Algebraic flows and their entropy

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A *flow* in a category $\mathfrak{X}$ is an object X of $\mathfrak{X}$ provided with an endomorphism $T : X \to X$ in $\mathfrak{X}$. In most of the cases $\mathfrak{X}$ will be the category of (topological) groups and (continuous) group homomorphisms, the category of right modules over a ring R and the R-module homomorphisms, or just the category of topological (resp., measure) spaces and continuous (resp., measure preserving) maps. An isomorphism between two flows $T : X \to X$ and $S : Y \to Y$ is an isomorphism $\xi : X \to Y$ in $\mathfrak{X}$ such that $\xi^{-1} \circ S \circ \xi = T$.

A fundamental numerical invariant used to classify the flows up to isomorphism is the entropy. It was introduced in ergodic theory by Kolmogorov and Sinai in 1958, and in topological dynamics by Adler, Konheim, and McAndrew [1]. These authors proposed also a brief general scheme for defining *algebraic* entropy in the context of abelian groups, developed further in [11, 4]. Since this approach was appropriate only for torsion groups, a modification was proposed by Peters [7] in the case of non-torsion abelian groups. A second modification was proposed in [2], since Peters' approach works only for monomorphisms. Adjoint (dual) entropy in abelian groups was introduced in [3, 6]. The algebraic entropy was extended to the context of modules by Salce and Zanardo [9]. In all these cases the entropy is intended to measure the "chaos" or "disorder" created by the discrete dynamical system. Recently Salce, Vamos and Virili [8] found a fruitful connection between the algebraic entropy of module endomorphisms and multiplicities of length functions defined by Vámos in the sixties [10].

The aim of this lecture is to expose the unifying approach from [5] covering all notions of entropy mentioned above by using a single one $h_{\mathfrak{S}}$, defined for endomorphisms in he category $\mathfrak{S}$ of *normed commutative semigroups*. Once the entropy $h_{\mathfrak{S}}$ is defined, one can easily build a natural functor F from each of the above mentioned categories $\mathfrak{X}$ (assigning an appropriate normed semigroup FX to every object X of $\mathfrak{X}$), so that the specific entropy of a self-map T in $\mathfrak{X}$ can be obtained as the entropy $h_{\mathfrak{S}}(FT)$ in $\mathfrak{S}$. This approach simultaneously covers the existing notions of entropy in the various categories [1, 2, 3, 4, 6, **?**, 7, 8, 9, 11] and allows for a transparent uniform treatment of all these notions of entropy.

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