A SIMPLE PROOF OF THE COMPLETENESS THEOREM OF THE INTUITIONISTIC PREDICATE CALCULUS WITH RESPECT TO THE TOPOLOGICAL SEMANTICS

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Abstract. In this paper a simple proof of the completeness theorem of the intuitionistic predicate calculus with respect to the topological semantics is shown. From a technical point of view the proof of the completeness theorem is based on a Rasiowa-Sikirski-like theorem for the countable Heyting algebras which allows to embadd any countable Heyting algebra into a suitable topology in a such way that a countable quantity of the existing suprema are respected.

Introduction

The aim of this paper is to show a simple proof of the completeness theorem of the intuitionistic predicate calculus with respect to the usual topological semantics.

In the first section we are going to recall the main definitions of the intuitionistic predicate calculus; moreover an interpretation of the formulas of the intuitionistic predicate calculus into the open sets of a topological space is given in such a way that it will be possible to show a validity theorem.

Then, in the second section, after some introductory remarks on the Heyting algebras, we will prove a Rasiowa-Sikorski-like theorem for the countable Heyting algebras; it will be usefull to embad a countable Heyting algebra into a suitable topology in such a way that a countable quantity of the existing suprema is respected.

Finally, in the third section, the algebric results of the second section will be used to prove the completeness theorem for the topological semantics: first one constructs the Lindembaum algebra of the formulas of the intuitionistic predicate calculus, which turn out to be a countable Heyting algebra with a

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countable quantity of suprema in correspondence with the existential quantifiers, and then one embads it into a suitable topology.

The main definitions

In this section we are going to recall the main definitions on the syntactical aspects of the intuitionistic predicate calculus and on the interpretation of its formulas into the open sets of a topological space.

Hence let us suppose to construct the formulas by means of a language $\mathcal{L}$ which contains a countable quantity of variables, a (possibly empty) set of signs for functions and for constants, starting from the usual inductive construction of the terms and going on with the construction of the formulas by using the propositional connetives $\bot, \wedge, \vee, \rightarrow$ and the quantifiers $\forall, \exists$.

Finally, let us introduce a sequent calculus for the intuitionistic predicate logic, where by Γ we mean a set of formulas and A, B and C are formulas [Takeuti 75]:

(axioms) $$\frac{}{\Gamma, A \vdash A} \qquad \frac{}{\Gamma, \bot \vdash A}$$

(and) $$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C} \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

(or) $$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

(implication) $$\frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \rightarrow B \vdash C} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$$

(univ. quant.) $$\frac{\Gamma, A[x := t] \vdash C}{\Gamma, \forall x.A \vdash C} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \forall x.A}$$

(esist. quant.) $$\frac{\Gamma, A \vdash C}{\Gamma, \exists x.A \vdash C} \qquad \frac{\Gamma \vdash A[x := t]}{\Gamma \vdash \exists x.A}$$

where, as usual, in the universal quantification introduction rule and in the existential quantification elimination rule we mean that the quantified variable does not appear in the conclusion.

We can now define a valuation $V(-)$ of the formulas of the intuitionistic predicate calculus into the open sets of a topological space τ in such a way that if $A_1, \ldots, A_n \vdash B$ then $V(A_1) \cap \cdots \cap V(A_n) \subseteq V(B)$.

In fact, let us consider any structure suitable for the language $\mathcal{L}$, $\mathcal{D} \equiv \langle D, \mathcal{R}, \mathcal{F}, \mathcal{C}, \tau \rangle$ where τ is a topological space which we will use to assign the *truth values* when interpreting of the formulas, D is a set on which the terms of $\mathcal{L}$ will be interpreted, $\mathcal{R}$ is a set of funtions, each one of a suitable arity n, from D^n to the open sets of τ on which the atomic predicates of $\mathcal{L}$ will be interpreted, $\mathcal{F}$ is a set of functions, each one of the suitable arity n, from D^n to D on which the signs for functions of $\mathcal{L}$ will be interpreted and, finally,

$\mathcal{C}$ is a subset of the set D on which the constants of the language $\mathcal{L}$ will be interpreted.

Then we can define the interpretation $V(-)$ as follows. Let us suppose we already have given an interpretation for all the terms of $\mathcal{L}$, i.e. $V(x_i) \in D$ for any variable x_i, $V(c_i) \in \mathcal{C}$ for any constant c_i and $V(f(t_1, \ldots, t_m)) = V(f)(V(t_1), \ldots, V(t_m)$, where $V(f) \in \mathcal{F}$, for any sign for function of $\mathcal{L}$, and let us show how $V(-)$ can be extended to all the formulas, provided that $V(P) \in \mathcal{R}$.

$$\begin{aligned}
V(P(t_1, \ldots, t_m)) &= V(P)(V(t_1), \ldots, V(t_m)) \\
V(\bot) &= \emptyset \\
V(A \wedge B) &= V(A) \cap V(B) \\
V(A \vee B) &= V(A) \cup V(B) \\
V(A \to B) &= \cup\{\mathcal{O} \in \tau : V(A) \cap \mathcal{O} \subseteq V(B)\} \\
V(\forall x.A) &= Int(\cap_{d \in D} V^{[x:=d]}(A)) \\
V(\exists x.A) &= \cup_{d \in D} V^{[x:=d]}(A)
\end{aligned}$$

where by $V^{[x:=d]}$ we mean the valuation which coincides with the valuation $V(-)$ almost everywhere but for the variable x which is interpreted in the element $d \in D$.

It is immediate to prove, by induction on the complexity of the formula A, that the valuation of A only depends on the valuation of the variables which appear in A and that $V(A[x := t]) = V^{[x:=V(t)]}(A)$. Moreover we have the following theorem.

Theorem: Validity. *For any valuation $V(-)$ of the formulas of the intuitionistic predicate calculus in a structure $\mathcal{D}$, if*

$$A_1, \ldots, A_n \vdash B$$

then

$$V(A_1) \cap \cdots \cap V(A_n) \subseteq V(B)$$

Proof. The proof is by induction on the derivation of $A_1, \ldots, A_n \vdash B$. Most of the cases are trivial, so here we only show the the inductive step for the universal quantifier rules. Let us suppose that $\Gamma, A[x := t] \vdash C$ then, by inductive hypothesis, $V(\Gamma) \cap V(A[x := t]) \subseteq V(C)$ hence $V(\Gamma) \cap V(\forall x.A) \subseteq V(C)$ since $V(\forall x.A) = Int(\cap_{d \in D} V^{[x:=d]}(A)) \subseteq V^{[x:=V(t)]}(A) = V(A[x := t])$. On the other hand if $\Gamma \vdash A$ then, by inductive hypothesis, for any $d \in D$, $V^{[x:=d]}(\Gamma) \subseteq V^{[x:=d]}(A)$, but $V^{[x:=d]}(\Gamma) = V(\Gamma)$, since, by hypothesis, x does not appear in Γ, and hence $V(\Gamma) \subseteq \cap_{d \in D} V^{[x:=d]}(A)$, so $V(\Gamma) \subseteq Int(\cap_{d \in D} V^{[x:=d]}(A)) = V(\forall x.A)$ since $V(\Gamma)$ is an open set.

It is convenient to observe that in the interpretation of the connective $\rightarrow$ it is not necessary to consider *all* of the open sets of the topology τ but it is sufficient to consider only those of any of its base $\mathcal{B}_\tau$, i.e. we can put

$$V(A \rightarrow B) = \cup\{b \in \mathcal{B}_\tau : V(A) \cap b \subseteq V(B)\}.$$

In fact $\cup\{\mathcal{O} \in \tau : V(A) \cap \mathcal{O} \subseteq V(B)\} = \cup\{b \in \mathcal{B}_\tau : V(A) \cap b \subseteq V(B)\}$: to prove that $\cup\{\mathcal{O} \in \tau : V(A) \cap \mathcal{O} \subseteq V(B)\} \subseteq \cup\{b \in \mathcal{B}_\tau : V(A) \cap b \subseteq V(B)\}$ it is sufficient to note that if $x \in \cup\{\mathcal{O} \in \tau : V(A) \cap \mathcal{O} \subseteq V(B)\}$ then there exists $\mathcal{O} \in \tau$ such that $x \in \mathcal{O}$ and $V(A) \cap \mathcal{O} \subseteq V(B)$, but $\mathcal{O} = \cup\{b \in \mathcal{B}_\tau : b \subseteq \mathcal{O}\}$ and hence there is $b \in \mathcal{B}_\tau$ such that $x \in b$ and $b \subseteq \mathcal{O}$, so that $V(A) \cap b \subseteq V(A) \cap \mathcal{O} \subseteq V(B)$, and hence $x \in \cup\{b \in \mathcal{B}_\tau : V(A) \cap b \subseteq V(B)\}$; the other inclusion is straightforword since any element of the base $\mathcal{B}_\tau$ is an open set.

A similar property holds for the interpretation of the universal quantifier. In fact let U be any subset of the set X on which the topology τ is defined; moreover let $\mathcal{B}_\tau$ be one of the base of τ, then

$$\begin{aligned} Int(U) &= \cup\{\mathcal{O} \in \tau : \mathcal{O} \subseteq U\} \\ &= \cup\{b \in \mathcal{B}_\tau : b \subseteq U\} \end{aligned}$$

can be proved as above and hence we can put

$$V(\forall x.A) = \cup\{b \in \mathcal{B}_\tau : b \subseteq \cap_{d \in D} V^{[x:=d]}(A)\}.$$

A Rasiowa-Sikorski-like theorem for the countable Heyting Algebras.

In this paragraph we will set up the technical device that we need in order to prove the completeness theorem. In particular we will show how any countable Heyting algebra, $\mathcal{H} \equiv \langle H, 0, 1, +, \bullet, \rightarrow\rangle$, can be embedded into a suitable topology in such a way that a countable quantity of suprema are respected.

Let us begin by recalling some standard definitions and results. A Heyting algebra $\mathcal{H} \equiv \langle H, 0, 1, +, \bullet, \rightarrow\rangle$ is a structure such that $\langle H, 0, 1, +, \bullet\rangle$ is a distributive lattice with 0 and 1 and that for all $x, y, z \in H, x \bullet y \leq z$ iff $x \leq y \rightarrow z$, where the order relation $\leq$ is defined by putting $x \leq y \equiv x = x \bullet y$.

Note that, because of the implication operation, $\bullet$ is distributive over all the existing suprema; in fact $x \bullet \vee\{t : t \in T\} \leq \vee\{x \bullet t : t \in T\}$ since, for any $t \in T$, $x \bullet t \leq \vee\{x \bullet t : t \in T\}$ which gives $t \leq x \rightarrow \vee\{x \bullet t : t \in T\}$, so that $\vee\{t : t \in T\} \leq x \rightarrow \vee\{x \bullet t : t \in T\}$ and hence $x \bullet \vee\{t : t \in T\} \leq \vee\{x \bullet t : t \in T\}$; the other implication is trivial.

A straightforword set-theoretic example of Heyting algebra is obtained by considering the open sets of a topological space τ over a set X, whose base is $\mathcal{B}_\tau$. In fact we can put

$$\begin{aligned} 0_\tau &\equiv \emptyset \\ 1_\tau &\equiv X \\ \mathcal{O}_1 \bullet_\tau \mathcal{O}_2 &\equiv \mathcal{O}_1 \cap \mathcal{O}_2 \\ \mathcal{O}_1 +_\tau \mathcal{O}_2 &\equiv \mathcal{O}_1 \cup \mathcal{O}_2 \\ \mathcal{O}_1 \to_\tau \mathcal{O}_2 &\equiv \cup\{\mathcal{O} \in \tau : \mathcal{O}_1 \cap \mathcal{O} \subseteq \mathcal{O}_2\} = \cup\{b \in \mathcal{B}_\tau : \mathcal{O}_1 \cap b \subseteq \mathcal{O}_2\} \end{aligned}$$

Our aim in this work is to show that, at least for the countable Hayting algebras, topologies are the paradigmatic example of Heyting algebras as well as families of subsets are the paradigmatic example of Boolean algebras, i.e. any countable Heyting algebra can be embedded into a suitable topology.

Definition 2.1: Filter. Let $\mathcal{H}$ be a Heyting algebra. Then a subset F of H is called a *filter* if:

$$1 \in F \qquad \frac{x \in F \quad x \leq y}{y \in F} \qquad \frac{x \in F \quad y \in F}{x \bullet y \in F}.$$

Definition 2.2: Prime filter. Let $\mathcal{H}$ be a Heyting algebra and F one of its filters. Then F is called a *prime filter* if whenever $x + y \in F$ then $x \in F$ or $y \in F$.

In this work we are interested in a particular class of prime filters, i.e. those which respect a countable quantity of subsets of H.

Definition 2.3. Let $\mathcal{H}$ be a Heyting algebra, F one of its filters and T a subset of H which has a supremum in $\mathcal{H}$. Then F *respects* T if whenever $\vee T \in F$ there exists $b \in T$ such that $b \in F$.

Now we want to show that there exist prime filters which respect a countable quantity of subsets $T_1, \ldots, T_n, \ldots$. To this purpose we can prove the following theorem which guarantees the existence in any Heyting algebra of a filter (not necessarily a prime filter) which respects all of the subsets $T_1, \ldots, T_n, \ldots$.

Theorem 2.4. *Let $\mathcal{H}$ be a Heyting algebra, $x, y \in H$ such that $x \not\leq y$ and $T_1, \ldots, T_n, \ldots$ a countable quantity of subsets of H which have a supremum in H. Then there exists a filter F of $\mathcal{H}$ which contains x, does not contain y and respects all of the subsets $T_1, \ldots, T_n, \ldots$.*

Proof. We can classically construct such a filter in a countable number of steps, starting from the filter[1] $F_0 = \uparrow x \equiv \{z \in H : x \leq z\}$ and going on by extending it to a filter which respects all of the subsets $T_1, \ldots, T_n, \ldots$. Let

[1] It is easy to check that F_0 is indeed a filter of $\mathcal{H}$.

us first construct a new countable list $W_1, \ldots, W_m, \ldots$ of subsets of H out of the list $T_1, \ldots, T_n, \ldots$ in such a way that any subset T_i appears a countable number of times among $W_1, \ldots, W_m, \ldots$; one can for instance consider the list $W_1 = T_1, W_2 = T_1, W_3 = T_2, W_4 = T_1, W_5 = T_2, W_6 = T_3, \ldots$.

Now put $c_0 = x$, and hence $F_0 = \uparrow c_0$, and suppose, by inductive hypothesis that we have constructed an element c_n such that $c_n \not\leq y$ and we have defined $F_n = \uparrow c_n$, then put

$$c_{n+1} = \begin{cases} c_n & if \ \vee W_n \notin F_n \\ c_n \bullet b_n & if \ \vee W_n \in F_n \end{cases}$$

where b_n is an element of W_n such that $c_n \bullet b_n \not\leq y$; in fact such an element exists because in the latter case $\vee W_n \in F_n = \uparrow c_n$, and hence $c_n \leq \vee W_n$, then if for all $b \in W_n, c_n \bullet b \leq y$ then $c_n = c_n \bullet \vee W_n = \vee_{b \in W_n} c_n \bullet b \leq y$ which is contrary to the inductive hypothesis.

Now we can define the filter $F_{n+1} = \uparrow c_{n+1}$ and it is immediate to check that $F_n \subseteq F_{n+1}$ and that $y \notin F_{n+1}$ since $c_{n+1} \not\leq y$. Finally we put $F = \cup_{i \in \omega} F_i$. Then F is a filter since it is the union of a chain of filters which are contained one into another; moreover $x \in F$ since $x \in F_0 \subseteq F$ while $y \notin F = \cup_{i \in \omega} F_i$ otherwise there would be an $i \in \omega$ such that $y \in F_i$ which is contrary to the way we have constructed the filters; moreover F respects all of the subsets $T_1, \ldots, T_n, \ldots$ since if $\vee T_n \in F = \cup_{i \in \omega} F_i$ then there is an $i \in \omega$ such that $\vee T_n \in F_i$ and hence, since any T_n appears a countable number of times in the list $W_1, \ldots, W_m, \ldots$, for some $h \geq i$ it happens that $W_h = T_n$ and hence $\vee W_h = \vee T_n \in F_i \subseteq F_h$ and so there exists $b_h \in T_n$ such that $b_h \in F_{h+1} \subseteq F$.

It should be noted that the theorem ensures the existence of a proper filter which respects a countable quantity of suprema without any condition at all, since $1 \not\leq 0$ always holds and 1 is contained in any filter.

The previous theorem can be used to construct a *prime* filter which respects the countable quantity of suprema of the subsets $T_1, \ldots, T_n, \ldots$ in case we are dealing with a *countable* Heyting algebra.

Corollary 2.5. *Let $\mathcal{H}$ be a countable Heyting algebra, $x, y \in H$ such that $x \not\leq y$ and $T_1, \ldots, T_n, \ldots$ a countable quantity of subsets of H which have a supremum in H. Then there exists a prime filter which contains x, does not contain y and respects all of the subsets $T_1, \ldots, T_n, \ldots$.*

Proof. We have only to observe that in this case there is a countable quantity of binary suprema and hence we can use the previous theorem in order to obtain a filter which respects the countable quantity of binary suprema and the countable quantity of subsets $T_1, \ldots, T_n, \ldots$. It is then obvious that this filter is prime because it respects all the binary suprema.

The existence of prime filters which respect a given countable quantity of suprema is the key point to construct a topology in which a countable

Heyting algebra $\mathcal{H}$ can be embedded using a morphism which respects all of the countable quantity of suprema of the subsets $T_1, \ldots, T_n, \ldots$. In fact, let us consider the collection

$$\begin{aligned} Pt(\mathcal{H}) = \{P : & P \text{ prime filter of } \mathcal{H} \\ & \text{which respects all of the subsets } T_1, \ldots, T_n, \ldots\} \end{aligned}$$

and the topology $\tau_{\mathcal{H}}$ on $Pt(\mathcal{H})$ whose base $\mathcal{B}_{\tau_{\mathcal{H}}}$ are the subsets $ext(x) = \{P \in Pt(\mathcal{H}) : x \in P\}$ for $x \in H$. It is easy to see that $\mathcal{B}_{\tau_{\mathcal{H}}}$ is a base for a topology since

$$\begin{aligned} ext(0) &= \{P \in Pt(\mathcal{H}) : 0 \in P\} = \emptyset \\ ext(1) &= \{P \in Pt(\mathcal{H}) : 1 \in P\} = Pt(\mathcal{H}) \\ ext(x \bullet y) &= \{P \in Pt(\mathcal{H}) : x \bullet y \in P\} \\ &= \{P \in Pt(\mathcal{H}) : x \in P\} \cap \{P \in Pt(\mathcal{H}) : y \in P\} = ext(x) \cap ext(y) \end{aligned}$$

since $x \bullet y$ is contained in a filter P if and only if $x \in P$ and $y \in P$.

In order to extend the map $ext(-)$ to a full morphism of the Heyting algebra $\mathcal{H}$, it is convinient to show first that

$$ext(x) \subseteq ext(y) \text{ iff } x \leq y.$$

In fact if $x \leq y$ then, for any filter $P \in ext(x)$, i.e. such that $x \in P$, we have that $y \in P$, i.e. $P \in ext(y)$; on the other hand if $x \not\leq y$ then corollary 2.5 shows that there exists a prime filter P which respects all the suprema we are considering and such that it contains x and does not contain y, i.e. such that $P \in ext(x)$ but $P \notin ext(y)$.

We are now in the position to prove that $ext(-)$ is an embadding of the Heyting algebra $\mathcal{H}$ into the topology $\tau_{\mathcal{H}}$, in fact

$$\begin{aligned} ext(x + y) &= \{P \in Pt(\mathcal{H}) : x + y \in P\} \\ &= \{P \in Pt(\mathcal{H}) : x \in P\} \cup \{P \in Pt(\mathcal{H}) : y \in P\} = ext(x) \cup ext(y) \end{aligned}$$

since $x + y$ is contained in a prime filter P if and only if $x \in P$ or $y \in P$, and

$$\begin{aligned} ext(x \to y) &= \{P \in Pt(\mathcal{H}) : x \to y \in P\} \\ &= \cup\{ext(z) : z \leq x \to y\} \\ &= \cup\{ext(z) : z \bullet x \leq y\} \\ &= \cup\{ext(z) : ext(z \bullet x) \subseteq ext(y)\} \\ &= \cup\{ext(z) : ext(z) \cap ext(x) \subseteq ext(y)\} = ext(x) \to_{\tau_{\mathcal{H}}} ext(y) \end{aligned}$$

Moreover, due to our choice in the definition of the points in $Pt(\mathcal{H})$, we can show that $ext(-)$ also respects all the suprema of the subsets $T_1, \ldots, T_n, \ldots$ that we are considering.

$$\begin{aligned} ext(\vee_{b \in T_i} b) &= \{P \in Pt(\mathcal{H}) : \vee_{b \in T_i} b \in P\} \\ &= \cup_{b \in T_i} \{P \in Pt(\mathcal{H}) : b \in P\} = \cup_{b \in T_i} ext(b) \end{aligned}$$

since $\vee_{b \in T_i} b$ is contained in a prime filter P which respects all of the sets $T_1, \ldots, T_n, \ldots$ if and only if there is an element $b \in T_i$ such that $b \in P$.

Finally, not only the suprema of the subsets $T_1, \ldots, T_n, \ldots$ are respected but also all the existing infima since

$$\begin{aligned} ext(\wedge_{i \in I} x_i) &= \{P \in Pt(\mathcal{H}) : \wedge_{i \in I} x_i \in P\} \\ &= \cup\{ext(z) : z \leq \wedge_{i \in I} x_i\} \\ &= \cup\{ext(z) : z \leq x_i, \text{ for any } i \in I\} \\ &= \cup\{ext(z) : ext(z) \subseteq ext(x_i), \text{ for any } i \in I\} \\ &= \cup\{ext(z) : ext(z) \subseteq \cap_{i \in I} ext(x_i)\} = Int(\cap_{i \in I} ext(x_i)) \end{aligned}$$

To conclude the proof that ext is an embedding of $\mathcal{H}$ into $\tau_{\mathcal{H}}$ we have to show that it is injective, i.e. that if $x \neq y$ then $ext(x) \neq ext(y)$. But this is straightforword since if $x \neq y$ then $x \not\leq y$ or $y \not\leq x$ and hence $ext(x) \not\subseteq ext(y)$ or $ext(y) \not\subseteq ext(x)$.

The completeness theorem

Now we are going to apply the algebric results of the previous section to show the completeness theorem for the topological semantics of the intuitionistic predicate calculus.

First of all one observes that

$$A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$$

is an equivalence relation between the formulas of the intuitionistic predicate calculus and that the rules we have stated in the first section are exactly what is needed to prove that $\leftrightarrow$ is a congruence with respect to all the propositional connectives and the quantifiers. Hence by putting

$$[A] \equiv \{B : \vdash A \leftrightarrow B\}$$

we obtain a countable quantity of equivalence classes which we can turn into a Heyting algebra $\mathcal{P}$ by putting:

$$\begin{aligned} O_{\mathcal{P}} &\equiv [\bot] \\ 1_{\mathcal{P}} &\equiv [\bot \rightarrow \bot] \\ [A] \bullet_{\mathcal{P}} [B] &\equiv [A \wedge B] \\ [A] +_{\mathcal{P}} [B] &\equiv [A \vee B] \\ [A] \rightarrow_{\mathcal{P}} [B] &\equiv [A \rightarrow B] \end{aligned}$$

so that there are only a countable quantity of suprema and infima in correspondence with the existential and universal quantifiers

$$\wedge_{t\in Term}[A[x := t]] \equiv [\forall x.A]$$
$$\vee_{t\in Term}[A[x := t]] \equiv [\exists x.A]$$

where we mean that $Term$ is the set of all the terms of the language we are considering.

Since $\mathcal{P}$ is a countable Heyting algebra it is possible to embad it into the topology $\tau_{\mathcal{P}}$ in such a way that the countable quantity of existing suprema, i.e. those corresponding to the existential quantifiers, are respected.

Hence we can define a valuation $V^*(-)$ of the formulas of the intuitionistic predicate calculus by using the topology $\tau_{\mathcal{P}}$. In fact, let us consider the structure $\mathcal{D}_{\mathcal{P}} = \langle Term, \mathcal{R}_{\mathcal{P}}, \mathcal{F}_{\mathcal{P}}, \mathcal{C}_{\mathcal{P}}, \tau_{\mathcal{P}}\rangle$, where $\mathcal{R}_{\mathcal{P}}$ is the set of the functions from $Term^n$ to $\tau_{\mathcal{P}}$, defined in correspondence to the atomic predicate $P_i^{(n)}$ of the language we are considering by putting $V^*(P_i^{(n)})(t_1,\ldots,t_n) = ext([P_i^{(n)}(t_1,\ldots,t_n)])$, $\mathcal{F}_{\mathcal{P}}$ is the set of the functions from $Term^n$ to $Term$, defined in correspondence with the sign for function $f_i^{(n)}$ of the language we are considering by putting $V^*(f_i^{(n)})(t_1,\ldots,t_n) = f_i^{(n)}(t_1,\ldots,t_n)$ and $\mathcal{C}_{\mathcal{P}}$ is the set of the constants c of the language we are considering so that $V^*(c) = c$.

Moreover let us suppose that, for any variable x_i, we put $V^*(x_i) = x_i$ then, for any term $t \in Term$ we obtain that $V^*(t) = t$ and hence we can define the valuation $V^*(-)$ on all the formulas simply by putting

$$V^*(A) \equiv ext([A]).$$

In fact it is easy to show that this position define an interpretation: all the inductive steps for the propositional connectives are straightforward so let us here verify only those for the quantifiers (note that $V^*(A[x := t]) = V^{*[x:=t]}(A)$).

$$\begin{aligned}
V^*(\exists x.A) &= ext([\exists x.A]) \\
&= ext(\vee_{t\in Term}[A[x := t]]) \\
&= \cup_{t\in Term} ext([A[x := t]]) \\
&= \cup_{t\in Term} V^*(A[x := t]) = \cup_{t\in Term} V^{*[x:=t]}(A) \\
V^*(\forall x.A) &= ext([\forall x.A]) \\
&= ext(\wedge_{t\in Term}[A[x := t]]) \\
&= Int(\cap_{t\in Term} ext([A[x := t]])) \\
&= Int(\cap_{t\in Term} V^*(A[x := t])) = Int(\cap_{t\in Term} V^{*[x:=t]}(A))
\end{aligned}$$

Now we can conclude the proof of the completeness theorem since

$$\begin{aligned} A_1, \ldots, A_n \vdash B & \text{ iff } A_1 \wedge \cdots \wedge A_n \vdash B \\ & \text{ iff } [A_1 \wedge \cdots \wedge A_n] \leq [B] \\ & \text{ iff } [A_1] \bullet \cdots \bullet [A_n] \leq [B] \\ & \text{ iff } ext([A_1]) \cap \cdots \cap ext([A_n]) \subseteq ext([B]) \\ & \text{ iff } V^*(A_1) \cap \cdots \cap V^*(A_n) \subseteq V^*(B) \end{aligned}$$

and so we have proved that there is a structure $\mathcal{D}_{\mathcal{P}}$, based on the topology $\tau_{\mathcal{P}}$, and a valuation $V^*(-)$ such that $V^*(A_1) \cap \cdots \cap V^*(A_n) \not\subseteq V^*(B)$ if $A_1, \ldots, A_n \not\vdash B$ which is classically equivalent to the completeness theorem.

Theorem 3.3: Completeness. *If, for any structure $\mathcal{D}$ and for any valuation $V(-)$ of the formulas of the intuitionistic predicate calculus into $\mathcal{D}$, $V(A_1) \cap \cdots \cap V(A_n) \subseteq V(B)$ holds then $A_1, \ldots, A_n \vdash B$.*

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