrace{\frac{\partial F}{\partial z}(x, u')}_{=0} - \frac{d}{dx}\frac{\partial F}{\partial \xi}(x, u') - \lambda \frac{\partial G}{\partial z}(x, u) = 0.$$

which leads to the following boundary value problem:

$$\begin{cases} u''(x) = \lambda u^{2k}(x) - \lambda x^{2h} & \text{for all } x \in I, \\ u(-1) = u(1) = 1. \end{cases} \tag{3.2}$$

Now the most straight-forward approach to prove Theorem 6 would be to compare analytically regular and singular extremals. If we can show that for any singular extremal u_{sing} we have $\mathcal{F}(u_{\text{reg}}) < \mathcal{F}(u_{\text{sing}})$ we are finished. But it turns out that this is more difficult then it may look like. In fact, it is in general impossible to bound a regular extremal. To see this we recall that the Lagrange multiplier λ is given by $\lambda := \partial_\tau \Phi(0,0)^T \partial_\tau \Psi(0,0)^{-1}$ where $\Phi(\epsilon, \tau) = \mathcal{F}(u + \epsilon\varphi + \tau\psi)$ and $\Psi(\epsilon, \tau) = \mathcal{G}(u + \epsilon\varphi + \tau\psi)$ (see Thm. 4). In our setting this means that

$$\lambda := \frac{\int_I u'(x)\psi'(x)dx}{\int_I \left(x^{2h} - u^{2k}(x)\right)\psi(x)dx}.$$

Now $u^{2k}(x)$ might be arbitrarily close to x^{2h}. Therefore it is impossible to find a general bound on λ and this approach will fail. So we will turn our focus on the study of singular extremals.

3.2 Singular extremals

If u is a singular extremal of (3.1) then for all $\psi \in C_c^\infty(\mathring{I})$ (where $\mathring{I} = (-1, 1)$ denotes the interior of I) we have

$$\frac{d}{d\epsilon}\, \mathcal{G}\left(u + \epsilon\psi\right)\big|_{\epsilon=0} = 0.$$

This implies that

$$\frac{\partial}{\partial u}\left(x^{2h}u - \frac{u^{2k+1}}{2k+1}\right) = 0, \qquad \text{for all } x \in I.$$

So $u^{2k} = x^{2h}$ and therefore

$$u(x) = |x|^{h/k} = \begin{cases} x^{h/k} & \text{if } x \geq 0; \\ (-x)^{h/k} & \text{if } x < 0. \end{cases} \tag{3.3}$$

We see that this is not a smooth function for all choices of h and k. For example if $h = 2$ and $k = 1$, u is smooth but if $h = k = 1$ then $u(x) = |x|$ which is not in $C^1(I)$. However it is Lipschitz-continuous on I. Another obvious example of a non-smooth version is when $h = 1$ and $k = 2$. Then $u(x) = \sqrt{|x|}$ which is not even Lipschitz in $x = 0$. The question is, whether one of these non-smooth candidates is actually a minimizer of Problem (3.1).

Let us compute $\mathcal{G}(u)$. Observe, that for a singular u we have $G(-x, u(-x) = G(x, u(x))$ for all $x \in [-1, 1]$, meaning that G is an even function for all $x \in [-1, 1]$. This helps us to compute the integral of G:

$$\begin{aligned}
\mathcal{G}(u) &= 2\int_0^1 G(x, u(x))dx \\
&= 2\int_0^1 \left(x^{(2hk+h)/k} - \frac{x^{(2hk+h)/k}}{2k+1} \right) dx \\
&= 2\left(1 - \frac{1}{2k+1}\right) \int_0^1 x^{(2hk+h)/k} dx \\
&= 2\left(\frac{2k}{2k+1}\right) \frac{k}{2hk+h+k} \\
&= V_{h,k}.
\end{aligned}$$

Now we see where the definition of the value of the constraint comes from. The only possibility to take is $V_{h,k}$, otherwise a singular extremal u can not be a solution of (3.1).

3.3 Proof of Theorem 6

Let u be a singular extremal. As mentioned above u is an even function for all natural numbers h and k. We will therefore prove the Theorem for $I = [0, 1]$ to make things easier. The idea is the following: We fix $\delta > 0$ and modify u in $[0, \delta]$ by setting $u_\varepsilon(x) = \varepsilon > 0$ for $x \leq \delta$ with $u_\varepsilon(\delta) = \varepsilon$. Basically we remove a part where the L^2 norm of the derivative is big. We then modify u_ε in the interval $[\frac{1}{2}, \frac{3}{4}]$ in such a way that we have a gain of L^2 derivative in this part. We call this modified function u_ε^η, where $\eta \in \mathbb{R}$ is fixed so that $\mathcal{G}(u_\varepsilon^\eta) = \mathcal{G}(u)$. Roughly speaking we take more derivative away than we add. We then show that $\mathcal{F}(u_\varepsilon^\eta) < \mathcal{F}(u)$ for $\varepsilon > 0$ small, meaning that u can't be a minimizer of (3.1).

So let $\varepsilon > 0$ and $\eta \in \mathbb{R}$ be fixed. We define

$$u_\varepsilon^\eta(x) := \begin{cases} \varepsilon & : x \leq \delta \\ u(x) & : x \in [\delta, \frac{1}{2}] \\ m_1 x + d_1 & : x \in [\frac{1}{2}, \frac{1}{2} + |\eta|] \\ u(x) + \eta & : x \in [\frac{1}{2} + |\eta|, \frac{3}{4} - |\eta|] \\ m_2 x + d_2 & : x \in [\frac{3}{4} - |\eta|, \frac{3}{4}] \\ u(x) & : x \geq \frac{3}{4}, \end{cases} \tag{3.4}$$

where

$$m_1(\eta) := \frac{(\frac{1}{2} + |\eta|)^{h/k} + \eta - (\frac{1}{2})^{h/k}}{|\eta|} \qquad \text{and} \qquad d_1(\eta) := \left(\frac{1}{2}\right)^{h/k} - \frac{m_1(\eta)}{2}$$

and

$$m_2(\eta) := \frac{(\frac{3}{4})^{h/k} - (\frac{3}{4} - |\eta|)^{h/k} - \eta}{|\eta|} \qquad \text{and} \qquad d_2(\eta) := \left(\frac{3}{4}\right)^{h/k} - \frac{3m_2(\eta)}{4}.$$

The picture is the following:

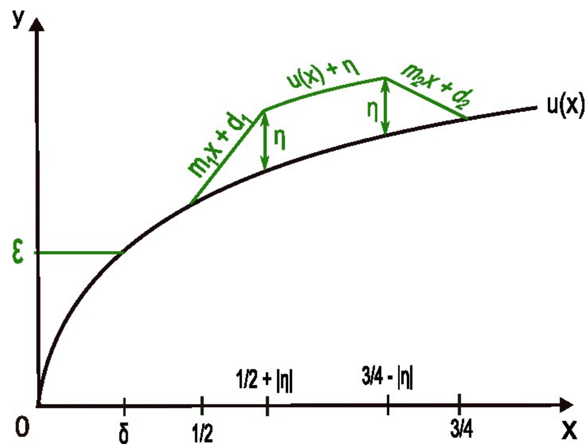

Figure 3.1: Sketch for $\eta > 0$ where the modifications for u_ε^η are drawn in green.

Since we want u_ε^η to be continuous we choose $\delta = \varepsilon^{k/h}$.

Remark 12. We have $u_\varepsilon^\eta \in H^1(I)$. In fact, u_ε^η is Lipschitz continuous on I and it is bounded so in L^∞. Therefore we have $u \in H^{1,\infty}(I) \subseteq H^1(I)$. (See [2, Prop. 8.4] for a proof)

Proof of Thm 6. Step 1: We would like to express η in terms of ε. For this we set

$$\begin{aligned}\Theta_\eta(\varepsilon) &:= \mathcal{G}(u) - \mathcal{G}(u_\varepsilon^\eta) \\ &= \int_0^\delta (G(x, u(x)) - G(x, u_\varepsilon^\eta(x))\, dx + \int_{\frac{1}{2}}^{\frac{3}{4}} (G(x, u(x)) - G(x, u_\varepsilon^\eta(x))\, dx \\ &\equiv 0.\end{aligned}$$

To compute these integrals we notice that for $0 \leq a < b \leq 1$ we have

$$\begin{aligned}\int_a^b G(x, u(x))dx &= \left(1 - \frac{1}{2k+1}\right) \int_a^b x^{\frac{2kh+h}{k}}\, dx \\ &= \underbrace{\left(\frac{2k}{2k+1}\right)\left(\frac{k}{2kh+h+k}\right)}_{=\frac{V_{h,k}}{2}} \left[x^{\frac{2hk+h+k}{k}}\right]_a^b.\end{aligned}$$

For the first term we have

$$\begin{aligned}\int_0^{\varepsilon^{k/h}} (G(x,u)-G(x,u_\varepsilon^\eta)) &= \int_0^{\varepsilon^{k/h}} G(x,u)dx - \int_0^{\varepsilon^{k/h}} \left(x^{2h}\varepsilon - \frac{\varepsilon^{2k+1}}{2k+1}\right)dx \\ &= \frac{V_{h,k}}{2}\left(\varepsilon^{k/h}\right)^{\frac{2kh+h+k}{k}} - \frac{(\varepsilon^{k/h})^{2h+1}}{2h+1}\varepsilon + \frac{\varepsilon^{2k+1}}{2k+1}\varepsilon^{k/h} \\ &= \underbrace{\left(\frac{2k}{(2k+1)}\frac{k}{(2kh+h+k)} - \frac{1}{2h+1} + \frac{1}{2k+1}\right)}_{C_1}\varepsilon^{\frac{2kh+h+k}{h}}.\end{aligned}$$

Now

$$C_1 := \frac{2k}{(2k+1)}\frac{k}{(2kh+h+k)} - \frac{1}{2h+1} + \frac{1}{2k+1} = \frac{2h^2(2k+1)}{(2k+1)(2h+1)(2kh+h+k)}$$

and therefore we have $C_1 > 0$ for all choices of $k,\ h \in \mathbb{N}$.

For the second term we write

$$\begin{aligned}\int_{\frac12}^{\frac34} (G(x,u(x))-G(x,u_\varepsilon^\eta(x)))\,dx = &\underbrace{\int_{\frac12}^{\frac12+|\eta|} (G(x,u(x))-G(x,u_\varepsilon^\eta(x)))\,dx}_{:=I} + \underbrace{\int_{\frac12+|\eta|}^{\frac34-|\eta|} (G(x,u(x))-G(x,u_\varepsilon^\eta(x)))\,dx}_{:=II} \\ &+ \underbrace{\int_{\frac34-|\eta|}^{\frac34} (G(x,u(x))-G(x,u_\varepsilon^\eta(x)))\,dx}_{:=III}.\end{aligned}$$

Our goal is to express these three terms as a part depending linearly on η and a rest $o(|\eta|)$ for η small.

<u>Estimates for I:</u> We have

$$\int_{\frac12}^{\frac12+|\eta|} G(x,u)dx = \frac{V_{h,k}}{2}\left[x^{\frac{2kh+h+k}{k}}\right]_{\frac12}^{\frac12+|\eta|}$$

and for η small enough we can do a Taylor expansion:

$$\begin{aligned}\frac{V_{h,k}}{2}\left[x^{\frac{2kh+h+k}{k}}\right]_{\frac12}^{\frac12+|\eta|} &= \frac{V_{h,k}}{2}\left(\left(\frac12\right)^{\frac{2kh+h+k}{k}} + \frac{2kh+h+k}{k}\left(\frac12\right)^{2h+1}\underbrace{\frac{d}{d\eta}|\eta||\eta|}_{=\eta \text{ for } \eta>0 \text{ and } \eta<0} + o(|\eta|) - \left(\frac12\right)^{\frac{2kh+h+k}{k}}\right) \\ &= \frac{2k}{2k+1}\frac{k}{2kh+h+k}\frac{2kh+h+k}{k}\left(\frac12\right)^{2h+1}\eta + o(|\eta|) \\ &= \frac{2k}{2k+1}\left(\frac12\right)^{2h+1}\eta + o(|\eta|).\end{aligned}$$

Now we have

$$\int_{\frac{1}{2}}^{\frac{1}{2}+|\eta|} G(x,u_\varepsilon^\eta)dx = \int_{\frac{1}{2}}^{\frac{1}{2}+|\eta|}\left(x^{2h}\left(m_1x+d_1\right)-\frac{(m_1x+d_1)^{2k+}}{2k+1}\right)dx$$
$$= \frac{m_1}{2h+2}\left[x^{2h+2}\right]_{\frac{1}{2}}^{\frac{1}{2}+|\eta|}+\frac{d_1}{2h+1}\left[x^{2h+1}\right]_{\frac{1}{2}}^{\frac{1}{2}+|\eta|}-\frac{1}{(2k+1)(2k+2)m_1}\left[(m_1x+d_1)^{2k+2}\right]_{\frac{1}{2}}^{\frac{1}{2}+|\eta|}.$$

For every term we do again a Taylor expansion:

- $\frac{m_1}{2h+2}\left[x^{2h+2}\right]_{\frac{1}{2}}^{\frac{1}{2}+|\eta|} = m_1\left(\frac{1}{2}\right)^{2h+1}\underbrace{\left(\frac{d}{d\eta}|\eta|\right)|\eta|}_{=\eta}+o(|\eta|),$
- $\frac{d_1}{2h+1}\left[x^{2h+1}\right]_{\frac{1}{2}}^{\frac{1}{2}+|\eta|} = d_1\left(\frac{1}{2}\right)^{2h}\underbrace{\left(\frac{d}{d\eta}|\eta|\right)|\eta|}_{=\eta}+o(|\eta|),$
- $\left[\frac{(m_1x+d_1)^{2k+2}}{(2k+2)(2k+1)m_1}\right]_{\frac{1}{2}}^{\frac{1}{2}+|\eta|} = \frac{(\frac{m_1}{2}+d_1)^{2k+1}}{(2k+1)}\underbrace{\left(\frac{d}{d\eta}|\eta|\right)|\eta|}_{=\eta}+o(|\eta|).$

Recall that we have $d_1 := \left(\frac{1}{2}\right)^{h/k}-\frac{m_1}{2}$ and therefore $\left(\frac{m_1}{2}+d_1\right)^{2k+1} = \left(\frac{1}{2}\right)^{(h/k)(2k+1)} = \left(\frac{1}{2}\right)^{\frac{2kh+h}{k}}$. Adding everything together we get

$$I = \left\{\frac{2k}{2k+1}\left(\frac{1}{2}\right)^{2h+1}-m_1\left(\frac{1}{2}\right)^{2h+1}-d_1\left(\frac{1}{2}\right)^{2h}+\frac{1}{2k+1}\left(\frac{1}{2}\right)^{\frac{2kh+h}{k}}\right\}\eta+o(|\eta|).$$

Now $-m_1\left(\frac{1}{2}\right)^{2h+1}-d_1\left(\frac{1}{2}\right)^{2h} = -\left(\frac{1}{2}\right)^{2h}\left(\frac{m_1}{2}+d_1\right) = -\left(\frac{1}{2}\right)^{2h}\left(\frac{1}{2}\right)^{h/k}$ and so

$$I = \underbrace{\left\{\left(\frac{1}{2}\right)^{2h}\left[\frac{2k}{2(2k+1)}-\left(\frac{1}{2}\right)^{h/k}+\frac{1}{2k+1}\left(\frac{1}{2}\right)^{h/k}\right]\right\}}_{:=C_I}\eta+o(|\eta|).$$

We have

$$C_I = \left(\frac{1}{2}\right)^{2h}\frac{2k(2^{\frac{h-k}{k}})-(2k+1)+1}{2^{h/k}(2k+1)} = \left(\frac{1}{2}\right)^{2h}\frac{2k(2^{\frac{h-k}{k}}-1)}{2^{h/k}(2k+1)}$$

and notice that $C_I = 0$ if and only if $h = k$. Moreover, for $h > k$ we have $C_I > 0$ and for $h < k$ we have $C_I < 0$.

III:

The calculations for part III are almost the same as for part I but with other factors contributing to the constants. We first note that for a term of the form x^a we have the following Taylor expansion for :

$$\left[x^a\right]_{\frac{3}{4}-|\eta|}^{\frac{3}{4}} = \left(\frac{3}{4}\right)^a-\left(\left(\frac{3}{4}\right)^a+a\left(\frac{3}{4}\right)^{a-1}\underbrace{\left(-\frac{d}{d\eta}|\eta|\right)|\eta|}_{=-\eta\text{ for }\eta>0\text{ and }\eta<0}+o(|\eta|)\right) = a\left(\frac{3}{4}\right)^{a-1}\eta+o(|\eta|),$$

so we can adapt the calculations from I and get

$$III = \underbrace{\left\{ \frac{2k}{2k+1} \left(\frac{3}{4}\right)^{2h+1} - m_2 \left(\frac{3}{4}\right)^{2h+1} - d_2 \left(\frac{3}{4}\right)^{2h} + \frac{1}{2k+1} \left(\frac{3m_2}{4} + d_2\right)^{2k+1} \right\}}_{=:C_{III}} \eta + o(|\eta|).$$

This time we have $\left(\frac{3m_2}{4} + d_2\right)^{2k+1} = \left(\frac{3}{4}\right)^{(h/k)(2k+1)}$ and $-\left(\frac{3}{4}\right)^{2h} \left(\frac{3m_2}{4} + d_2\right) = -\left(\frac{3}{4}\right)^{2h} \left(\frac{3}{4}\right)^{h/k}$ so

$$\begin{aligned} C_{III} &= \left(\frac{3}{4}\right)^{2h} \left\{ \frac{3}{4}\frac{2k}{2k+1} - \left(\frac{3}{4}\right)^{h/k} + \frac{1}{2k+1}\left(\frac{3}{4}\right)^{h/k} \right\} \\ &= \left(\frac{3}{4}\right)^{2h} \frac{6k(4^{\frac{h-k}{k}}) - 3^{h/k}(2k+1) + 3^{h/k}}{4^{h/k}(2k+1)} \\ &= \left(\frac{3}{4}\right)^{2h} \frac{6k(4^{\frac{h-k}{k}} - 3^{\frac{h-k}{k}})}{4^{h/k}(2k+1)}. \end{aligned}$$

We notice that we have $C_{III} = 0$ if and only if $h = k$, $C_{III} > 0$ if $h > k$ and $C_{III} < 0$ if $h < k$.

II:

For $x \in [\frac{1}{2} + |\eta|, \frac{3}{4} - |\eta|]$ we have $u_\varepsilon^\eta(x) = x^{h/k} + \eta$. So

$$\begin{aligned} II &:= \int_{\frac{1}{2}+|\eta|}^{\frac{3}{4}-|\eta|} \left(x^{2h} x^{h/k} - \frac{(x^{h/k})^{2k+1}}{2k+1} - x^{2h}(x^{h/k} + \eta) + \frac{(x^{h/k} + \eta)^{2k+1}}{2k+1} \right) dx \\ &= -\int_{\frac{1}{2}+|\eta|}^{\frac{3}{4}-|\eta|} \frac{x^{\frac{2kh+h}{k}}}{2k+1} dx - \eta \int_{\frac{1}{2}+|\eta|}^{\frac{3}{4}-|\eta|} x^{2h} + \int_{\frac{1}{2}+|\eta|}^{\frac{3}{4}-|\eta|} \frac{(x^{h/k} + \eta)^{2k+1}}{2k+1} dx \\ &= o(|\eta|) \end{aligned}$$

since, again by a Taylor expansion, we have $(x^{h/k}+\eta)^{2k+1} = (x^{h/k})^{2k+1} + (2k+1)(x^{h/k})^{2k}\eta + o(|\eta|)$. This is enough to know if we have $h \neq k$ since we already know that in this case $C_I, C_{III} \neq 0$.

Now we have two cases:

- If $h \neq k$:

$$\Theta_\eta(\varepsilon) = C_1 \varepsilon^\alpha + C_I \eta + C_{III} \eta + o(|\eta|) = 0$$

  where $\alpha := \frac{2kh+h+k}{h}$ and so

$$\eta = -\frac{C_1}{C_I + C_{III}} \varepsilon^\alpha + o(\varepsilon^\alpha). \tag{3.5}$$

  - If $h > k$ we have $C_I + C_{III} > 0$ so $-\frac{C_1}{C_I + C_{III}} < 0$ and therefore $\eta < 0$.
  - If $h < k$ we have $C_I + C_{III} < 0$ so $-\frac{C_1}{C_I + C_{III}} > 0$ and therefore $\eta > 0$.

- If $h = k$ we have

$$\Theta_\eta(\varepsilon) = C_I \varepsilon^\alpha + o(|\eta|) = 0$$

meaning $\varepsilon^\alpha = o(|\eta|)$ and this approach will not work. One could think in this case we expand the integrals I, II, III again to get a quadratic equation for η. But this equation will have no real solution if we don't have an additional assumption on G. We will study this phenomena in the next chapter.

Step 2): Now we define $\phi(\varepsilon) := \mathcal{F}(u^\eta_\varepsilon) - \mathcal{F}(u)$. If we have $\phi(\varepsilon) < 0$ we are finished. Notice that $u'(x) = \frac{h}{k} x^{\frac{h}{k}-1}$ and

$$\frac{d}{dx} u^\varepsilon_\eta(x) = \begin{cases} 0 & : x \le \varepsilon^{k/h} \\ u'(x) & : x \in [\varepsilon^{k/h}, \frac{1}{2}] \\ m_1 & : x \in [\frac{1}{2}, \frac{1}{2} + |\eta|] \\ u'(x) & : x \in [\frac{1}{2} + |\eta|, \frac{3}{4} - |\eta|] \\ m_2 & : x \in [\frac{3}{4} - |\eta|, \frac{3}{4}] \\ u'(x) & : x \ge \frac{3}{4}. \end{cases}$$

Consequently we have

$$\phi(\varepsilon) = -\frac{1}{2}\int_0^{\varepsilon^{k/h}} u'(x)^2 dx + \frac{1}{2}\int_{\frac{1}{2}}^{\frac{1}{2}+|\eta|} \left(m_1^2 - u'(x)^2\right) dx + \frac{1}{2}\int_{\frac{3}{4}-|\eta|}^{\frac{3}{4}} \left(m_2^2 - u'(x)^2\right) dx.$$

Note that for $0 \le a < b \le 1$ we have

$$\int_a^b u'(x)^2 dx = \frac{h^2}{k^2}\int_a^b x^{\frac{2h-2k}{k}} dx = \frac{h^2}{k}\frac{1}{2h-k}\left[x^{\frac{2h-k}{k}}\right]_a^b \tag{3.6}$$

and so

$$\begin{aligned}\phi(\varepsilon) &= -\frac{h^2}{2k}\frac{1}{2h-k}\left(\varepsilon^{k/h}\right)^{\frac{2h-k}{k}} + \frac{m_1^2}{2}|\eta| - \underbrace{\frac{h^2}{2k}\frac{1}{2h-k}\left[x^{\frac{2h-k}{k}}\right]_{\frac{1}{2}}^{\frac{1}{2}+|\eta|}}_{=\frac{h^2}{2k^2}(\frac{1}{2})^{\frac{2h-2k}{k}}|\eta| + o(|\eta|)} + \frac{m_2^2}{2}|\eta| - \underbrace{\frac{h^2}{2k}\frac{1}{2h-k}\left[x^{\frac{2h-k}{k}}\right]_{\frac{3}{4}-|\eta|}^{\frac{3}{4}}}_{\frac{h^2}{2k^2}(\frac{3}{4})^{\frac{2h-2k}{k}}|\eta| + o(|\eta|)} \\ &= -\frac{h^2}{2k}\frac{1}{2h-k}\left(\varepsilon^{k/h}\right)^{\frac{2h-k}{k}} + \frac{1}{2}\left\{m_1^2 + m_2^2 - \frac{h^2}{k^2}\left(\left(\frac{1}{2}\right)^{\frac{2h-2k}{k}} + \left(\frac{3}{4}\right)^{\frac{2h-2k}{k}}\right)\right\}|\eta| + o(|\eta|).\end{aligned}$$

Conclusion: Recall that if $h \ne k$ we have $\eta = -\frac{C_1}{C_I + C_{III}}\varepsilon^\alpha + o(\varepsilon^\alpha)$ by (3.5) and $\alpha := \frac{2kh+k+h}{h}$.

- If $2h - k > 0$ then $\frac{2h-k}{h} < \frac{2kh+k+h}{h}$ and therefore, if $h \ne k$, $\eta = o(\varepsilon^{\frac{2h-k}{h}})$ for $\varepsilon \to 0$ so

$$\phi(\varepsilon) = -\frac{h^2}{2k}\frac{1}{2h-k}\varepsilon^{\frac{2h-k}{h}} + o(\varepsilon^{\frac{2h-k}{h}}).$$

  This means $\mathcal{F}(u^\eta_\varepsilon) < \mathcal{F}(u)$ so u can't be a minimizer of (3.1).

- If $2h - k \le 0$ we have:

$$\lim_{\gamma\to 0}\int_\gamma^1 u'(x)^2 dx = \frac{h^2}{k}\frac{1}{2h-k}\left[x^{\frac{2h-k}{k}}\right]_\gamma^1 = -\infty,$$

  so in this case we see that $u' \notin L^2([0,1])$ and so u can't be a minimizer of (3.1).

Therefore for all k, $h \in \mathbb{N}$ with $h \neq k$ we see that $u(x) = |x|^{\frac{h}{k}}$ can't be a minimizer of (3.1) which concludes the proof of Theorem 6. $\square$

If $h \neq k$ Theorem 6 tells us that a minimizer of Problem (3.1) is always as regular as F and G. We present now two Corollaries.

3.4 Corollaries

For both Corollaries we change the functional $\mathcal{F}$ while $\mathcal{G}$ stays the same.

3.4.1 Extension to $p > 1$

We still consider our minimization problem

$$\min_{u \in H^1(I)} \{ \mathcal{F}_p(u) \,|\, u' \in L^p(I)\, u(\pm 1) = 1 \text{ and } \mathcal{G}(u) = V_{h,k} \} \tag{3.7}$$

where we generalize the functional $\mathcal{F}_p$ as follows: For every $p > 1$ and for all $u \in H^1(I)$ with the additional assumption that $u' \in L^p(I)$ we define

$$\mathcal{F}_p(u) := \frac{1}{p} \int_I |u'(x)|^p \, dx.$$

Let us mention that we do not now if there exists a minimizer of this new problem. In Chapter 2 we worked in the space H^1. But nevertheless the theory can be generalized so that we have existence for all $p > 1$. A detailed analysis of this can be found in [1, Chapter 3]. We will therefore assume existence for the following Corollary.

Corollary 1. *For all $k, h \in \mathbb{N}$ such that $h \neq k$ there exists no singular minimizer of (3.7).*

Proof. The idea is the same as before. Since we did not change $\mathcal{G}$ we adapt the very same computation as in Theorem 6 to get an ε dependence for η. Now we again show that $\mathcal{F}_p(u_\varepsilon^\eta) < \mathcal{F}_p(u)$. We have

$$\begin{aligned}
\phi_p(\varepsilon) &:= \mathcal{F}_p(u_\varepsilon^\eta) - \mathcal{F}_p(u) \\
&= -\frac{1}{p} \int_0^{\varepsilon^{k/h}} |u'(x)|^p dx + \frac{1}{p} \int_{\frac{1}{2}}^{\frac{1}{2}+|\eta|} \left(|m_1|^p - |u'(x)|^p\right) dx + \frac{1}{p} \int_{\frac{3}{4}}^{\frac{3}{4}+|\eta|} \left(|m_2|^p - |u'(x)|^p\right) dx.
\end{aligned}$$

The only positive terms are the ones involving m_1 and m_2. So if we can show that they go faster to zero than the first term we are done. Note that $|u'(x)|^p = \frac{h^p}{k^p} x^{p\frac{h-k}{k}}$ and so

$$\frac{1}{p} \int_0^1 |u'(x)|^p dx = \underbrace{\frac{1}{p} \frac{h^p}{k^p} \frac{k}{ph + (1-p)k}}_{=:C} \left[x^{\frac{ph+(1-p)k}{k}} \right].$$

Therefore if $ph + (1-p)k \leq 0$ we have that $u' \notin L^P(I)$ and so u can't be a solution of (3.7). So we consider the case $ph + (1-p)k > 0$. This leads to

$$k < h\frac{p}{p-1}. \tag{3.8}$$

Now

- $\frac{1}{p}\int_0^{\varepsilon^{k/h}} |u'(x)|^p dx = C\varepsilon^{\frac{ph+(1-p)k}{h}}$
- $\frac{1}{p}\int_{\frac{1}{2}}^{\frac{1}{2}+|\eta|} |m_1|^p = \frac{1}{p}|m_1|^p|\eta| + o(|\eta|)$

Recall that $\eta = C''\varepsilon^\alpha + o(\varepsilon^\alpha)$ with $\alpha := \frac{2kh+h+k}{h}$ and the constant $C'' = -\frac{C_1}{C_I+C_{III}}$ depending on $h, k \in \mathbb{N}$ with $h \neq k$ from the proof of Theorem 6. Now we are interested for which $k, h \in \mathbb{N}$ we have $\frac{ph+(1-p)k}{h} < \frac{2kh+h+k}{h}$ because then we have for $\varepsilon > 0$ small that $\phi_p(\varepsilon) < 0$. This is equivalent to $ph + k - pk < 2kh + h + k$ which leads to

$$h\frac{p-1}{2h+p} < k. \tag{3.9}$$

Now together with (3.8) we have

$$h\frac{p-1}{2h+p} < k < h\frac{p}{p-1}$$

and so $(p-1)^2 < 2hp + p^2$. This is equivalent to

$$1 < 2p(h+1)$$

and since $p > 1$ we see that this is true for all $h \in \mathbb{N}$. It remains to check that (3.9) also holds true for all $k \in \mathbb{N}$. To see this we use that (3.9) holds for all $h \in \mathbb{N}$. Since

$$\frac{h(p-1)}{2h+p} = \frac{h}{h}\frac{p-1}{2+p/h} = \frac{p-1}{2+p/h}$$

and $\frac{p-1}{2+p/h} \leq \frac{p-1}{2+p} < 1$ we have that

$$\frac{p-1}{2+p} < 1 \leq k$$

since $k \geq 1$. This means that for all $p > 1$ and for all $k, h \in \mathbb{N}$ with $k < \frac{p}{p-1}$ we have $\phi_p(\varepsilon) < 0$ and so $\mathcal{F}_p(u_\varepsilon^\eta) < \mathcal{F}(u)$, meaning that u is not a minimizer of (3.7). Therefore we see that for all $k, h \in \mathbb{N}$ and for all $p > 1$ (3.7) has no singular solution. □

3.4.2 Theorem 6 for the Length

Another important example for $\mathcal{F}$ is length. We look at the following problem

$$\min_{u \in H^1(I)} \{ \mathcal{F}_L(u) \,|\, u(\pm 1) = 1 \text{ and } \mathcal{G}(u) = V_{h,k} \} \tag{3.10}$$

where

$$\mathcal{F}_L(u) := \int_I \sqrt{1 + |u'(x)|^2} dx.$$

Corollary 2. *For all $k, h \in \mathbb{N}$ such that $h \neq k$ there exists no singular solution of (3.10).*

Proof. Again we adapt the computations from Theorem 6 and define

$$
\begin{aligned}
\phi_L(\varepsilon) &:= \mathcal{F}_L(u_\varepsilon^\eta) - \mathcal{F}_L(u) \\
&= -\int_0^{\varepsilon^{k/h}} \left(1 + \frac{h^2}{k^2} x^{\frac{2h-2k}{k}}\right)^{\frac{1}{2}} dx + \int_{\frac{1}{2}}^{\frac{1}{2}+|\eta|} \left(1+m_1^2\right)^{\frac{1}{2}} dx - \int_{\frac{1}{2}}^{\frac{1}{2}+|\eta|} \left(1 + \frac{h^2}{k^2} x^{\frac{2h-2k}{k}}\right)^{\frac{1}{2}} dx \\
&\quad + \int_{\frac{3}{4}-|\eta|}^{\frac{3}{4}} \left(1+m_2^2\right)^{\frac{1}{2}} dx - \int_{\frac{3}{4}-|\eta|}^{\frac{3}{4}} \left(1 + \frac{h^2}{k^2} x^{\frac{2h-2k}{k}}\right)^{\frac{1}{2}} dx
\end{aligned}
$$

Again the only positive terms are the ones involving m_1 and m_2. We have $\int_{\frac{1}{2}}^{\frac{1}{2}+|\eta|} \left(1+m_1^2\right)^{\frac{1}{2}} dx = \left(1+m_1^2\right)^{\frac{1}{2}} |\eta|$ and $\int_{\frac{3}{4}-|\eta|}^{\frac{3}{4}} \left(1+m_2^2\right)^{\frac{1}{2}} dx = \left(1+m_2^2\right)^{\frac{1}{2}} |\eta|$. Now both of these terms go faster to zero than the first term. To see this it is enough to do a Taylor expansion for $x \geq 0$ small:

$$
\left(1 + \frac{h^2}{k^2 x^{\frac{2h-2k}{k}}}\right)^{\frac{1}{2}} = 1 + \underbrace{\frac{d}{dx}\left(1 + \frac{h^2}{k^2} x^{\frac{2h-2k}{k}}\right)^{\frac{1}{2}}\Bigg|_{x=0}}_{=0} x + o(x).
$$

So

$$
\int_0^{\varepsilon^{k/h}} \left(1 + \frac{h^2}{k^2} x^{\frac{2h-2k}{k}}\right)^{\frac{1}{2}} dx = \varepsilon^{k/h} + o(\varepsilon^{k/h})
$$

and since $|\eta| = C\varepsilon^\alpha + o(\varepsilon^\alpha)$ where $\alpha := \frac{2kh+h+k}{h} > \frac{k}{h}$ for all $k, h \in \mathbb{N}$ we have $\phi_L(\varepsilon) < 0$ for all $k, h \in \mathbb{N}$. $\square$

Chapter 4

Non-minimality of corners

4.1 Introduction

In this chapter we apply the construction of u_ε^η of Chapter 3 to $u(x) = |x|$ and a general constraint $\mathcal{G}$. The goal is to prove that if u is a singular extremal it cannot be a solution of the minimization problem for the energy.

We fix the following setting:
Let $I = [-1, 1]$ and $G : I \times \mathbb{R} \to \mathbb{R}$ be smooth. Let

$$\mathcal{F}(u) := \frac{1}{2} \int_I |u'(x)|^2 dx \qquad \text{and} \qquad \mathcal{G}(u) := \int_I G(x, u(x)) dx.$$

The minimum problem is the following:

$$\min_{u \in H^1(I)} \{ \mathcal{F}(u) \,|\, u(\pm 1) = 1 \text{ and } \mathcal{G}(u) = V \} \tag{4.1}$$

where $V \in \mathbb{R}$. We choose $V \in \mathbb{R}$ such that $\mathcal{G}(|x|) = V$. For this whole chapter we assume that $u(x) := |x|$ is a $\mathcal{G}$-singular extremal. We recall that this means that for all $\psi \in C_c^\infty(\mathring{I})$ we have

$$0 = \frac{d}{d\tau} \left(\int_I G(x, |x| + \tau\psi) dx \right) \Big|_{\tau=0}$$

and so

$$\partial_z G(x, |x|) \equiv 0 \qquad \text{on } I. \tag{4.2}$$

The result of this chapter is the following:

Theorem 7. *Let $u(x) = |x|$ and $G \in C^\infty(I \times \mathbb{R}; \mathbb{R})$ be such that there exists $\delta \in (0, 1)$ with*

$$\partial_z^2 G(0, 0) \partial_z^2 G(\delta, \delta) < 0. \tag{4.3}$$

If u is a $\mathcal{G}$-singular extremal then u is not a solution of Problem (4.1).

Remark 13. Let $h, k \in \mathbb{N}$ and $G(x, z) = \left(x^{2h} u - \frac{z^{2k+1}}{2k+1} \right)$ be the constraint of (3.1) from Chapter 3. We have $\partial_z^2 G(x, z) = -2kz^{2k-1}$ so assumption 4.3 fails to hold. So the question, whether $u(x) = |x|$ can be a minimizer of (3.1) is still open.

4.2 Proof of Theorem 7

Proof. The idea of the proof is the same as in Chapter 3. We modify u to u_ε^η and show that $\mathcal{F}(u_\varepsilon^\eta) < \mathcal{F}(u)$. As before it is enough to do the computations in $[0,1]$. Since $\partial_z^2 G$ is continuous there exists $\sigma > \delta$ such that

$$\partial_z^2 G(0,0)\partial_z^2 G(x,x) < 0$$

holds for all $x \in [\delta, \sigma)$. We define u_ε^η as follows: For all $\varepsilon > 0$ with $\varepsilon < \delta$ we fix $\eta \in \mathbb{R}$ such that $\sigma - \delta > 2|\eta|$ and

$$u_\varepsilon^\eta(x) := \begin{cases} \varepsilon & : x \leq \varepsilon \\ u(x) & : x \in [\varepsilon, \delta] \\ m_1 x + d_1 & : x \in [\delta, \delta + |\eta|] \\ u(x) + \eta & : x \in [\delta + |\eta|, \sigma - |\eta|] \\ m_2 x + d_2 & : x \in [\sigma - |\eta|, \sigma] \\ u(x) & : x \geq \sigma \end{cases}$$

where

$$m_1(\eta) := \frac{|\eta| + \eta}{|\eta|} \qquad \text{and} \qquad d_1(\eta) := \delta - \delta m_1(\eta)$$

and

$$m_2(\eta) := \frac{|\eta| - \eta}{|\eta|} \qquad \text{and} \qquad d_2 := \sigma - \sigma m_2(\eta).$$

Remark 14. For all values of η we have that $m_1, m_2 \in \{0, 2\}$. In particular, if $\eta > 0$ then $m_1 = 2$ and $m_2 = 0$ and if $\eta < 0$ then $m_1 = 0$ and $m_2 = 2$.

As a first step, we want to express η in terms of ε. For this we compute again the asymptotic behaviour of the difference of the constraint: $\mathcal{G}(u) - \mathcal{G}(u_\varepsilon^\eta)$. We start expanding the integrals in $[0, \varepsilon]$. We define

$$\theta_\varepsilon(\varepsilon) := \int_0^\varepsilon \left(G(x, u(x)) - G(x, u_\varepsilon^\eta(x))\right) dx = \int_0^\varepsilon \left(G(x,x) - G(x,\varepsilon)\right) dx.$$

We have $\theta_\varepsilon(0) = 0$ and

$$\frac{d}{d\varepsilon}\theta_\varepsilon(\varepsilon) = G(\varepsilon,\varepsilon) - G(\varepsilon,\varepsilon) + \int_0^\varepsilon \frac{d}{d\varepsilon}\left(G(x,x) - G(x,\varepsilon)\right) dx = -\int_0^\varepsilon \partial_z G(x,\varepsilon) dx,$$

were we used Leibniz's rule for the differentiation under the integral sign. Therefore we have $\theta_\varepsilon'(0) = 0$. Now

$$\theta_\varepsilon''(\varepsilon) = -\partial_z G(\varepsilon,\varepsilon) - \int_0^\varepsilon \partial_z^2 G(x,\varepsilon) dx = -\int_0^\varepsilon \partial_z^2 G(x,\varepsilon) dx$$

since $\partial_z G(\varepsilon, \varepsilon) = 0$ for all ε by (3.3). Again we have $\theta_\varepsilon''(0) = 0$, and so we compute

$$\theta_\varepsilon'''(\varepsilon) = -\partial_z^2 G(\varepsilon,\varepsilon) - \int_0^\varepsilon \partial_z^3 G(x,\varepsilon) dx.$$

Now by assumption (4.3) of the Theorem we have $\theta_\varepsilon'''(0) = -\partial_z^2 G(0,0) \neq 0$ so for all $\varepsilon > 0$ small enough we have

$$\theta_\varepsilon(\varepsilon) = \underbrace{-\frac{1}{6}\partial_z^2 G(0,0)}_{C_1}\varepsilon^3 + o(\varepsilon^3).$$

Next, we expand the integral on the interval $[\delta, \sigma]$.

This time it turns out to be convenient to compute first the asymptotic behaviour in the middle part $[\delta + |\eta|, \sigma - |\eta|]$. We define

$$\widehat{\theta}(\eta) := \int_{\delta+|\eta|}^{\sigma-|\eta|} (G(x, u(x)) - G(x, u_\varepsilon^\eta(x)))\, dx = \int_{\delta+|\eta|}^{\sigma-|\eta|} (G(x,x) - G(x, x+\eta))\, dx.$$

Let us first assume that $\eta > 0$. We have $\widehat{\theta}(0) = 0$ and

$$\begin{aligned}\frac{d}{d\eta}\widehat{\theta}(\eta) =& G(\sigma-\eta, \sigma-\eta) - G(\sigma-\eta, \sigma-\eta+\eta) - G(\delta+\eta, \delta+\eta) \\ &+ G(\delta+\eta, \delta+\eta+\eta) - \int_{\delta+\eta}^{\sigma-\eta} \partial_z G(x, x+\eta) dx.\end{aligned}$$

So

$$\widehat{\theta}'(0) = -\int_\delta^\sigma \partial_z G(x,x) dx = 0$$

by (4.2). Now

$$\begin{aligned}\widehat{\theta}''(\eta) =& -\partial_x G(\sigma-\eta, \sigma-\eta) - \underbrace{\partial_z G(\sigma-\eta, \sigma-\eta)}_{=0 \text{ by } (4.2)} + \partial_x G(\sigma-\eta, \sigma-\eta+\eta) \\ &- \partial_z G(\sigma-\eta, \sigma-\eta+\eta) - \partial_x G(\delta+\eta, \delta+\eta) - \underbrace{\partial_z G(\delta+\eta, \delta+\eta)}_{=0 \text{ by } (4.2)} \\ &+ \partial_x G(\delta+\eta, \delta+\eta+\eta) + \partial_z G(\delta+\eta, \delta+\eta+\eta) - \partial_z G(\sigma-\eta, \sigma-\eta+\eta) \\ &+ \partial_z G(\delta+\eta, \delta+\eta+\eta) - \int_{\delta+\eta}^{\sigma-\eta} \partial_z^2 G(x, x+\eta) dx\end{aligned}$$

So we see, using again (4.2), that

$$\widehat{\theta}''(0) = -\partial_x G(\sigma,\sigma) + \partial_x G(\sigma,\sigma) - \partial_x G(\delta,\delta) + \partial_x G(\delta,\delta) - \int_\delta^\sigma \partial_z^2 G(x,x) dx,$$

If $\eta < 0$ we have a minus sign in front of the remaining terms since $\frac{d}{d\eta}|\eta| = -1$ and so

$$\widehat{\theta}''(0) = \partial_x G(\sigma,\sigma) - \partial_x G(\sigma,\sigma) + \partial_x G(\delta,\delta) - \partial_x G(\delta,\delta) - \int_\delta^\sigma \partial_z^2 G(x,x) dx.$$

Therefore for all $\eta \in \mathbb{R}$ with $2|\eta| < \sigma - \delta$ we have

$$\widehat{\theta}''(0) = -\int_\delta^\sigma \partial_z^2 G(x,x)dx \neq 0$$

by assumption (4.3) of the Theorem and so

$$\widehat{\theta}(\eta) = -\underbrace{\left(\frac{1}{2}\int_\delta^\sigma \partial_z^2 G(x,x)dx\right)}_{:=C_2}\eta^2 + o(|\eta|^2).$$

We compute the asymptotic behaviour in $[\delta, \delta + |\eta|]$: Let

$$\tilde{\theta}(\eta) := \int_{\frac{1}{2}}^{\frac{1}{2}+|\eta|} \left(G(x,u(x)) - G(x,u_\varepsilon^\eta(x))\right)dx = \int_{\frac{1}{2}}^{\frac{1}{2}+|\eta|} \left(G(x,x) - G(x,m_1x+d_1)\right)dx.$$

It is enough to show that for $\eta \to 0$ we have $\tilde{\theta}(\eta) = o(\eta^2)$. Let $x \in [\delta, \delta + |\eta|]$. We have by Taylor's Theorem that

- $G(x,x) = G(\delta,\delta) + \partial_x G(\delta,\delta)(x-\delta) + \underbrace{\partial_z G(\delta,\delta)}_{=0}(x-\delta) + \partial_x^2 G(\delta,\delta)\frac{(x-\delta)^2}{2} + \partial_x \underbrace{\partial_z G(\delta,\delta)}_{=0}(x-\delta)^2 + \partial_z^2 G(\delta,\delta)\frac{(x-\delta)^2}{2} + o((x-\delta)^2)$
- $G(x,m_1x+d_1) = G(\delta,\delta) + \partial_x G(\delta,\delta)(x-\delta) + \underbrace{\partial_z G(\delta,\delta)}_{=0} m_1(x-\delta) + \partial_x^2 G(\delta,\delta)\frac{(x-\delta)^2}{2} + \partial_x \underbrace{\partial_z G(\delta,\delta)}_{=0} m_1(x-\delta)^2 + \partial_z^2 G(\delta,\delta)m_1^2\frac{(x-\delta)^2}{2} + o((x-\delta)^2)$

Notice that we used Schwarz Theorem to change the order of differentiation. Therefore

$$|G(x,x) - G(x,m_1x+d_1)| = \left|\pm\partial_z^2 G(\delta,\delta)\frac{(x-\delta)^2}{2} + o((x-\delta)^2)\right| \leq C_3^{(1)}\eta^2 + o(\eta^2)$$

on $[\delta, \delta+|\eta|]$ where $C_3^{(1)} > 0$ and the sign in front of $\partial_z G(\delta,\delta)$ depends on the value of m_1. So

$$\tilde{\theta}(\eta) = o(\eta^2).$$

For the interval $[\sigma - |\eta|, \sigma]$ the computations are almost the same. We have

$$|G(x,x) - G(x,m_2x+d_2)| = \left|\pm\partial_z^2 G(\sigma,\sigma)\frac{(\sigma-x)^2}{2} + o((\sigma-x)^2)\right| \leq C_3^{(2)}\eta^2 + o(\eta^2).$$

Now we express η in terms of ε: Let

$$\Theta(\varepsilon,\eta) := \int_0^1 \left(G(x,u(x)) - G(x,u_\varepsilon^\eta(x))\right)dx.$$

We just showed that $\Theta(\varepsilon,\eta) = -C_1\varepsilon^3 + o(\varepsilon^3) - C_2\eta^2 + o(|\eta|^2)$. Define $C := -\frac{C_1}{C_2}$. With $\Theta(\varepsilon,\eta) = 0$ we get $C_2\eta^2 + o(|\eta|^2) = -C_1\varepsilon^3 + o(\varepsilon^3)$ and so

$$\eta(\varepsilon) = \left(\pm\sqrt{C}\right)\varepsilon^{3/2} + o(\varepsilon^{3/2}). \tag{4.4}$$

Notice that by assumption (4.3) we have that $\operatorname{sgn}(C_1) = -\operatorname{sgn}(C_2)$ so $C > 0$ and therefore $\eta > 0$.

The second step is the same as in Theorem 6. We define $\phi(\varepsilon) := \mathcal{F}(u_\varepsilon^\eta) - \mathcal{F}(u)$ so

$$\begin{aligned}
\phi(\varepsilon) &= -\frac{1}{2}\int_0^\varepsilon u'(x)^2 dx + \frac{1}{2}\int_\delta^{\delta+|\eta|} \left(m_1^2 - u'(x)^2\right) dx + \int_{\sigma-|\eta|}^{\sigma} \left(m_2^2 - u'(x)^2\right) dx \\
&= -\frac{1}{2}\varepsilon + \frac{1}{2}m_1^2|\eta(\varepsilon)| - \frac{1}{2}|\eta(\varepsilon)| + \frac{1}{2}m_2^2|\eta(\varepsilon)| - \frac{1}{2}|\eta(\varepsilon)| \\
&= -\frac{1}{2}\varepsilon + 2|\eta(\varepsilon)| - \frac{1}{2}|\eta(\varepsilon)| = -\frac{1}{2}\varepsilon + \frac{3}{2}|\eta(\varepsilon)|,
\end{aligned}$$

where we used the observations made in Remark 14. Together with (4.4) we have $\phi(\varepsilon) = -\frac{1}{2}\varepsilon + o(\varepsilon)$ for $\varepsilon > 0$ small enough so we proved $\mathcal{F}(u_\varepsilon^\eta) < \mathcal{F}(u)$ meaning that u is not a minimizer of (4.1). $\square$

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