Minimizing capital injections by investment and reinsurance for a piecewise deterministic reserve process model

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Abstract

We consider the possibility for an insurance company to rely on capital injections to bring the reserve back to a given level if it has fallen below it and study the problem of dynamically choosing the reinsurance level and the investment in the financial market in order to minimize the expected discounted total amount of capital injection. The reserve process is described by a piecewise deterministic process, where the random discontinuities are triggered by the arrival of a claim or by a change in the prices of the risky assets in which the company invests. The capital injections, combined with the specific model, make the problem non-linear and difficult to solve via an HJB approach. The emphasis here is on making the actual computation of a solution possible by value iteration combined with an approximation based on discretization. This leads to a nearly optimal solution with an approximation that can be made arbitrarily precise. Numerical results show the feasibility of the proposed approach.

Keywords: Reinsurance and investment, Minimizing capital injections, Value iteration, Approximation methods.

1 Introduction

A traditional criterion in Insurance is the minimization of the ruin probability and, among the tools to achieve this, there is the choice of the reinsurance level and, possibly the investment in the financial market. If one considers the possibility of capital injections, for example by the shareholders, to bring the level of the reserve (below we shall refer to it as wealth process) back to a given level if it has fallen below it, then a more recent criterion is (see e.g.[10],[11]) the minimization of the expected, discounted cumulative amount of capital injections. In this paper we shall consider the dynamic stochastic optimization problem consisting of minimizing this latter criterion by an optimal choice of the reinsurance level and of the investment in the

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financial market; our study could be extended to include also other control variables, such as dividend payments. Without loss of generality we shall assume a zero-level, below which to inject capital in order to bring the wealth process back to this level. We do not consider also a possible dual approach for the case of negative risk sums (see [2], [16]).

As dynamic model for the wealth process we consider a model that was inspired by [24] and then further developed in [9] and [23]. It leads to wealth processes of the type of piecewise deterministic processes (see e.g. [8], [5]). We shall in fact assume that the wealth evolves as a process with piecewise deterministic trajectories, where the random discontinuities are triggered either by the arrival of a claim or by a change in the prices of the assets in which the company invests. It corresponds to assuming that the asset prices do not follow continuous trajectories, but rather change at random points in time and by tick size. This happens for example for high frequency trading with frequent and small price changes, but we may also assume that a price change is registered if it exits from a given band around some predetermined reference values, which leads to less frequent and larger price changes. For such a model it follows that the control actions (level of reinsurance, rebalancing of the investment portfolio) have to be made only at the jump times as a consequence of a claim or a change in the asset prices.

To solve the resulting dynamic optimization problem, we may view our model either in discrete or continuous time. In the latter case one may consider an HJB approach using either a diffusion approximation or viscosity solutions (e.g. [3]). An HJB appproach is however not particularly suitable in our case (see also the discussion in section 2.1.7 in [9]) and turns out to be computationally rather demanding. Remaining always in the context of the Dynamic Programming (DP) principle, here we opt for a more direct approach, based on the contraction property of the DP operator and leading to a Value Iteration algorithm (VI) of the type used in discrete time, infinite-horizon Markovian decision problems. In fact, although for our problem we shall consider a finite horizon, as it is more natural in economic activities, the number of possible jump times is not bounded from above. On the other hand, viewing our problem in discrete time, the time instants are not deterministically given, but random. A Value Iteration algorithm for a model analogous to the one in this paper was already studied in [9], but for a more standard objective function. Assuming, as we shall do without loss of generality, a zero level to trigger the capital injections, the amount of the injection is given by the negative part of the wealth process. This induces a non-linearity in the problem that is not easy to deal with.

The Value Iteration algorithm, that we shall derive, is quite general and can include problems with additional control variables such as payments of dividends. To allow for feasible computations, in our Value Iteration algorithm we shall use an approximating model based on the discretization of time and of the values of the wealth process and show the convergence of the approximations. There is a well-established procedure to this effect (see e.g. [18]): one starts by restricting the variables to a compact set so that the discretization leads to a finite number of possible values. The specific structure of our problem leads however to various difficulties if one tries a direct application of this procedure. Overcoming the difficulties requires various results that make up a great portion of the technical part of the paper. Although there are some recent more sophisticated methods to improve the discretization in view of a better convergence, namely the so-called optimal quantization (see e.g.[21]), for our problem it can not be applied straightforwardly and the simpler uniform discretization turned out to be satisfactory. Our approach could however be extended to include also optimal quantization and an attempt in this direction has been made in ([4]). Finally, in order to show the feasibility of our approach and to draw some qualitative conclusions, we also show examples of numerical calculations where, in addition to a time and space discretization, we perform also a control discretization.

In the next Section 2 we present a formal description of the model and of the problem. Section 3 contains preliminary definitions and results. Section 4 is intended to set the stage for the application of the Value Iteration algorithm to our problem. In Section 5 we then derive the approximate Value Iteration approach, by which the solution to the given problem becomes feasible. Numerical results are reported in Section 6 and an Appendix contains technical and longer proofs.

2 Model and objective

We consider a risk/reserve process X_t of a company, typically an insurance company, that here we shall call wealth process and that results from deterministic premia collected by the company, from stochastic claim payments and from investment in the financial market where, for simplicity, we assume investment in a single risky asset only. We allow the evolution of the wealth process to be controlled by reinsurance in addition to the investment. A standard criterion for the choice of the controls is the minimization of the ruin probability (for a model analogous to the one studied here, this was considered in [23]). Here, following some more recent literature (see e.g. [10],[11]) we do not consider the ruin probability but do not allow the wealth process to become negative by assuming that, as soon as it becomes negative, the shareholders inject sufficient capital to bring it back to zero. The zero level is chosen for convenience, but one may consider any other solvency margin. Given a finite time horizon $T > 0$, our criterion will then be the minimization of the expected value of the (discounted) cumulative injected capital up to the horizon T.

Claims occur at random points in time and also their sizes are random, while asset prices are usually modeled as continuous time processes. As mentioned in the Introduction, here we let the prices change at discrete random points in time (see e.g.[12],[14],[22]), with their sizes being random as well. This will allow us to consider the timing of the events (arrivals of claims and price changes) to be triggered by a single continuous-time semi-Markov process. Such a formulation was already considered in Schäl (see [24],[25]), where the total number of random events is fixed so that the horizon is random as well. Following [9] and [23] we consider here a fixed finite time horizon as a more natural choice, which on the other hand implies that also the number and not only the timing of the events becomes random and this makes the problem non-standard.

Since between event times the situation for the company does not change, we shall determine the controls (level of reinsurance and amount of investment) only at the event times.

On a more formal basis we model the event times with a point process $\{T_n\}_{n\geq 0}$, where T_n denotes the time of the nth event with $T_0 = 0$. We assume them to be non-explosive, namely $\lim_{n\to\infty} T_n = +\infty$ a.s. We denote by $S_n := T_n - T_{n-1}$ the waiting time between the $(n-1)$st and the nth event, assuming that $\{S_n\}_{n\geq 0}$ are i.i.d. with distribution function $G(s)$ having support in the positive half line and such that $P\{T_n \leq T\} > 0$ for all $n \geq 0$. Put

$$N_t = \max\{n : T_n \leq t\} = \sum_{n=1}^{+\infty} 1_{\{T_n \leq t\}}, \ \forall t \in [0, T] \tag{1}$$

and consider (see [25]) also a piecewise constant and right continuous process $\{K_t\}_{t\in[0,T]}$ which identifies the two types of events: if at T_n a claim arrives, then $K_{T_n} = 0$; if a price change occurs, then $K_{T_n} = 1$. This process is supposed to be a Markov process that results as a special case of a more general semi-Markov process, where the transition matrix of the embedded Markov chain $\{K_{T_n}\}_{n\geq 1}$ is of the form

$$\Pi = \begin{pmatrix} p_{0,0} & p_{0,1} \\ p_{1,0} & p_{1,1} \end{pmatrix} \tag{2}$$

and such that

$$\mathbb{P}\left\{T_n - T_{n-1} \leq s, K_{T_n} = j | K_{T_{n-1}} = i\right\} = p_{i,j}\mathbb{P}\left\{T_n - T_{n-1} \leq s\right\} = p_{i,j}\mathbb{P}\left\{S_n \leq s\right\} =: p_{i,j}G(s),$$

for all $s \geq 0$, $i, j \in \{0, 1\}$. In a general semi-Markov process, the $G(s)$ may depend on i, j. However, many cases of interest can be modeled also with a $G(s)$ that is independent of i, j (see e.g. [23] for the problem considered here) and so we stay with this Markovian model.

In addition to the i.i.d. sequence S_n consider two further i.i.d. sequences: $\{Z_n\}_{n\geq 1}$ representing the claim sizes at the various event times T_n and having distribution function $F(z)$ with support also in the positive half-line and with a finite second order moment; $\{W_n\}_{n\geq 1}$ representing the returns of the risky asset between T_{n-1} and T_n. More precisely, letting Π_t be the price of the risky asset at the generic time t, we have

$$\frac{\Pi_{T_n} - \Pi_{T_{n-1}}}{\Pi_{T_{n-1}}} =: W_n$$

For the i.i.d. sequence W_n we assume a distribution function $H(w)$ that has support in an interval of the form $[\underline{w}, \bar{w}]$ where $\underline{w} \in (-1, 0]$ and $\bar{w} > 0$. The random variables W_n may be continuous or discrete (prices changing by tick size).

All the various random variables are supposed to be defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ given by

$$\mathcal{F}_t = \sigma\left(N_s, K_s, S_{N_s}, Z_{N_s}, W_{N_s} : 0 \leq s \leq t\right), \ \forall t \in [0, T].$$

Concerning the controls, consider first the reinsurance/retention level b_t at the generic time t putting $b_n := b_{T_{n-1}}$. Among the various reinsurance schemes we choose here, to fix the ideas, a proportional reinsurance, by which the loss faced by the insurance company for a claim of size z is $h(b, z) = bz$ so that $b \in [0, 1]$ with $b = 0$ meaning full reinsurance and $b = 1$ no reinsurance (the results below all hold also for more general $h(b, z)$ provided it is continuous and monotonically increasing in both variables). Let then $B = \{b_t\}_{t\in[0,T]}$ be the predictable *reinsurance control process* that we assume to belong to the set of piecewise constant processes

$$\mathcal{U}_{re} = \left\{B = \{b_t\}_{t\in[0,T]} : b_t = b_n \text{ on } [T_{n-1}, T_n), \forall n = 1, \ldots, N_T + 1;\ \ b_t \in [0, 1]\right\} \tag{3}$$

where it is intended that $T_{N_T+1} = T$. Recall now that the company collects premium payments and we denote by c this premium rate. On the other hand, the company has to pay to the reinsurer a premium that depends on the chosen level b. This leads to define the net premium rate $c(b)$ that can be determined on the basis of the so-called *expected value principle with safety loading*. For this we base ourselves on a general definition of *safety loading parameter* in section I.1 of [2], where the safety loading parameter $\theta > 0$ of the re-insurer satisfies $\theta = \frac{p-\rho}{\rho}$. In our case of proportional reinsurance the parameters p and ρ can be determined as $p = c - c(b)$, $\rho = \frac{(1-b)\, E\{Z_1\}}{E\{S_1\}}$, with S_1 representing the time between two events and not just between two claims. This leads to $c(b) = c - (1+\theta)(1-b)\frac{E\{Z_1\}}{E\{S_1\}}$. To determine the net premium rate c, the insurer may in turn apply a safety loading principle with parameter $\eta > 0$ where, this time, we put $p = c$ and $\rho = \frac{E\{Z_1\}}{E\{S_1\}}$. It follows that $c = (1+\eta)\frac{E\{Z_1\}}{E\{S_1\}}$ and, replacing this latter expression into the previous expression for $c(b)$, we finally end up with

$$c(b) = [b(1+\theta) - (\theta - \eta)] \frac{E\{Z_1\}}{E\{S_1\}} \tag{4}$$

that is continuous and increasing and, assuming $\theta > \eta$, one has $c(0) < 0$. (See [23] for a comment on using $E\{Z_1\}$ also when, as in our case, Z_1 refers to the random time between events and not necessarily just claims).

In addition to the reinsurance control we consider the predictable *investment control process* $D = \{\delta_t\}_{t\in[0,T]}$ where δ_t represents the monetary amount invested at the generic time t in the risky asset and that we also assume to belong to the set of piecewise constant processes

$$\mathcal{U}_{inv} = \Big\{D = \{\delta_t\}_{t\in[0,T]} : \delta_t = \delta_n \text{ on } [T_{n-1}, T_n), \forall n = 1, \dots, N_T + 1;\; \delta_t \in [-C_1, C_2]\Big\} \tag{5}$$

where both, C_1 and C_2, are positive constants. This means that the company cannot sell short more than a monetary amount C_1 of the risky asset, nor buy more than C_2.

Any pair (B, D) of reinsurance and investment control processes belonging to $\mathcal{U}_{re} \times \mathcal{U}_{inv}$ reduces to a sequence (b_n, δ_n) belonging to $U := [0,1] \times [-C_1, C_2]$ which is a compact set.

Definition 2.1. *We shall call the pair (b_n, δ_n) a control action over the period $[T_{n-1}, T_n)$ and call admissible policy a policy $\pi = (B, D) \in \mathcal{U}_{re} \times \mathcal{U}_{inv}$ (consists of a predictable sequence of control actions $(b, \delta) \in U = [0,1] \times [-C_1, C_2]$).*

Finally, the evolution of the wealth process will also be affected by the capital injections that take place only when X_t drops below zero. By their definition, injections can take place only at event times. Denoting by $Y_n \geq 0$ the amount injected at T_n, for the wealth process we have

$$X_t = x + \int_0^t c(b_s)ds + \sum_{n=1}^{N_t} \left(K_{T_n}\delta_n W_n - (1 - K_{T_n})b_n Z_n + Y_n\right). \tag{6}$$

This leads to the following one-step dynamics between the $(n-1)$st and the nth event

$$X_{T_n} = X_{T_{n-1}} + c(b_n)S_n + K_{T_n}\delta_n W_n - (1 - K_{T_n})\, b_n Z_n + Y_n\,, \quad T_n \leq T \tag{7}$$

where (b_n, δ_n) are the control actions at $t = T_{n-1}$.

Remark 2.2. *If one starts at a generic $t \in [T_{n-1}, T_n)$ with a wealth $x \geq 0$, then the one-step dynamics up to $T_n = T_{N_t+1}$ can still be represented by (7) provided one replaces $X_{T_{n-1}}$ by $x - c(b_n)(t - T_{n-1})$, which is a known quantity at any time $t \in [T_{n-1}, T_n)$.*

Without loss of generality we can thus limit ourselves to time instants $t = T_{N_t} = T_n$ for some $n \geq 0$ and, replacing for simplicity of notation the pedexes T_n by n, write (7) equivalently as

$$X_n = X_{n-1} + c(b_n)S_n + K_n\delta_n W_n - (1 - K_n)\, b_n Z + Y_n\,, \quad n \leq N_T \tag{8}$$

Below we shall follow a Dynamic Programming approach that leads to optimal controls of the Markov type. For such controls, the pairs (X_{T_n}, K_{T_n}) form a Markov process and so we may consider as state space triples of the form (t, x, k) (see Section 4.1 below). Since our objective is the minimization of the expected total discounted capital injections, we let the objective criterion be given by

$$V^{B,D}(0, x; k) := E^{B,D}_{0,x;k} \left\{ \sum_{j=1}^{N_T} e^{-\gamma T_j} Y^{B,D}_j \right\} \tag{9}$$

where, assuming that we start at $t = T_0 = 0$, $x = X^{B,D}_0 > 0$, $k = K_0$ and γ is a discount factor. Put then

$$V^*(0, x; k) := \inf_{(B,D)\in\mathcal{U}_{re}\times\mathcal{U}_{inv}} V^{B,D}(0, x; k) \tag{10}$$

The discounting implies that the necessary capital should be injected as late as possible and never exceed the amount required to restore the wealth to zero. This gives also a further justification for (11) below.

3 Preliminary definitions and results

As mentioned, we assume that, as soon as the wealth process becomes negative, the shareholders inject sufficient capital to bring it back to zero. We thus have, at $t = T_n$,

$$Y_n = (X_{n-1} + c(b_n)S_n + K_n\delta_n\, W_n - (1 - K_n)\, b_n Z_n)^- \tag{11}$$

and, by (8),

$$X_n = (X_{n-1} + c(b_n)S_n + K_n\delta_n\, W_n - (1 - K_n)\, b_n Z_n)^+ \tag{12}$$

It follows from (11) that the only control processes are B and D. To stress this fact we shall write $X^{B,D}$ and $Y^{B,D}$. For convenience define

$$\Psi^{B,D}_n := c(b_n)S_n + K_n\delta_n W_n - (1 - K_n)b_n Z_n \quad \text{and put} \quad \hat{X}^{B,D}_n := X^{B,D}_{n-1} + \Psi^{B,D}_n \tag{13}$$

We thus obtain the following representation

$$X^{B,D}_n = \left(\hat{X}^{B,D}_n\right)^+ \quad ; \quad Y^{B,D}_n = \left(\hat{X}^{B,D}_n\right)^- \tag{14}$$

and from $|\hat{X}^{B,D}_n| = \left(\hat{X}^{B,D}_n\right)^+ + \left(\hat{X}^{B,D}_n\right)^- = X^{B,D}_n + Y^{B,D}_n$ we also have

$$0 \le X^{B,D}_n \le |\hat{X}^{B,D}_n|,\ \forall n \le N_T,\ \forall (B, D) \in \mathcal{U}_{re} \times \mathcal{U}_{inv} \quad \text{and} \quad Y^{B,D}_n \le |\hat{X}^{B,D}_n|,\ \forall n \ge 1 \tag{15}$$

We shall now show that, under our model assumptions, the criterion (10) is well defined with the expectations in (9) being finite for all $(B, D) \in \mathcal{U}_{re} \times \mathcal{U}_{inv}$. To this effect notice that by (13) and (15) and recalling that $X^{B,D}_0 = x$, we have

$$|\hat{X}_n| \le x + \sum_{j=1}^{n} |\Psi^{B,D}_j| \quad \text{as well as} \quad Y^{B,D}_n \le x + \sum_{j=1}^{n} |\Psi^{B,D}_j| \tag{16}$$

Defining

$$\begin{aligned} Q^{B,D} &:= \textstyle\sum_{j=1}^{N_T} |\Psi^{B,D}_j| \\ G_1 &:= T\,|c(1)| + \max[C_1, C_2]\, E\{|W_1|\} + E\{Z_1\} < \infty \\ G_2 &:= 3\, (T\, c(1))^2 + 3\, (\max[C_1, C_2])^2\, E\{(W_1)^2\} + 3E\{Z_1^2\} < \infty \end{aligned} \tag{17}$$

we have the following Lemma with proof in the Appendix.

Lemma 3.1. *We have that, uniformly in* $(B,D) \in \mathcal{U}_{re} \times \mathcal{U}_{inv}$,

$$E\{Q^{B,D} \mid N_T\} \leq G_1 N_T \quad ; \quad E\{(Q^{B,D})^2 \mid N_T\} \leq G_2 (N_T)^2$$

Corollary 3.2. *Uniformly in* $(B,D) \in \mathcal{U}_{re} \times \mathcal{U}_{inv}$, *and with* $R > 0$ *and* $0 < x < R$, *we have*

$$\begin{aligned} i) \quad & E_{0,x;k}^{B,D}\left\{|\hat{X}_n^{B,D}| \mid N_T\right\} \leq x + G_1 N_T \quad ; \quad E_{0,x;k}^{B,D}\left\{Y_n^{B,D} \mid N_T\right\} \leq x + G_1 N_T, \\ & E_{0,x;k}^{B,D}\left\{(Y_n^{B,D})^2 \mid N_T\right\} \leq 2x^2 + 2(G_1)^2 (N_T)^2, \quad \forall n \leq N_T \end{aligned}$$

$$ii) \quad P_{0,x;k}^{B,D}\left\{\sup_{n \leq N_T} |\hat{X}_n^{B,D}| > R\right\} \leq \tfrac{G_1 E\{N_T\}}{R-x}$$

Proof. Given Lemma 3.1, the proof becomes immediate noticing that from (16) we have that $x + \sum_{j=1}^{N_T} |\Psi_j^{B,D}| = x + Q^{B,D}$ is a common bound for $|\hat{X}_n^{B,D}|$ with $n \leq N_T$, as well as for $\sup_{n \leq N_T} |\hat{X}_n^{B,D}|$. For *ii*) one applies furthermore the Markov inequality. □

Corollary 3.2 implies that, for all $x > 0, k \in \{0,1\}$, and uniformly in the policy $(B,D) \in \mathcal{U}_{re} \times \mathcal{U}_{inv}$, one has that

$$V^{B,D}(0,x;k) = E_{0,x;k}^{B,D}\left\{\sum_{j=1}^{N_T} E_{0,x;k}^{B,D}\left\{e^{-\gamma T_j} Y_j^{B,D} \mid N_T\right\}\right\} \leq x\, E\{N_T\} + G_1 E\{(N_T)^2\}$$

For later use, we conclude this section with a further corollary of Lemma 3.1.

Corollary 3.3. *For* $R > 0$, $x \in [0,R]$ *and uniformly in the policy* $(B,D) \in \mathcal{U}_{re} \times \mathcal{U}_{inv}$ *we have*

$$P_{0,x;k}^{B,D}\left\{\sup_{n \leq N_T} |\hat{X}_n^{B,D}| > R \mid N_T\right\} \leq \frac{G_2 (N_T)^2}{(R-x)^2}$$

Proof. Recalling from the proof of Corollary 3.2 that $\sup_{n \leq N_T} |\hat{X}_n^{B,D}| \leq x + Q^{B,D}$ and using the Markov inequality as well as Lemma 3.1, we obtain

$$\begin{aligned} & P_{0,x;k}^{B,D}\left\{\sup_{n \leq N_T} |\hat{X}_n^{B,D}| > R \mid N_T\right\} \leq P\left\{Q^{B,D} > R - x \mid N_T\right\} \\ & \quad = P\left\{(Q^{B,D})^2 > (R-x)^2 \mid N_T\right\} \leq \tfrac{E\left\{(Q^{B,D})^2 \mid N_T\right\}}{(R-x)^2} \leq \tfrac{G_2 (N_T)^2}{(R-x)^2} \end{aligned}$$

□

4 Value Iteration

In this section we shall discuss some basic issues relating to a direct application of a Value Iteration approach to the problem formulated in the previous Section 2 and we do it here for the objective criterion $V^*(0,x;k)$ in (10), which then serves also to motivate the remaining part of the paper, in particular the next Section 5 where we study an approximate Value Iteration approach and show its convergence. For the solution of our problem one may in fact consider various approaches, among which also an HJB approach which, as mentioned in the Introduction, turns out to be not particularly suitable for the given problem. We shall therefore concentrate on a Dynamic Programming Approach leading to Value Iteration (see e.g. [18], [19], [20], [5]).

4.1 The Dynamic Programming methodology and Value Iteration

As it will follow from the Dynamic Programming (DP) setup, we may restrict ourselves to Markov controls (B, D), (see Proposition 4.4 below) i.e. such that the induced control actions (b_n, δ_n) are of the form

$$b_n = b(T_{n-1}, X_{n-1}; K_{n-1}) \quad ; \quad \delta_n = \delta(T_{n-1}, X_{n-1}; K_{n-1}) \tag{18}$$

As already mentioned, with Markov controls the pair $(X_n, K_n) = (X_{T_n}, K_{T_n})$ is Markov.

In view of applying the Dynamic Programming methodology we first extend the definition of the objective criterion (9) to an arbitrary $t \in [0, T]$ with $t = T_{N-t} = T_n$ and consider as state space the set

$$E := \{(t, x, k) \mid t \in [0, T],\ x \in \mathbb{R}^+,\ k \in \{0, 1\}\} = [0, T] \times [0, \infty) \times \{0, 1\}, \tag{19}$$

Definition 4.1. *For $(t, x, k) \in E$ with $t = T_n < T$, $(N_t = n)$, and relying on the Markovianity of the controls $(B, D) \in \mathcal{U}_{re} \times \mathcal{U}_{inv}$, let the value function (cost-to-go function) be given by*

$$V^{B,D}(t, x; k) := E^{B,D}_{t,x;k}\left\{\sum_{j=n+1}^{N_T} e^{-\gamma(T_j - t)} Y_j^{B,D}\right\} = E^{B,D}_{t,x;k}\left\{\sum_{j=n+1}^{\infty} e^{-\gamma(T_j - t)} Y_j^{B,D} \mathbf{1}_{\{j \le N_T\}}\right\}$$

The optimal value (cost-to-go) function then is

$$V^*(t, x; k) := \inf_{(B,D) \in \mathcal{U}_{re} \times \mathcal{U}_{inv}} V^{B,D}(t, x; k)$$

In line with Remark 2.2 we shall evaluate the value function for (random) triples $(t, x, k) \in E$ of the form $t = T_{N_t} = T_n$ for some $n \ge 0$, $x = X_{T_n}$, $k = K_{T_n}$, i.e. we may put $V^{B,D}(T_n, X_{T_n}, K_{T_n}) = V^{B,D}(t, x, k)_{|t=T_n, x=X_{T_n}, k=K_{T_n}}$.

Next, denote by $B(E)$ the Banach space of bounded functions on E with respect to the sup-norm, namely $B(E) = \mathcal{L}^\infty(E)$, and let $C(E)$ be

$$C(E) := \{v \in B(E) \mid (t, x) \to v(t, x; k) \text{ is continuous } \forall k \in \{0, 1\}\} \tag{20}$$

It can be shown (see e.g. [1]) that also $C(E)$ is a Banach space in the sup-norm. Noticing now that, given x, k at the event time $t = T_{N_t} = T_n$, the values of $X^{B,D}$ and $Y^{B,D}$ at the next event time $T_{n+1} = T_n + S_{n+1}$ depend on B, D only through $b = b_{n+1} = b(T_n, X_n, K_n)$ and $\delta = \delta_{n+1} = \delta(T_n, X_n, K_n)$, in line with the Dynamic Programming principle we introduce the following

Definition 4.2. *For $v \in B(E)$, $(b, \delta) \in U$ and with $t = T_{N_t} = T_n$ an event time, define an operator, also called Bellman operator, by (being S_n i.i.d. we use simply the symbol S)*

$$\mathcal{T}^{b,\delta}[v](t, x; k) := E^{b,\delta}_{t,x;k}\left\{e^{-\gamma S} \mathbf{1}_{\{t+S \le T\}} \left[Y^{b,\delta}_{n+1} + v\left(t + S, X^{b,\delta}_{n+1}; K_{n+1}\right)\right]\right\} \tag{21}$$

and let

$$\mathcal{T}^*[v](t, x; k) := \inf_{(b,\delta) \in U} \mathcal{T}^{b,\delta}[v](t, x; k) \tag{22}$$

Remark 4.3. *For our model it follows easily that, for $v \in B(E)$, also $\mathcal{T}[v] \in B(E)$ and it can be shown (see [1]) that, for $v \in C(E) \subset B(E)$ and any given $(t, x; k) \in E$, the mapping $(b, \delta) \to \mathcal{T}^{b,\delta}[v](t, x; k)$ is continuous so that the* inf *in (22) is a* min.

Invoking Michael's theorem (see e.g. Proposition D.3 in [19]) one can immediately deduce the following

Proposition 4.4. *Given $v \in C(E)$, there exist continuous selectors $b^* : E \to [b_0, 1]$, $\delta^* : E \to [-C_1, C_2]$ such that*

$$\mathcal{T}^*[v](t, x; k) := \min_{(b,\delta)\in U} \mathcal{T}^{b,\delta}[v](t, x; k) = \mathcal{T}^{b^*(t,x;k),\delta^*(t,x;k)}[v](t, x; k)$$

implying that we can indeed limit ourselves to Markov controls. Furthermore, $\mathcal{T}^*[v] \in C(E)$.

Following are three preliminary Lemmas in view of the main result of this section, namely Theorem 4.8 below, and their proofs are in the Appendix.

Lemma 4.5. *(contraction property of $\mathcal{T}^*$)*
Assuming $G(0) = \mathbb{P}\{S = 0\} < 1$ and putting $\alpha := E\left\{e^{-\gamma S}\right\} < 1$, for $v, v' \in B(E)$ we have

$$||\mathcal{T}^*[v] - \mathcal{T}^*[v']||_\infty \leq \alpha||v - v'||_\infty$$

Lemma 4.6. *For given $(B, D) \in \mathcal{U}_{re} \times \mathcal{U}_{inv}$, which we have assumed to be Markovian as in (18), and $(t, x; k) \in E$ (with $t = T_{N_t} = T_n$ for some n) we have the following fixed-point property*

$$\mathcal{T}^{b,\delta}[V^{B,D}](t, x; k) = V^{B,D}(t, x; k)$$

where (b, δ) are the control actions corresponding to (B, D) evaluated at $(t, x; k)$, i.e. if $t = T_{N_t}$, then $b = b_{n+1} = b(T_n, X_n; K_n)$ and, analogously, for $\delta = \delta_{n+1}$.

Lemma 4.7. *V^* is a fixed point of the operator $\mathcal{T}^* : B(E) \to B(E)$.*

We come now to the main result of this section, namely

Theorem 4.8. *Let V^* and $\mathcal{T}^*$ be as defined above. Then*

i) V^* is the unique fixed point of the operator $\mathcal{T}^*$ in $C(E)$.

ii) Define the sequence $\{v^*_m\}_{m\geq 0}$ as

$$v^*_m(t, x; k) := \begin{cases} 0 & \text{for } m = 0 \\ \mathcal{T}^*[v^*_{m-1}](t, x; k) & \text{for } m > 0 \end{cases}$$

then $v^*_m \in C(E)$ and $\lim_{m\to\infty} ||v^*_m - V^*||_\infty = 0$.

iii) The stationary policy $(\tilde{B}, \tilde{D})$, corresponding to a control action $(\tilde{b}, \tilde{\delta})$ given by

$$(\tilde{b}, \tilde{\delta}) := argmin_{(b,\delta)\in U}\mathcal{T}^{b,\delta}[V^*],$$

is an optimal control policy and can be chosen to be Markovian.

Proof. Concerning point i), from Lemma 4.5 we have that $\mathcal{T}^*$ is a contraction operator in $B(E)$ and so V^* is by Lemma 4.7 the unique fixed point of $\mathcal{T}^*$ in $B(E)$. For the restriction $\mathcal{T}^*_{|C(E)}$ of $\mathcal{T}^*$ to $C(E)$ we have by Proposition 4.4 that $\mathcal{T}^*_{|C(E)} : C(E) \to C(E)$ and, by the contraction property, has a unique fixed point V^*_c, namely $\mathcal{T}^*_{|C(E)}(V^*_c) = \mathcal{T}^*(V^*_c) = V^*_c \in C(E)$. Being $C(E) \subset B(E)$, it follows that $V^* = V^*_c \in C(E)$ is indeed the unique fixed point of $\mathcal{T}^*$ in $C(E)$.

Coming to $ii)$ notice that, always by Proposition 4.4, we have $v_m^* \in C(E)$. By the uniqueness of V^* as fixed point of the contraction mapping $\mathcal{T}^*$ in $C(E)$ we obtain $\lim_{m\to\infty} ||v_m^* - V^*||_\infty = 0$.

Concerning point $iii)$ recall first that, since $V^* \in C(E)$ (see point $i)$), the *argmin* exists by virtue of Proposition 4.4 and, among the minimizers, there are continuous selectors in the form of Markovian controls (see (18)) . Using also Lemma 4.7, we then have $\forall(t,x;k) \in E$ (assume $t = T_{N_t} = T_n$ for some n)

$$V^{\tilde{B},\tilde{D}}(t,x;k) = \min_{(b,\delta)\in U} \mathcal{T}^{b,\delta}[V^*](t,x;k) = \mathcal{T}^*[V^*](t,x;k) = V^*(t,x;k)$$

and $(\tilde{B},\tilde{D})$ is Markovian. □

Remark 4.9. *It can be shown (see e.g. [17]) that v_m^* coincides with the optimal value V^* in case when there are at most m events.*

By point $iii)$ of the above Theorem 4.8, we could be able to obtain a stationary optimal policy but this requires the knowledge of the fixed point V^* of the operator $\mathcal{T}^*$ that is not easy to determine directly. On the other hand, point $ii)$ of that Theorem shows that we can approximate V^* arbitrarily closely by successively iterating $\mathcal{T}^*$ and this leads us to the Value Iteration.

4.2 The value iteration algorithm

Before presenting the main result of this subsection in Proposition 4.11 below, which can be seen as a corollary to Theorem 4.8, we recall that $V^{B,D}$ is the fixed point of $\mathcal{T}^{b,\delta}$ in $B(E)$ in the sense of Lemma 4.6; furthermore, it can be easily shown that, analogously to the contracting properties of $\mathcal{T}^*$ (see Lemma 4.5), also $\mathcal{T}^{b,\delta}$ is, for all $(b,\delta) \in U$, contracting and has the same contracting factor α.

Introduce next the following

Definition 4.10. *For $m > 0$ a given integer, define $\mathcal{U}_{re}^m \subset \mathcal{U}_{inv}$ [respectively $\mathcal{U}_{inv}^m \subset \mathcal{U}_{inv}$] as the set of admissible Markov policies with only m components, i.e. where at most m event times are considered. Denote its elements by B^m [resp. D^m], namely $B^m := (b_1^m, \cdots, b_m^m) \in \mathcal{U}_{re}^m$ [(resp.) $D^m := (\delta_1^m, \cdots, \delta_m^m) \in \mathcal{U}_{inv}^m$].*

The main result here is then

Proposition 4.11. *Given $\varepsilon > 0$, let m_ε be such that (for v_m^* see Theorem 4.8, point ii))*

$$||V^* - v_m^*||_\infty < \varepsilon, \quad \forall m > m_\varepsilon \tag{23}$$

For $m > m_\varepsilon$ let $(B^{*m}, D^{*m}) \in \mathcal{U}_{re}^m \times \mathcal{U}_{inv}^m$ be a (non-stationary) policy obtained from the first m iterations of $\mathcal{T}^*$ when starting from $v = 0$ so that

$$(B^{*m}, D^{*m}) = argmin_{(B^m,D^m)\in\mathcal{U}_{re}^m\times\mathcal{U}_{inv}^m} E_{0,x;k}^{B^m,D^m}\left\{\sum_{j=1}^m e^{-\gamma T_j} Y_j^{B^m,D^m} \mathbf{1}_{\{j\le N_T\}}\right\}$$

and let $(B^*, D^*) \in \mathcal{U}_{re} \times \mathcal{U}_{inv}$ be an arbitrary extension of (B^{*m}, D^{*m}) beyond the first m jumps. Then

$$||V^* - V^{B^*,D^*}||_\infty < 2\varepsilon$$

Proof. From the given definition of (B^*, D^*) one has $v^*_m = v^{B^*,D^*}_m$ and, since $\mathcal{T}^{B^*,D^*}$ has the same contraction factor as $\mathcal{T}^*$, we also have

$$||V^{B^*,D^*} - v^{B^*,D^*}_m||_\infty < \varepsilon, \quad \forall m > m_\varepsilon \tag{24}$$

From (23) and (24) one then obtains

$$||V^* - V^{B^*,D^*}||_\infty \leq ||V^* - v^*_m||_\infty + ||v^*_m - V^{B^*,D^*}||_\infty = ||V^* - v^*_m||_\infty + ||v^{B^*,D^*}_m - V^{B^*,D^*}||_\infty \leq 2\varepsilon$$

□

The contracting properties of the operators $\mathcal{T}^*$ and $\mathcal{T}^{B^*,D^*}$ have allowed us to approximate arbitrarily closely the optimal value V^* by V^{B^*,D^*} where (B^*, D^*) is the strategy defined in Proposition 4.11 for a sufficiently large m. At the same time this allows us to determine also a nearly optimal (in general non-stationary) Markovian policy as specified in the following subsection.

4.2.1 Algorithm to determine the nearly optimal policy

Assume the company stands at the beginning of the planning period, namely at time $t = T_0 = 0$ with an initial surplus of $x > 0$. Having chosen an m sufficiently large so that $||V^* - v^*_m||_\infty < \varepsilon$ (such an m can be explicitly determined as a function of the contraction factor α and of the parameters of the model, see [1], [7]) the nearly optimal strategy (B^*, D^*) that, according to Proposition 4.11, guarantees that $||V^* - V^{B^*,D^*}||_\infty < 2\varepsilon$, is an arbitrary extension beyond m of the strategy $(B^m, D^m) \in \mathcal{U}^m_{re} \times \mathcal{U}^m_{inv}$ given by

$$B^m := \Big(b^m_1(0, x; K_0), \cdots, b^m_m(T_{m-1}, X^{B^m,D^m}_{m-1}; K_{m-1})\Big)$$

$$D^m := \Big(\delta^m_1(0, x; K_0), \cdots, \delta^m_m(T_{m-1}, X^{B^m,D^m}_{m-1}; K_{m-1})\Big)$$

where, recalling the definition of v^*_m (see point *ii*) of Theorem 4.8), the individual action pairs $(b^m_i(\cdot), \delta^m_i(\cdot))$ are, for $i = 1, \cdots, m$, determined as follows:

1. From the first iteration of $\mathcal{T}^*$ with $v^*_0 = 0$ we obtain

   $$(b^m_m, \delta^m_m)(T_{m-1}, X^{B^m,D^m}_{m-1}; K_{T_{m-1}}) = argmin_{(b,\delta)\in U} \mathcal{T}^{b,\delta}[v^*_0](T_{m-1}, X^{B^m,D^m}_{m-1}; K_{m-1})$$

   and this control action is applied when $m - 1 < N_T$ events have already happened, i.e. the company stands at time T_{m-1} with a surplus equal to X^{B^m,D^m}_{m-1} (resulting from having applied in the first $m-1$ steps the policy (B^m, D^m)) and having observed K_{m-1}.

2. From the last iteration of $\mathcal{T}^*$ we obtain

   $$(b^m_1, \delta^m_1)(0, x; K_0) = argmin_{(b,\delta)\in U} \mathcal{T}^{b,\delta}[v^*_{m-1}](0, x; K_0)$$

   and it is applied at the initial time $T_0 = 0$.

3. For a generic $i \in \{2, \cdots, m-1\}$

   $$(b^m_i, \delta^m_i)(T_{i-1}, X^{B^m,D^m}_{i-1}; K_{i-1}) = argmin_{(b,\delta)\in U} \mathcal{T}^{b,\delta}[v^*_{m-i}](T_{i-1}, X^{B^m,D^m}_{i-1}; K_{i-1})$$

   which results from the $(m-i)$th iteration of $\mathcal{T}^*$ and is applied at time $t = T_{i-1}$.

4. If more than m events happen, the strategy (B^m, D^m) is completed in an arbitrary way, for example as follows: for $m < j$ let

$$b_j^m(T_{j-1}, X_{j-1}^{B^m,D^m}; K_{j-1}) = 1\,;\;\; \delta_j^m(T_{j-1}, X_{j-1}^{B^m,D^m}; K_{j-1}) = 0\,;\;\; \forall j \in \{m+1, \cdots, N_T\}$$

.

5 Approximations for the Value Iteration approach

We already mentioned that, on the basis of point *iii*) of Theorem 4.8, one could obtain a stationary optimal policy, but this requires the knowledge of V^* that, in general, is not available. On the other hand, by the value iteration algorithm of subsection 4.2.1 one can obtain a policy (B^*, D^*) leading to a value V^{B^*,D^*} that (see Proposition 4.11) differs from V^* by at most $2\,\varepsilon$ for a sufficiently large m, but this algorithm too requires the knowledge, this time, of the functions $v_m^*(t, x; k)$ obtained from the successive applications of the operator $\mathcal{T}^*$ according to point *ii*) of Theorem 4.8. It is however clear that, unless there exists a finitely parameterized family of functions $v(t, x; k)$ that is closed under the operator $\mathcal{T}^*$, the expressions for the successive functions v_m^* may become more and more analytically intractable (the functions v_m^* have in fact different analytic expressions on different portions of the state space E and these portions become more numerous as m increases, see [1]).

To overcome this difficulty, in [1] an approach has been proposed that relies on a weaker concept of optimality, which however overcomes only partially the difficulty. On the other hand, accepting an at least approximate solution, one may perform a discretization in order to end up with finite numbers of values for the various variables, so that computation becomes feasible in any case. This concerns the values of the time variable $t \in [0, T]$, the values x of the wealth process and, possibly, also the values $(b, \delta) \in U$ of the control actions. To obtain a finite number of discretized values of the variables, one has to consider compact sets that contain the various variables with high probability and perform the discretizations within these sets. The time and control variables are already on a compact set, but $X_t^{B,D}$ is not. We shall thus restrict $X_t^{B,D}$ to a compact set (see sub-section 5.1). More precisely, we shall do it for the sequence $X_n^{B,D}$ of the values of $X_t^{B,D}$ at the various event times T_n which are also the time instants when the control actions and the capital injections take place and that are therefore those that matter for the optimization problem. We shall more specifically restrict the associated sequence $\hat{X}_n^{B,D}$ (see (13)) to a compact interval $[-R, R]$ with $R > 0$ sufficiently large so that this interval contains $\hat{X}_n^{B,D}$ with large probability, independently of the choice of the control policy. By Corollary 3.2 one may choose R large enough so that $\frac{G_1 E\{N_T\}}{R-x}$ becomes small as desired. The fact that G_1 depends on the parameters of the model and the distributions of the random quantities, implies that in this way R is chosen endogenously and not exogenously. Notice also that the bounds in Corollary 3.2 and those to be derived in sub-section 5.2 below are essentially only theoretical bounds intended to show that the approximations indeed converge. In reality, as it is usual in such situations, the approximations turn out to be much more precise.

Remark 5.1. *In practice (see also the numerical results in section 6 below) a reasonable choice for the value of R appears to be a fraction of the maximal value that the various $X_n^{B,D}$ can achieve over the given horizon when there are no claims nor investments in the financial market. Also such a choice of R turns thus out to be endogenously given.*

In sub-section 5.1 we shall consider the discretization of the variables and define then the approximating problem. In sub-section 5.2 we shall show the convergence of the approximations and thereby also show that an optimal/nearly optimal control for the approximating problem, when suitably extended, is nearly optimal in the original problem. Finally, in sub-section 5.3 we describe the implementation of the value iteration approach for the approximating problem.

5.1 The approximating problem

We shall first consider the restriction of the $x-$variable to a compact set and then perform a discretization within the compact sets.

For $R > 0$ we shall restrict $\hat{X}_n^{B,D}$ to $[-R, R]$. Our problem concerns then the three state variables $(t, x; k)$ on the compact state space $E_R := [0,T] \times [0,R] \times \{0,1\}$, and the two control variables (b, δ), defined on the compact set $U = [b_0, 1] \times [-C_1, C_2]$ (see Definition 2.1). Except, possibly, for the control variables, in order to actually compute a solution, in our case via a Value Iteration algorithm as described in Section 4, the state variables should be discrete and finite valued. Since $k \in \{0,1\}$ is already discrete, we shall next proceed with the discretization of the time and space variables (t, x) and call the resulting discretized problem the *approximating problem*.

We shall consider an uniform grid on the compact set $[0,T] \times [-R, R]$. Given two positive integers N and M, possibly so that T and R are multiples of N and M respectively, consider the following partitions of $[0,T]$ and $[-R, R]$ (since the restriction to $[-R, R]$ concerns actually the associated sequence $\hat{X}_n^{B,D}$, we allow x to take values also in $[-R, 0)$)

$$\Gamma_i = \left[i\,\tfrac{T}{N}, (i+1)\,\tfrac{T}{N}\right) \text{ for } i \in \{0, \cdots, N-1\} \text{ with } \Gamma_N = \{T\}$$

$$\Xi_j = \left[j\,\tfrac{R}{M}, (j+1)\,\tfrac{R}{M}\right) \text{ for } j \in \{-M, \cdots, M-1\} \text{ with } \Xi_M = [R, +\infty)$$

and choose the following representative elements for each set in the partitions

$$\bar{t}_i = i\,\frac{T}{N} \text{ for } i = 0, \cdots, N \quad ; \quad \bar{x}_j = j\,\frac{R}{M} \text{ for } j = -M, \cdots, M$$

.

Remark 5.2. *For the convergence results in section 5.2 below we could choose as representative elements any point in the various subintervals. Having chosen the lower endpoints, we can include in the partition the upper half line $[R, +\infty)$, but have to exclude the lower half-line $(-\infty, -R]$. We shall thus take $\bar{x}_{-M}$ as representative element of $\left(-\infty, -\frac{M-1}{M}\,R\right)$.*

Define the grids

$$\bar{G} := \{\bar{t}_i \mid i = 0, \cdots, N\} \times \{\bar{x}_j \mid j = -M, \cdots, M\} \text{ as well as } G := \bar{G} \cap ([0,T] \times [0, +\infty)) \quad (25)$$

that depend on N and M and, implicitly, also on R and consider the projection operators

$$\bar{T}(t) := \sum_{i=0}^{N} \bar{t}_i \mathbf{1}_{\{\Gamma_i\}}(t)\,, \quad t \in [0,T]\,; \qquad \bar{X}(x) := \sum_{j=0}^{M} \bar{x}_j \mathbf{1}_{\{\Xi_j\}}(x)\,, \quad x \in \mathbb{R} \quad (26)$$

Notice that $\bar{T}(\cdot)$ and $\bar{X}(\cdot)$ are step functions with the discontinuities only in $\bar{G}$ and we shall occasionally use the notation $(\bar{T}, \bar{X})$ to denote a process on $\bar{G}$.

In order to define the approximating problem, given any $(B,D) \in \mathcal{U}_{re} \times \mathcal{U}_{inv}$, we shall discretize the processes $X_n^{B,D}$ and $Y_n^{B,D}$ so that they take values on the grid G and denote them by $X_{G,n}^{B,D}$ and $Y_{G,n}^{B,D}$ respectively. We shall assume that $x = X_0^{B,D} \in [-R,R]$ so that also $X_{G,0}^{B,D} = \bar{x} = \bar{X}(X_0^{B,D}) \in [-R,R]$. For $T_n > 0$ let then

$$\begin{cases} X_{G,n}^{B,D} & = & \bar{X}\left(X_{G,n-1}^{B,D} + c(b_n)S_n + K_n\delta_n W_n - (1-K_n)b_n Z_n\right) + Y_{G,n}^{B,D} \\ Y_{G,n}^{B,D} & = & \left(\bar{X}\left(X_{G,n-1}^{B,D} + c(b_n)S_n + K_n\delta_n W_n - (1-K_n)b_n Z_n\right)\right)^- \end{cases} \tag{27}$$

Notice that the pedix G here and in further quantities below refers synthetically also to the dependence on N and M as well as on R. For convenience of notation let, analogously to (13) and for given $X_{G,n-1}^{B,D}$,

$$\hat{X}_{G,n}^{B,D} := X_{G,n-1}^{B,D} + \Psi_n^{B,D} \tag{28}$$

so that, by analogy with (14), we may write more concisely

$$X_{G,n}^{B,D} = \left(\bar{X}(\hat{X}_{G,n}^{B,D})\right)^+ \quad , \quad Y_{G,n}^{B,D} = \left(\bar{X}(\hat{X}_{G,n}^{B,D})\right)^- \tag{29}$$

By construction we have

$$X_{G,n}^{B,D} \in \{\bar{x}_j | j = 0, \cdots, M\} \quad ; \quad Y_{G,n}^{B,D} \in \{-\bar{x}_j | j = -M, \cdots, -1\}$$

so that both $X_{G,n}^{B,D}$ as well as $Y_{G,n}^{B,D}$ belong to $[0,R]$.

Remark 5.3. *The amounts of capital injections to be actually applied are the values* $Y_n^{B,D} = \left(\hat{X}_n^{B,D}\right)^-$*, where* $\hat{X}_n^{B,D}$ *are the actually observed values of the wealth process before the injections. The quantities* $Y_{G,n}^{B,D}$ *are introduced only to define the value function of the approximating problem in analogy to that of the original problem (see Definition 5.5 below).*

For the approximating problem with the discretized values of (t,x), a Markov control policy $(B,D) \in \mathcal{U}_{re} \times \mathcal{U}_{inv}$ corresponds to control actions that are defined on $\bar{G}$ and are of the form

$$b_n^G = b_n^G(\bar{T}_{n-1}, \bar{X}_{n-1}; K_{n-1}) \in [0,1] \quad ; \quad \delta_n^G = \delta_n^G(\bar{T}_{n-1}, \bar{X}_{n-1}; K_{n-1}) \in [-C_1, C_2]$$

This Markov policy on $\bar{G}$ can be extended to become a Markov policy in E (state space of the original problem) according to the following

Definition 5.4.

i) Given a Markov control policy $(B,D) \in \mathcal{U}_{re} \times \mathcal{U}_{inv}$ *for the approximating problem on* $\bar{G}$*, we shall consider as its extension to* E *the one, where at the generic event time* T_{n-1}

$$b_n(T_{n-1}, X_{n-1}; K_{n-1}) = \begin{cases} b_n^G(\bar{T}(T_{n-1}), \bar{X}(X_{n-1}); K_{n-1}) & \text{for } X_{n-1} \le R \\ b_n^G(\bar{T}(T_{n-1}), R; K_{n-1}) & \text{for } X_{n-1} > R \end{cases} \tag{30}$$

and, analogously, for $\delta_n(\cdot)$.

ii) Given a control policy $(B,D) \in \mathcal{U}_{re} \times \mathcal{U}_{inv}$ *on the state space* E*, one can consider its restriction to* $\bar{G}$ *by simply evaluating it at the elements of* $\bar{G}$.

We shall use the symbols (B, D) and (b, δ) for policies and control actions both on E as well as on $\bar{G}$.

Concerning the objective criterion for the approximating problem, recalling Definition 4.1 introduce, for $t = 0$, the following

Definition 5.5. *Given $t = T_0 = 0$, $x = X_0 \in [0, R], k \in \{0, 1\}$, consider the following initial (for $t = 0$) value function for the approximating problem where, letting $(B, D) \in \mathcal{U}_{re} \times \mathcal{U}_{inv}$ denote any control policy also in $\bar{G}$,*

$$V_G^{B,D}(0, x; k) := E_{0,\bar{X}(x);k}^{B,D} \left\{ \sum_{j=1}^{N_T} e^{-\gamma \bar{T}_j} Y_{G,j}^{B,D} \right\} \tag{31}$$

and define as optimal value (cost-to-go) function for the approximating problem

$$V_G^*(0, x; k) := \inf_{(B,D) \in \mathcal{U}_{re} \times \mathcal{U}_{inv}} V_G^{B,D}(0, x; k) \tag{32}$$

Notice that, whenever for a given policy $(B, D) \in \mathcal{U}_{re} \times \mathcal{U}_{inv}$ on E we write $V_G^{B,D}$, we actually consider the restriction of this policy to $\bar{G}$. Analogously, when for a Markov policy $(B, D) \in \mathcal{U}_{re} \times \mathcal{U}_{inv}$, considered on $\bar{G}$, we write $V^{B,D}$, we mean the extension of this policy to E according to (30).

Remark 5.6. *Analogously to the cost-to-go function $V^{B,D}(t, x; k)$ in Definition 4.1, we may consider also a generalization of (31) to arbitrary $t < T$ with $t = T_n < T$, namely*

$$V_G^{B,D}(t, x; k) := E_{t,x;k}^{B,D} \left\{ \sum_{j=n+1}^{N_T} e^{-\gamma (T_j - t)} Y_{G,j}^{B,D} \right\} = E_{t,x;k}^{B,D} \left\{ \sum_{j=n+1}^{\infty} e^{-\gamma (T_j - t)} Y_{G,j}^{B,D} \mathbf{1}_{\{j \le N_T\}} \right\} \tag{33}$$

5.2 Convergence of the approximating problem

We shall now show that by choosing sufficiently large values of R, N and M , the solution of the approximating problem leads to a value for the original problem on the state space E that is arbitrarily close to its optimal value.

First we need the following preliminary Lemma, for which the proof is in the Appendix.

Lemma 5.7. *Assuming $x = X_0 \in [0, R]$, on the events $X_n^{B,D} \in [-R, R]$ we have for all $(B, D) \in \mathcal{U}_{re} \times \mathcal{U}_{inv}$*

$$|X_n^{B,D} - X_{G,n}^{B,D}| \le (n+1) \frac{R}{M} \quad \text{for } n \ge 0 \quad ; \quad |Y_n^{B,D} - Y_{G,n}^{B,D}| \le (n+1) \frac{R}{M}, \quad \text{for } n \ge 1 \tag{34}$$

where M is the discretization parameter for the wealth process.

We come now to the main result of this subsection, namely the following Proposition and its Corollary. The proof of the Proposition is in the Appendix, while that of the Corollary is immediate.

Proposition 5.8. *For $t = \bar{t} = 0$, for $\bar{x} = \bar{X}(x)$ with $x = X_0 \in [0, R]$, for all $k \in \{0, 1\}$ as well as all policies $(B, D) \in \mathcal{U}_{re} \times \mathcal{U}_{inv}$ on E we have*

i) $\quad |V^{B,D}(0, x; k) - V_G^{B,D}(0, x; k)| \le B(R, N, M)$

ii) $\quad |\inf_{(B,D)\in\mathcal{U}_{re}\times\mathcal{U}_{inv}} V^{B,D}(0, x; k) - \inf_{(B,D)\in\mathcal{U}_{re}\times\mathcal{U}_{inv}} V_G^{B,D}(0, x; k)| \le B(R, N, M)$

where $B(R, N, M)$, the same in both cases, is given by

$$B(R,N,M) := \left[\frac{3R}{2M} + \gamma T\, \frac{R}{N}\right] E\{N_T\} + \frac{R}{2M}\, E\{N_T^2\} + \frac{x\sqrt{2G_2}\, E\{(N_T)^2\} + \sqrt{2G_1G_2}\, E\{(N_T)^3\}}{(R-x)} \tag{35}$$

with G_1, G_2 as defined in (17). This quantity can be made arbitrarily small by choosing R sufficiently large so that, for a given inital wealth x, the last term in (35) is small. Given R, also the discretization parameters N, M have then to be chosen sufficiently large so that also the first two terms are small.

Corollary 5.9. *For $t = \bar{t} = 0$, for $x = X_0 \in [0, R]$ and for all $k \in \{0, 1\}$ we have*

i) If (B_G^*, D_G^*) is an optimal policy for the approximating problem, namely

$$V_G^{B_G^*, D_G^*}(0, x; k) = \inf_{(B,D)\in\mathcal{U}_{re}\times\mathcal{U}_{inv}} V_G^{B,D}(0, x; k)$$

then

$$|V^{B_G^*, D_G^*}(0, x; k) - \inf_{(B,D)\in\mathcal{U}_{re}\times\mathcal{U}_{inv}} V^{B,D}(0, x; k)| \le 2\,(B(R, N, M)$$

where $B(R, N, M)$ is as in Proposition 5.8.

ii) If (B_G^θ, D_G^θ) is an $\theta-$optimal policy for the approximating problem, namely

$$V_G^{B_G^\theta, D_G^\theta}(0, x; k) \le \inf_{(B,D)\in\mathcal{U}_{re}\times\mathcal{U}_{inv}} V_G^{B,D}(0, x; k) + \theta$$

then

$$|V^{B_G^\theta, D_G^\theta}(0, x; k) - \inf_{(B,D)\in\mathcal{U}_{re}\times\mathcal{U}_{inv}} V^{B,D}(0, x; k)| \le 2\, B(R, N, M) + \theta$$

We have now obtained that, if we determine an optimal or nearly optimal policy for the approximating problem, then it results in being nearly optimal also in the original problem on the state space E, provided the discretization parameters N, M as well as R are sufficiently large. It remains thus to obtain for the approximating problem an optimal or also only nearly optimal policy. For this, as mentioned, we shall use a Value Iteration algorithm in line with the description in section 4. Notice also that, using the Value Iteration algorithm (Dynamic Programming) to determine (B_G^*, D_G^*) or (B_G^θ, D_G^θ), this policy can be obtained as a Markov policy.

5.3 Value iteration in the discretized (approximating) problem

The value iteration algorithm has been described in section 4 (see in particular point *ii)* of Theorem 4.8 and subsection 4.2.1). The algorithm is based on the operators $\mathcal{T}^{b,d}$ and $\mathcal{T}^*$ and so we have first to particularize these operators to the discretized approximating problem. To this effect consider the space $D(E_R)$ of cadlag functions on $E_R := [0,T] \times [0,R] \times \{0,1\}$ and let $D_G(E_R) \subset D(E_R)$ be the subset of functions with the property that

$$v(t,x;k) = v(\bar{T}(t), \bar{X}(x); k) \quad \forall\, (t,x;k) \in E_R$$

where $T(\cdot)$ and $\bar{X}(\cdot)$ are the projection operators defined in (26). These cadlag functions are bounded with discontinuities only in $\bar{G} \times \{0,1\}$.

By analogy to Definition 4.2 and taking into account the objective function (31) with (32) as well as (33), for the approximating problem we introduce the following

Definition 5.10. *Let $(t,x;k) \in E_R$ with t an event time, namely $t = T_{N_t} = T_n$ for some $n \geq 0$. For $v \in B(E_R)$, the space of bounded functions on E_R, and any $(b,\delta) \in U$ define the operator $\mathcal{T}_G^{b,\delta} : B(E_R) \to B(E_R)$ as*

$$\mathcal{T}_G^{b,\delta}[v](t,x;k) := E^{b,\delta}_{\bar{T}(t),\bar{X}(x);k}\left\{ e^{-\gamma(\bar{T}(t+S)-\bar{T}(t))} \mathbf{1}_{\{t+S \leq T\}} \left[Y^{b,\delta}_{G,n+1} + v\left(\bar{T}(t+S), X^{b,\delta}_{G,n+1}; K_{n+1}\right)\right]\right\} \tag{36}$$

Furthermore,

$$\mathcal{T}_G^{*}[v](t,x;k) := \inf_{(b,\delta)\in U} \mathcal{T}_G^{b,\delta}[v](t,x;k) \tag{37}$$

Remark 5.11. *By its definition in (36) we have that $\mathcal{T}_G^{b,\delta}[v](t,x;k) = \mathcal{T}_G^{b,\delta}[v](\bar{T}(t),\bar{X}(x);k)$ so that, as a function of $(t,x;k)$, it belongs to $D_G(E_R)$. Furthermore $(\bar{T}(t+S), X^{b,\delta}_{G,n+1}; K_{n+1}) \in G \times \{0,1\}$ so that we may also consider $\mathcal{T}_G^{b,\delta}$ as an operator*

$$\mathcal{T}_G^{b,\delta} \; : \; D_G(E_R) \;\to\; D_G(E_R)$$

The same with $\mathcal{T}_G^{*}$.

In Section 4 we had seen that the Value Iteration algorithm builds on the sequence $v_m^*(t,x;k)$ of the iterates of the operator $\mathcal{T}^*$ as specified in point *ii*) of Theorem 4.8. Here we shall therefore consider iterates of the operator $\mathcal{T}_G^*$ on E_R according to

$$v^*_{G,m}(t,x;k) := \begin{cases} 0 & \text{for } m = 0 \\ \mathcal{T}_G^*[v^*_{G,m-1}](t,x;k) & \text{for } m > 0 \end{cases} \tag{38}$$

In Section 4 the convergence of v_m^* to V^* was obtained on the basis of the contraction property of $\mathcal{T}^*$ on the Banach space $C(E)$. Here we are on the space $D(E_R)$ and so we could try to proceed analogously by using the Skorokhod norm. To obtain just the convergence, which is basically what we need, we can however avoid passing to the Skorokhod norm and base ourselves more simply on the following Proposition of which the proof is in the Appendix.

Proposition 5.12. *Given $\varepsilon > 0$, there exists an integer $m_\varepsilon > 0$ such that*

$$\|v^*_{G,m} - V^*_G\|_\infty < \varepsilon \quad \textit{for} \quad m > m_\varepsilon$$

Also for the approximating problem we can now state a Proposition that corresponds to Proposition 4.11 and for which the proof is completely analogous.

Proposition 5.13. *Given $\varepsilon > 0$, let m_ε be such that $\|V_G^* - v_{G,m}^*\|_\infty < \varepsilon$ for $m > m_\varepsilon$. Having chosen an $m > m_\varepsilon$ and assuming that the* inf *in (37) is actually a* min*, let $(B_G^m, D_G^m) \in \mathcal{U}_{re}^m \times \mathcal{U}_{inv}^m$ be the (non-stationary) policy obtained from the first m iterations of $\mathcal{T}_G^*$ when starting from $v = 0$, namely*

$$(B_G^m, D_G^m) = argmin_{(B^m,D^m)\in\mathcal{U}_{re}^m\times\mathcal{U}_{inv}^m} E_{0,\bar{X}(x);k}^{B^m,D^m}\left\{\sum_{j=1}^{m} e^{-\gamma\bar{T}_j} Y_{G,j}^{B^m,D^m}\mathbf{1}_{\{j\le N_T\}}\right\}$$

and let $(B_G^*, D_G^*) \in \mathcal{U}_{re} \times \mathcal{U}_{inv}$ be an arbitrary extension of (B_G^m, D_G^m) beyond the first m jumps. Then

$$\|V_G^* - V_G^{B_G^*,D_G^*}\|_\infty < 2\,\varepsilon$$

The nearly optimal policy for the approximating problem can now be described by complete analogy to subsection 4.2.1. We limit ourselves to mention how Step 3, for a generic $i \in \{2, \cdots, m-1\}$ with $i \le N_T$, adapts to the present case, namely we have

$$(b_{G,i}^m, \delta_{G,i}^m)(T_{i-1}, X_{i-1}^{B_G^m,D_G^m}; K_{i-1}) = argmin_{(b,\delta)\in U}\mathcal{T}_G^{b,\delta}\left[v_{G,m-i}^*\right](T_{i-1}, X_{i-1}^{B_G^m,D_G^m}; K_{i-1})$$

in the sense that $(b_{G,i}^m, \delta_{G,i}^m)$ is the action-pair corresponding to the policy (B_G^m, D_G^m) at the $i-$th event time in the state $(T_{i-1}, X_{i-1}^{B_G^m,D_G^m}; K_{i-1}) \in E_R$.

5.3.1 Computational aspects

To complete the Value Iteration algorithm we need to be able to compute the expectation in (36) and to minimize the resulting expression with respect to $(b,\delta) \in U$. Notice that the expectation concerns the random variables $Y_{G,n+1}^{b,\delta}$ as well as $(\bar{T}(t+S), X_{G,n+1}^{b,\delta}, K_{n+1})$. Considering the definitions in (27) and (29), the random variables with respect to which to perform the expectation are thus only $\bar{T}(t+S), \bar{X}_{G,n+1}^{b,\delta}, K_{n+1}$ that take values in the finite set $\bar{G} \times \{0,1\}$. Furthermore, the expectation is conditional on $(\bar{T}(t), \bar{X}(x), k)$ that also take values in $\bar{G} \times \{0,1\}$. Denote then by $(\bar{t}_i, \bar{x}_j, k)$ the generic element in $\bar{G} \times \{0,1\}$, where $i \in \{0, \cdots, N\}$, $j \in \{-M, \cdots, M\}$, $k \in \{0,1\}$. Assuming that $(\bar{T}(t), \bar{X}(x), k) = (\bar{t}_i, \bar{x}_j, k)$, the expectation in (36) can be rewritten as

$$\begin{aligned} &E_{\bar{T}(t),\bar{X}(x);k}^{b,\delta}\left\{e^{-\gamma(\bar{T}(t+S)-\bar{T}(t))}\mathbf{1}_{\{t+S\le T\}}\left[Y_{G,n+1}^{b,\delta} + v\left(\bar{T}(t+S), X_{G,n+1}^{b,\delta}; K_{n+1}\right)\right]\right\} \\ &= \textstyle\sum_{\ell=0}^{N}\sum_{m=-M}^{M}\sum_{h=0}^{1}\left\{e^{-\gamma(\bar{t}_\ell-\bar{t}_i)}\left[(\bar{x}_m)^- + v\left(\bar{t}_\ell, (x_m)^+, h\right)\right]\right\} \\ &\qquad P_{\bar{t}_i,\bar{x}_j,k}^{b,\delta}\left\{\left(\bar{T}(t+S), \bar{X}_{G,n+1}^{b,\delta}; K_{n+1}\right) = (\bar{t}_\ell, x_m, h)\right\} \end{aligned} \tag{39}$$

which can be readily computed once, for each pair $(b,\delta) \in U$, we are given the matrix of transition probabilities $P^{b,\delta}$.

In view of the matrix of transition probabilities, recall that the only driving quantities in the model are the independent sequences of i.i.d. random variables (S_n, Z_n, W_n). This allows one to derive from the distributions $G(s), F(z), H(w)$ of (S_n, Z_n, W_n) a stationary transition matrix. More precisely, with $\bar{t}_i = \bar{T}(t)$ where t is an event time, namely $t = T_{N_t} = T_n$, one has for m with $-M < m < M$

$$\begin{aligned} &P_{\bar{t}_i,\bar{x}_j,k}^{b,\delta}\left\{\left(\bar{T}(t+S), X_{G,n+1}^{b,\delta}; K_{n+1}\right) = (\bar{t}_\ell, x_m, h)\right\} \\ &= p_{k,h}P\left\{\ell\,\tfrac{T}{N} \le \bar{t}_i + S < (\ell+1)\,\tfrac{T}{N}, m\,\tfrac{R}{M} \le \bar{x}_j + c(b)S + h\delta W - (1-h)bZ < (m+1)\,\tfrac{R}{M}\right\} \end{aligned} \tag{40}$$

while for $m = M$ we put

$$P^{b,\delta}_{\bar{t}_i,\bar{x}_j,k}\left\{\left(\bar{T}(t+S), X^{b,\delta}_{G,n+1}; K_{n+1}\right) = (\bar{t}_\ell, R, h)\right\} = \varepsilon \tag{41}$$

and for $m = -M$,

$$\begin{aligned} & P^{b,\delta}_{\bar{t}_i,\bar{x}_j,k}\left\{\left(\bar{T}(t+S), X^{b,\delta}_{G,n+1}; K_{n+1}\right) = (\bar{t}_\ell, -R, h)\right\} \\ & = p_{k,h} P\big\{\ell\,\tfrac{T}{N} \le \bar{t}_i + S < (\ell+1)\,\tfrac{T}{N}, -R \le \bar{x}_j + c(b)S + h\delta W - (1-h)bZ < (1-M)\,\tfrac{R}{M}\big\} + \varepsilon \\ & \qquad \forall i, \ell \in \{0, \cdots, N\},\ j \in \{-M, \cdots, M\},\ h \in \{0,1\} \end{aligned} \tag{42}$$

On the basis of the last term in (35) (see also Corollary 3.3), for a given initial condition $x = X_0^{B,D}$, the ε in the last two expressions can be related to R via $\frac{x\,\sqrt{2G_2}\,E\{(N_T)^2\}+\sqrt{2G_1G_2}\,E\{(N_T)^3\}}{(R-x)}$, where G_1 and G_2 depend on the characteristics of the model. In practice we shall choose independently a sufficiently large R so that ε is small enough to become negligible. Choosing R as specified in Remark 5.1, the probability in (41) will be practically zero.

The detailed expressions in (40) can be readily worked out and can be found in [7]. As an example, we mention here just the case when $h = 0$, for which

$$\begin{aligned} & p_{k,0} P\big\{\ell\,\tfrac{T}{N} \le \bar{t}_i + S < (\ell+1)\,\tfrac{T}{N}, m\,\tfrac{R}{M} \le \bar{x}_j + c(b)S - bZ < (m+1)\,\tfrac{R}{M}\big\} \\ & = p_{k,0} \int_{\ell\,\frac{T}{N}-\bar{t}_i}^{(\ell+1)\,\frac{T}{N}-\bar{t}_i} \int_{\frac{1}{b}\left(\bar{x}_j+c(b)s-(m+1)\,\frac{R}{M}\right)}^{\frac{1}{b}\left(\bar{x}_j+c(b)s-m\,\frac{R}{M}\right)} dF(z)\,dG(s) \end{aligned}$$

Since $c(b)$ is a continuous function of b, the transition probability for the above case of $h = 0$ is a continuous function of b on $[0,1]$. When $h = 1$, one has to distinguish between $\delta < 0, \delta = 0, \delta > 0$ but, again, for $\delta < 0$ and $\delta > 0$ one obtains a continuous dependence the transition probability on δ in $[-C_1, 0)$ and $(0, C_2]$ respectively. The existence of a minimizer in (37) and thus of an optimal policy for the approximating problem is therefore guaranteed. How easy it is to actually do the computations depends on the form of the distributions $G(s), F(z), H(w)$. In those cases for which this turns out too difficult, one can proceed with a further discretization of $(b,\delta) \in [0,1]\times[-C_1, C_2]$ that leads to a finite number of pairs (b_i, δ_i), for each of which one has then to compute the (stationary) transition probability matrix and the minimization reduces then to one over a finite set. This is the approach taken in [7], where convergence is shown when the control discretization becomes finer and finer and this is also the approach underlying the numerical examples in the next section. In this way one obtains also for the approximating problem only a nearly optimal policy that, as shown, can however be extended to become, for sufficiently fine discretizations and a sufficiently large value of R, nearly optimal in the original problem.

6 Numerical results

We present here some numerical results for which we consider a problem with the following data.

- The horizon is $T = 10$ (days, hours,....)
- The random time S between events is supposed to be an exponential random variable with parameter $\lambda = 1.8$. It implies that the *average number of events* is $E\{N_T\} = \lambda T = 18$.

- The random claim size Z is also supposed to be an exponential random variable with parameter $\mu = 0.002$ implying that the average claim size is $E\{Z\} = 500$ USD, Euro,...
- The random asset return W is here supposed to be a standard Gaussian random variable, restricted to the interval $[\underline{w}, \bar{w}]$ with $\underline{w} = -0.9$, $\bar{w} = 5$ (the price can decrease at most by 90% and increase at most by 500%).
- The amount δ_t invested at time t in the risky asset is supposed to take values in the compact set $[-C_1, C_2]$ with $C_1 = 500$, $C_2 = 1000$.
- The transition matrix for the identifier process K_t has as entries $p_{00} = 0.2, p_{01} = 0.8, p_{10} = 0.4, p_{11} = 0.6$.
- The discount factor is taken as $e^{-0.03}$.
- If not explicitly stated otherwise (shall do it for comparison reasons), the initial wealth is $X_0 = 25$.
- The number of iterations for the Value Iteration algorithm is taken to be $m = 40$, namely by the *expected number* 18 *of events in* $[0, 10]$ *plus a multiple of the standard deviation.*
- The safety loading parameters are chosen as $\eta = 0.01$ for the insurer and $\theta = 0.02$ for the re-insurer (again, only for comparison reasons, we shall consider also $\theta = 0.05$).
- The optimal values for b and δ in the different states (t, x, k) are determined on the basis of a simple comparison algorithm after having discretized the values of $(b, \delta) \in [0, 1] \times [-C_1, C_2]$. As discretization step Δ for $[0, 1]$ we take $\Delta = 0.1$, while for $[-C_1, C_2]$ we take $\Delta = 100$.

The above data imply that (see (4)) $c(b) = 900\,(1.02\,b - 0.01)$ so that $c(b) = 0$ for $b^* = 0.0098$ and $c(b) \leq 909$. Over the given horizon $T = 10$ the wealth/reserve $X_n^{B,D}$ thus grows on average at most up to 9090 (case of no re-investement nor claims). According to Remark 5.1 a reasonable choice for R thus turns out to be the value $R = 6000$, i.e. roughly two thirds of this maximal value 9090.

Table I below reports the numerical results obtained for the given data with the values of $X_0 = 25$ and $\theta = 0.02$. The first four columns show the values (event time, event type, claim size, asset return) obtained for a sample simulation run (a peculiarity here is the large value of the first two claims). The next four columns are intended as a benchmark showing the values of the wealth process X_n and of the capital injections Y_n obtained for the fixed strategy $b_n \equiv 1$, $\delta_n \equiv 0$ (no insurance and no investment in the financial market). The remaining two pairs of four columns each show the optimal values obtained for b_n and δ_n as well as the values of the corresponding wealth process X_n and capital injections Y_n obtained from solving the approximating problem when the time interval $[0, 10]$ is discretized into 20 subintervals (denoted as N20) and the wealth interval $[-R, R] = [-6000, 6000]$ into 2×15 and into 2×24 intervals respectively (denoted as M15 and M24). The row "Discounted total capital injections" in the bottom part of Table I shows the values, obtained for the given specific simulation run, that correspond to the fixed benchmark strategy and to the optimal strategies computed for the two discretization levels respectively. The last two rows show the empirical average and standard deviation for the total discounted capital injections, obtained in the three cases, when averaging over 100000 simulation runs. Figure I is a graphical representation for the evolution of the wealth/reserve process X_n in the three cases and for the specific simulation run.

From the results one may notice the considerable decrease in the total discounted capital injections when using the optimal strategy instead of the fixed benchmark strategy. Finer discretizations levels than $N20M24$ have, for the given data, not led to significant improvements (the optimal amount of the total discounted capital injections remains roughly at the same level).

Concerning the optimal strategy one may notice that the optimal choice for b is to re-insure as much as possible (proportional reinsurance leads in fact to linearity in b). Notice that 0.1 is the first discretized value for b after $b = b^* = 0.0098$ (below the level b^* the $c(b)$ becomes negative). The optimal values for b depend of course also on the discrepancy between the two safety loading parameters η and θ. For comparison purposes, in the further Table II we thus report the results, obtained for the specific simulation run as in Table I (for convenience they are repeated in the first four columns of Table II), in the case when the value of η remains the same, but that of θ is increased to $\theta = 0.05$ (second group of four columns of Table II). One may notice that, given the larger reinsurance cost, at the beginning the proportion b of non re-insured claims is increased from 0.1 to 0.2 and the optimal amount of discounted total capital injections augments from 43.67 to 116.71.

What may not appear to be obvious is that, at least for the given data, the optimal amount δ to be invested in the risky asset is basically always equal to zero. In this context it has to be noted though that our criterion concerns the minimization of the capital injections, not the maximization of wealth. Again for comparison purposes, in the last four columns of Table II we report the results for the case when the initial wealth X_0 is considerably increased from the value 25 to 5000, the simulation data are as in Table I and the number of subintervals of $[-R, R] = [-6000, 6000]$ is increased from 24 to 30 (N20 M30 R6000). In the latter case the optimal investment strategy δ is not anymore zero, but requires to go short in the risky asset and this the more so the closer one gets to maturity. This can be see to be in line with findings in [13] and [16] where it is shown that, for a diffusion-type model with a volatility of the wealth/reserve process large with respect to the drift, ruin is certain when one invests in the financial market even independently of the initial wealth. Here we see in fact that, for moderate values of the wealth/reserve process, one should not invest in the financial market as it might become dangerous, particularly if the market shows a negative tendency.

Simulation results				Fixed strategy b=1, d=0				N20 M15 R6000				N20 M24 R6000			
Event time T_{n-1}	Type of event	Claim size	Return of asset	b_n	delta_n	X_{n-1}	Y_{n-1}	b_n	delta_n	X_{n-1}	Y_{n-1}	b_n	delta_n	X_{n-1}	Y_{n-1}
0.000	0	0	0	1	0	25.00	0	0.2	0	25	0	0.1	0	25	0
0.162	0	297.28	0	1	0	0	124.66	0.2	0	0	6.10	0.1	0	8.72	0
0.365	0	696.62	0	1	0	0	512.30	0.2	0	0	103.92	0.1	0	0	44.15
0.591	1	0	1.484	1	0	205.44	0	0.2	0	39.46	0	0.1	0	18.71	0
1.232	0	97.16	0	1	0	690.65	0	0.2	0	131.89	0	0.1	0	62.04	0
1.616	1	0	0.352	1	0	1040.29	0	0.2	0	199.05	0	0.1	0	93.89	0
3.837	1	0	1.443	1	0	3059.14	0	0.1	0	586.83	0	0.1	0	277.79	0
4.114	0	482.92	0	1	0	2827.31	0	0.1	0	561.41	0	0.1	0	252.37	0
4.123	1	0	1.784	1	0	2835.4	0	0.1	0	562.14	0	0.1	0	253.11	0
4.303	0	587.28	0	1	0	2412.09	0	0.1	0	518.35	0	0.1	0	209.31	0
4.469	1	0	1.646	1	0	2563.26	0	0.1	0	532.12	0	0.1	0	223.08	0
5.869	0	494.29	0	1	0	3341.11	0	0.1	0	598.57	0	0.1	0	289.53	0
6.112	1	0	-0.361	1	0	3562.88	0	0.1	0	618.77	0	0.1	0	309.73	0
6.396	1	0	0.470	1	0	3820.31	0	0.1	0	642.22	0	0.1	0	333.18	0
6.754	0	790.71	0	1	0	3355.03	0	0.1	0	592.79	0	0.1	0	283.75	0
7.022	0	733.02	0	1	0	2865.78	0	0.1	0	541.70	0	0.1	0	232.66	0
8.281	1	0	0.676	1	0	4009.97	0	0.1	0	645.92	0	0.1	0	336.88	0
8.328	1	0	-0.738	1	0	4052.87	0	0.1	0	649.83	0	0.1	0	340.79	0
9.024	1	0	-0.770	1	0	4685.72	0	0.1	0	707.47	0	0.1	0	398.43	0
9.832	1	0	-0.040	1	0	5419.58	0	0.1	0	774.32	0	0.1	0	465.28	0
Discounted total capital injections				*630.77*				*108.86*				*43.67*			

Average discounted total capital injections over 100000 simulations	180.30	32.25	15.1
Standard deviation	455.66	87.44	43.1

Table I. Main numerical results

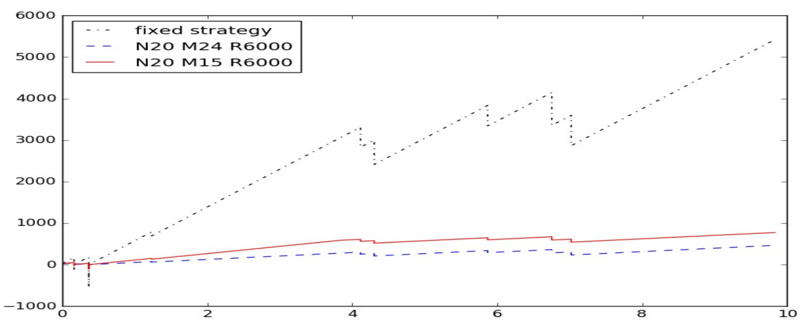

Figure I. Graphical representation

Simulation results as before				Theta=0.05, X_0=25, N20 M24 R6000				Theta=0.02, X_0=5000, N20 M30 R6000			
Event time T_{n-1}	Type of event	Claim size	Return of asset	b_n	delta_n	X_{n-1}	Y_{n-1}	b_n	delta_n	X_{n-1}	Y_{n-1}
0.000	0	0	0	0.2	0	25	0	0.1	0	5000	0
0.162	0	297.28	0	0.2	0	0	9.67	0.1	0	4983.69	0
0.365	0	696.62	0	0.2	0	0	108.27	0.1	0	4930.83	0
0.591	1	0	1.484	0.2	0	34.58	0	0.1	0	4949.54	0
1.232	0	97.16	0	0.2	0	113.22	0	0.1	0	4992.90	0
1.616	1	0	0.352	0.2	0	171.97	0	0.1	0	5024.70	0
3.837	1	0	1.443	0.1	0	511.78	0	0.1	0	5208.60	0
4.114	0	482.92	0	0.1	0	479.70	0	0.1	0	5183.24	0
4.123	1	0	1.784	0.1	0	480.22	0	0.1	0	5183.99	0
4.303	0	587.28	0	0.1	0	432.03	0	0.1	0	5140.16	0
4.469	1	0	1.646	0.1	0	441.74	0	0.1	0	5153.91	0
5.869	0	494.29	0	0.1	0	474.21	0	0.1	0	5220.40	0
6.112	1	0	-0.361	0.1	0	488.42	0	0.1	0	5240.52	0
6.396	1	0	0.470	0.1	0	505.04	0	0.1	0	5264.03	0
6.754	0	790.71	0	0.1	0	446.91	0	0	0	5214.61	0
7.022	0	733.02	0	0.1	0	389.28	0	0	0	5212.19	0
8.281	1	0	0.676	0.1	0	462.94	0	0	-100	5200.86	0
8.328	1	0	-0.738	0.1	0	465.69	0	0	-100	5274.24	0
9.024	1	0	-0.770	0.1	0	506.40	0	0	-100	5344.98	0
9.832	1	0	-0.040	0.1	0	553.67	0	0	-200	5337.70	0
Discounted total capital injections				116.71				0			

Table II. Results with changed data for comparison

Appendix.

Proof of Lemma 3.1

Proof. From the definition of $\Psi_n^{B,D}$ in (13) and the properties of the coefficients in its expression we obtain

$$|\Psi_n^{B,D}| \le |c(1)|\, S_n + \max[C_1, C_2]|W_n| + |Z_n|$$

Given that $S_n \ge 0$, from $\sum_{n=1}^{N_T} S_n = T_{N_T} \le T$ it follows that, for $n \le N_T$, one has $S_n \le T$. Considering also the i.i.d. property of W_n and Z_n and their independence on S_n, and thus on N_T, we then have $\forall n \le N_T$

$$E\{|\Psi_n^{B,D}| \mid N_T\} \le T\,|c(1)| + \max[C_1, C_2] E\{|W_1|\} + E\{Z_1\} = G_1$$

that does not depend on N_T. The first expression in the statement of the Lemma then follows from

$$E\{Q^{B,D} \mid N_T\} = \sum_{j=1}^{N_T} E\{|\Psi_j^{B,D}| \mid N_T\} \leq G_1 N_T$$

Next we have

$$\left(\Psi_n^{B,D}\right)^2 \leq 3\,(T\,c(1))^2 + 3\,(\max[C_1, C_2])^2 (W_n)^2 + 3\,(Z_n)^2 = G_2$$

from where, again by the i.i.d. and the independence property of S_n, W_n and Z_n, it follows that $E_{0,x;k}^{B,D}\left\{\left(\Psi_n^{B,D}\right)^2 \mid N_T\right\} \leq G_2$. The second statement now follows by

$$E_{0,x;k}^{B,D}\left\{(Q^{B,D})^2 \mid N_T\right\} = N_T \sum_{j=1}^{N_T} E_{0,x;k}^{B,D}\left\{\left(\Psi_j^{B,D}\right)^2 \mid N_T\right\} = G_2 (N_T)^2$$

□

Proof of Lemma 4.5

Proof. Let $(b^*, \delta^*) = argmin_{(b,\delta)\in U}\mathcal{T}^{b,\delta}[v](t,x;k)$ and $(b'^*, \delta'^*) = argmin_{(b,\delta)\in U}\mathcal{T}^{b,\delta}[v'](t,x;k)$, then

$$(\mathcal{T}^*[v] - \mathcal{T}^*[v'])(t,x;k) = \left(\mathcal{T}^{b^*,\delta^*}[v] - \mathcal{T}^{b'^*,\delta'^*}[v']\right)(t,x;k) \leq \left(\mathcal{T}^{b'^*,\delta'^*}[v] - \mathcal{T}^{b'^*,\delta'^*}[v']\right)(t,x;k)$$

$$\leq E_{t,x;k}^{b'^*,\delta'^*}\left\{e^{-\gamma S}\mathbf{1}_{\{t+S\leq T\}}\right\} ||v - v'||_\infty \leq \alpha\, ||v - v'||_\infty$$

Analogously

$$\left((\mathcal{T}^*[v'] - \mathcal{T}^*[v])(t,x;k)\right) \leq \left(\mathcal{T}^{b^*,\delta^*}[v'] - \mathcal{T}^{b^*,\delta^*}[v]\right)(t,x;k) \leq \alpha\, ||v - v'||_\infty$$

from which the conclusion follows. □

Proof of Lemma 4.6

Proof. With $t = T_{N_t} = T_n$ and $x = X_{T_n}, k = K_{T_n}$ we have

$$V^{B,D}(t,x;k) = E_{t,x;k}^{B,D}\left\{\textstyle\sum_{j=n+1}^{\infty} e^{-\gamma(T_j - t)} Y_j^{B,D}\mathbf{1}_{\{j\leq N_T\}}\right\}$$

$$= E_{t,x;k}^{B,D}\left\{e^{-\gamma S}\mathbf{1}_{\{n+1\leq N_T\}} Y_{n+1}^{B,D} + \textstyle\sum_{j=n+2}^{\infty} e^{-\gamma(T_j - t)} Y_j^{B,D}\mathbf{1}_{\{j\leq N_T\}}\right\}$$

$$= E_{t,x;k}^{b,d}\Big\{e^{-\gamma S}\mathbf{1}_{\{n+1\leq N_T\}}\Big(Y_{n+1}^{b,d} + E_{t,x;k}^{B,D}\left\{\textstyle\sum_{j=n+2}^{\infty} e^{-\gamma(T_j - T_{n+1})} Y_j^{B,D}\mathbf{1}_{\{j\leq N_T\}} \mid T_{n+1}, X_{T_{n+1}}^{B,D}; K_{T_{n+1}}\right\}\Big)\Big\}$$

$$= E_{t,x;k}^{b,d}\left\{e^{-\gamma S}\mathbf{1}_{\{n+1\leq N_T\}}\left(Y_{n+1}^{b,d} + V^{B,D}\left(T_{n+1}, X_{T_{n+1}}^{B,D}; K_{T_{n+1}}\right)\right)\right\} = \mathcal{T}^{b,\delta}[V^{B,D}](t,x;k)$$

□

Proof of Lemma 4.7

Proof. We use the fact that $V^{B,D}$ is a fixed point of the operator $\mathcal{T}^{B,D}$ in the sense of Lemma 4.6. In line with Remark 2.2 let $t = T_n$ for some n so that $t + S = T_{n+1}$. We then have (see also the proof of Lemma 4.6)

$$\begin{aligned}
&V^*(t,x;k) = \inf_{(B,D)\in\mathcal{U}_{re}\times\mathcal{U}_{inv}} V^{B,D}(t,x;k) = \inf_{(B,D)\in\mathcal{U}_{re}\times\mathcal{U}_{inv}} \mathcal{T}^{b_{n+1},\delta_{n+1}}[V^{B,D}](t,x;k)\\
&= \inf_{(b_{n+1},\delta_{n+1})\in U} E_{(t,x;k)}^{(b_{n+1},\delta_{n+1})}\Big\{e^{-\gamma S}\mathbf{1}_{\{n+1\le N_T\}}\Big[Y_{n+1}^{b_{n+1},\delta_{n+1}}\\
&\qquad\qquad + \inf_{(B,D)\in\mathcal{U}_{re}\times\mathcal{U}_{inv}} V^{B,D}(T_{n+1},X_{T_{n+1}}^{B,D};K_{T_{n+1}})\Big]\Big\}\\
&= \inf_{(b_{n+1},\delta_{n+1})\in U} E_{(t,x;k)}^{(b_{n+1},\delta_{n+1})}\left\{e^{-\gamma S}\mathbf{1}_{\{n+1\le N_T\}}\left[Y_{n+1}^{b_{n+1},\delta_{n+1}} + V^*\left(T_{n+1},X_{T_{n+1}}^{b_{n+1},\delta_{n+1}};K_{T_{n+1}}\right)\right]\right\}\\
&= \mathcal{T}^*[V^*](t,x;k)
\end{aligned}$$

where in the third equality we have used the DP principle. □

Proof of Lemma 5.7

Proof. We recall from (13) and (14) that $\hat{X}_n^{B,D} = X_{n-1}^{B,D} + \Psi_n^{B,D}$ and that $X_n^{B,D} = \left(\hat{X}_n^{B,D}\right)^+$ and $Y_n^{B,D} = \left(\hat{X}_n^{B,D}\right)^-$. We recall also from (28) and (29) that $\hat{X}_{G,n}^{B,D} = X_{G,n-1}^{B,D} + \Psi_n^{B,D}$ and that $X_{G,n}^{B,D} = \left(\bar{X}(\hat{X}_{G,n}^{B,D})\right)^+$, $Y_{G,n}^{B,D} = \left(\bar{X}(\hat{X}_{G,n}^{B,D})\right)^-$.

Given the assumptions, we may limit ourselves to the events $X_n^{B,D} \in [0,R]$. We start by showing the first relation in (34), which is immediately seen to be true for $n = 0$. For $n \ge 1$ we proceed by induction. For $n = 1$ we have

$$\begin{aligned}
|X_1^{B,D} - X_{G,1}^{B,D}| &= \left|\left(x+\Psi_1^{B,D}\right)^+ - \left(\bar{X}\left(\bar{x}+\Psi_1^{B,D}\right)\right)^+\right|\\
&\le \left|\left(x+\Psi_1^{B,D}\right)^+ - \left(\bar{x}+\Psi_1^{B,D}\right)^+\right| + \left|\left(\bar{x}+\Psi_1^{B,D}\right)^+ - \left(\bar{X}\left(\bar{x}+\Psi_1^{B,D}\right)\right)^+\right|\\
&\le |x-\bar{x}| + \frac{R}{M}
\end{aligned}$$

where the second summand in the last inequality follows from the fact that the corresponding previous expression concerns the difference between the positive part of a given quantity belonging to $[-R,R]$ and that of its projection on the grid G. The first summand results from the fact that the corresponding previous expressions are identical except for the initial condition. The result is immediate for the case when those expressions are either both positive or both negative. Otherwise we have

i) Case of $x+\Psi_1^{B,D} > 0$, $\bar{x}+\Psi_1^{B,D} < 0$:
$$\left|\left(x+\Psi_1^{B,D}\right)^+ - \left(\bar{x}+\Psi_1^{B,D}\right)^+\right| = x+\Psi_1^{B,D} < x-\bar{x}$$

ii) Case of $x+\Psi_1^{B,D} < 0$, $\bar{x}+\Psi_1^{B,D} > 0$:
$$\left|\left(x+\Psi_1^{B,D}\right)^+ - \left(\bar{x}+\Psi_1^{B,D}\right)^+\right| = |-\bar{x}-\Psi_1^{B,D}| = \bar{x}+\Psi_1^{B,D} < \bar{x}-x$$

Assuming then that the statement holds for $n-1$, we obtain

$$
\begin{aligned}
|X_n^{B,D} - X_{G,n}^{B,D}| &= \left| \left(\hat{X}_n^{B,D}\right)^+ - \left(\bar{X}(\hat{X}_{G,n}^{B,D})\right)^+ \right| \\
&\leq |\hat{X}_n^{B,D} - \bar{X}(\hat{X}_{G,n}^{B,D})| \leq |\hat{X}_n^{B,D} - \hat{X}_{G,n}^{B,D}| + |\hat{X}_{G,n}^{B,D} - \bar{X}(\hat{X}_{G,n}^{B,D})| \\
&\leq |X_{n-1}^{B,D} + \Psi_n^{B,D} - X_{G,n-1}^{B,D} - \Psi_n^{B,D}| + \tfrac{R}{M} \leq n\,\tfrac{R}{M} + \tfrac{R}{M} = (n+1)\,\tfrac{R}{M}
\end{aligned}
$$

where in the next-to-last inequality we have used the induction hypothesis.

Coming to the second relation in (34) notice that, since for $n \geq 1$,

$$
|Y_n^{B,D} - Y_{G,n}^{B,D}| = \left| \left(\hat{X}_n^{B,D}\right)^- - \left(\bar{X}(\hat{X}_{G,n}^{B,D})\right)^- \right| \leq |\hat{X}_n^{B,D} - \bar{X}(\hat{X}_{G,n}^{B,D})|,
$$

one can continue as above for the first relation. □

Proof of Proposition 5.8

Proof. Starting from $i)$ notice that, since (see (29)) $Y_{G,n}^{B,D} = \left(\bar{X}(\hat{X}_{G,n}^{B,D})\right)^-$ and $\hat{X}_{G,n}^{B,D}$ has as initial condition $\hat{X}_{G,0}^{B,D} = \bar{X}(x)$, the value function in Definition 5.5 can equivalently be expressed as $V_G^{B,D}(0,x;k) = E_{0,x;k}^{B,D}\left\{\sum_{j=1}^{N_T} e^{-\gamma \bar{T}_j} Y_{G,j}^{B,D}\right\}$ by conditioning simply on x instead of on $\bar{x} := \bar{X}(x)$. Introduce next the stopping time

$$
\nu_R^{B,D} := \min\left\{n \leq N_T \;|\; \hat{X}_n^{B,D} < -R\right\} \tag{43}
$$

and, for convenience, let $V_R^{B,D}(0,x;k) := E_{0,x;k}^{B,D}\left\{\sum_{j=1}^{\nu_R^{B,D}} e^{-\gamma T_j} Y_j^{B,D}\right\}$. We have now the following inequalities,

$$
\begin{aligned}
&|V^{B,D}(0,x;k) - V_G^{B,D}(0,x;k)| \leq |V^{B,D}(0,x;k) - V_R^{B,D}(0,x;k)| + |V_R^{B,D}(0,x;k) - V_G^{B,D}(0,x;k)| \\
&\quad \leq E_{0,x;k}^{B,D}\left\{\sum_{j=\nu_R^{B,D}+1}^{N_T} e^{-\gamma T_j} Y_j^{B,D}\right\} + E_{0,x;k}^{B,D}\left\{\sum_{j=1}^{N_T} e^{-\gamma T_j} |Y_j^{B,D} - Y_{G,j}^{B,D}|\right\} \\
&\qquad\qquad + E_{0,x;k}^{B,D}\left\{\sum_{j=1}^{N_T} |\, e^{-\gamma T_j} - e^{-\gamma \bar{T}_j}|\, Y_{G,j}^{B,D}\right\} \\
&\quad = I + II + III
\end{aligned} \tag{44}
$$

First, for I we have with the use of the Cauchy-Schwartz inequality and Corollaries 3.2 and 3.3

$$
\begin{aligned}
I &\le E^{B,D}_{0,x;k}\left\{\sum_{j=1}^{N_T} E^{B,D}_{0,x;k}\left\{\mathbf{1}_{\{\nu_R{}^{B,D}<N_T\}} Y_j^{B,D} \mid N_T\right\}\right\} \\
&\le E^{B,D}_{0,x;k}\left\{\sum_{j=1}^{N_T} \sqrt{E^{B,D}_{0,x;k}\{\mathbf{1}_{\{\nu_R{}^{B,D}<N_T\}} \mid N_T\}}\sqrt{E^{B,D}_{0,x;k}\{(Y_j^{B,D})^2 \mid N_T\}}\right\} \\
&\le E^{B,D}_{0,x;k}\left\{\sum_{j=1}^{N_T} \sqrt{P^{B,D}_{0,x;k}\left\{\sup_{i\le N_T} |\hat{X}_i^{B,D}| > R \mid N_T\right\}}\sqrt{E^{B,D}_{0,x;k}\{(Y_j^{B,D})^2 \mid N_T\}}\right\} \\
&\le E^{B,D}_{0,x;k}\left\{N_T \sqrt{\tfrac{G_2(N_T)^2}{(R-x)^2}}\sqrt{2x^2+2G_2(N_T)^2}\right\} \le E^{B,D}_{0,x;k}\left\{\tfrac{\sqrt{G_2}\,(N_T)^2}{R-x}\,\left(\sqrt{2}\,x+\sqrt{2G_1}\,N_T\right)\right\} \\
&= \tfrac{x\sqrt{2G_2}\,E\{(N_T)^2\}+\sqrt{2G_1G_2}\,E\{(N_T)^3\}}{R-x}
\end{aligned}
$$

Concerning II notice that, for $j \le \nu_R^{B,D}$ we have $X_j^{B,D} \in [-R,R]$ and so we can apply the second relation in (34) obtaining

$$II \le \frac{R}{M}\,E^{B,D}_{0,x;k}\left\{\sum_{j=1}^{N_T}(j+1)\right\} = \frac{R}{M}\,\left[\frac{3}{2}\,E\{N_T\}+\frac{1}{2}\,E\{N_T^2\}\right]$$

Coming to III recall that $Y_{G,j}^{B,D} \le R$. Since $\bar{T}_j \le T_j$, by Lagrange's theorem one has, with $c_j \in [T_j - \bar{T}_j]$,

$$III \le R\,E^{B,D}_{0,x;k}\left\{\sum_{j=1}^{N_T}\gamma e^{-\gamma c_j}\left(T_j-\bar{T}_j\right)\right\} \le \gamma R\,\frac{T}{N}\,E\{N_T\}$$

Putting all three parts together, we arrive at statement i).

Concerning statement ii), from i) we have that, for any given policy $(B^0,D^0) \in \mathcal{U}_{re}\times\mathcal{U}_{inv}$ on E,

$$\inf_{(B,D)\in\mathcal{U}_{re}\times\mathcal{U}_{inv}} V^{B,D}(0,x;k) \le V^{B^0,D^0}(0,x;k) \le V_{R,G}^{B^0,D^0}(0,x;k) + B(R,N,M)$$

and so

$$\inf_{(B,D)\in\mathcal{U}_{re}\times\mathcal{U}_{inv}} V^{B,D}(0,x;k) \le \inf_{(B,D)\in\mathcal{U}_{re}\times\mathcal{U}_{inv}} V_{R,G}^{B,D}(0,x;k) + B(R,N,M) \tag{45}$$

Analogously

$$\inf_{(B,D)\in\mathcal{U}_{re}\times\mathcal{U}_{inv}} V_{R,G}^{B,D}(0,x;k) \le \inf_{(B,D)\in\mathcal{U}_{re}\times\mathcal{U}_{inv}} V^{B,D}(0,x;k) + B(R,N,M) \tag{46}$$

From (45) and (46) the statement ii) follows immediately. □

Proof of Proposition 5.12

Proof. The cost-to-go function $V_G^{B,D}(t,x;k)$ in (33), evaluated at $t = T_{N_t} = T_n < T$, can be written for a given integer $m > 0$ as

$$
\begin{aligned}
V_G^{B,D}(t,x;k) = E^{B,D}_{\bar{T}(t),\bar{X}(x);k}&\left\{\sum_{j=n+1}^{N_T} e^{\gamma\,(\bar{T}_j-\bar{T}(t))}Y_{G,j}^{B,D}\,\mathbf{1}_{\{T_{n+1+m}<T\}}\right\} \\
&+E^{B,D}_{\bar{T}(t),\bar{X}(x);k}\left\{\sum_{j=n+1}^{N_T} e^{\gamma\,(\bar{T}_j-\bar{T}(t))}Y_{G,j}^{B,D}\,\mathbf{1}_{\{T_{n+1+m}\ge T\}}\right\}
\end{aligned}
$$

The second term on the right corresponds to $V_G^{B,D}(t,x;k)$ when one considers at most m event times after t and it can be obtained by iterating m times the operator $\mathcal{T}_G^{b,\delta}$ starting from $v=0$ and where (b,δ) is induced by (B,D) in the various current states $(t,x;k)$. It can be rewritten as $V_G^{B^m,D^m}(t,x;k)$ (for (B^m,D^m) see Definition 4.10 and subsection 4.2.1). Since all the terms in the sum are positive and $Y_{G,j}^{B,D}\leq R$, we have

$$E_{\bar{T}(t),\bar{X}(x);k}^{B^m,D^m}\left\{\sum_{j=n+1}^{N_T} e^{\gamma\,(\bar{T}_j-\bar{T}(t))}Y_{G,j}^{B^m,D^m}\mathbf{1}_{\{T_{n+1+m}\geq T\}}\right\}\leq E_{\bar{T}(t),\bar{X}(x);k}^{B,D}\left\{\sum_{j=n+1}^{N_T} e^{\gamma\,(\bar{T}_j-\bar{T}(t))}Y_{G,j}^{B,D}\right\}$$
$$\leq E_{\bar{T}(t),\bar{X}(x);k}^{B^m,D^m}\left\{\sum_{j=n+1}^{N_T} e^{\gamma\,(\bar{T}_j-\bar{T}(t))}Y_{G,j}^{B^m,D^m}\mathbf{1}_{\{T_{n+1+m}\geq T\}}\right\}+R\,E\left\{N_T\mathbf{1}_{\{N_T>n+1+m\}}\right\} \tag{47}$$

having used the fact that $\mathbf{1}_{\{T_{n+1+m}<T\}}=\mathbf{1}_{\{N_T>n+1+m\}}$. By the comments preceding (47) we may rewrite (47) as

$$V_G^{B^m,D^m}(t,x;k)\leq V_G^{B,D}(t,x;k)\leq V_G^{B^m,D^m}(t,x;k)+R\,E\left\{N_T\mathbf{1}_{\{N_T>n+1+m\}}\right\} \tag{48}$$

From (47) we also have

$$\begin{aligned}
&\inf_{(B^m,D^m)\in\mathcal{U}_{re}^m\times\mathcal{U}_{inv}^m} E_{\bar{T}(t),\bar{X}(x);k}^{B^m,D^m}\left\{\sum_{j=n+1}^{N_T} e^{\gamma\,(\bar{T}_j-\bar{T}(t))}Y_{G,j}^{B^m,D^m}\mathbf{1}_{\{T_{n+1+m}\geq T\}}\right\}\\
&\leq \inf_{(B,D)\in\mathcal{U}_{re}\times\mathcal{U}_{inv}} E_{\bar{T}(t),\bar{X}(x);k}^{B,D}\left\{\sum_{j=n+1}^{N_T} e^{\gamma\,(\bar{T}_j-\bar{T}(t))}Y_{G,j}^{B,D}\right\}\\
&\leq \inf_{(B^m,D^m)\in\mathcal{U}_{re}^m\times\mathcal{U}_{inv}^m} E_{\bar{T}(t),\bar{X}(x);k}^{B^m,D^m}\left\{\sum_{j=n+1}^{N_T} e^{\gamma\,(\bar{T}_j-\bar{T}(t))}Y_{G,j}^{B^m,D^m}\mathbf{1}_{\{T_{n+1+m}\geq T\}}\right\}\\
&\qquad\qquad +R\,E\left\{N_T\mathbf{1}_{\{N_T>n+1+m\}}\right\}
\end{aligned} \tag{49}$$

Based on the Dynamic Programming recursions one can furthermore deduce that (see e.g. [17])

$$\inf_{(B^m,D^m)\in\mathcal{U}_{re}^m\times\mathcal{U}_{inv}^m} E_{\bar{T}(t),\bar{X}(x);k}^{B^m,D^m}\left\{\sum_{j=n+1}^{N_T} e^{\gamma\,(\bar{T}_j-\bar{T}(t))}Y_{G,j}^{B^m,D^m}\mathbf{1}_{\{T_{n+1+m}\geq T\}}\right\}=v_{G,m}^*(t,x;k)$$

namely the $m-$th iterate of $\mathcal{T}_G^*$ starting from $v=0$. The inequalities (49) then lead to

$$v_{G,m}^*(t,x;k)\leq V_G^*(t,x;k)\leq v_{G,m}^*(t,x;k)+R\,E\left\{N_T\mathbf{1}_{\{N_T\geq n+1+m\}}\right\} \tag{50}$$

Next, using the Cauchy-Schwartz and Markov inequalities, one obtains

$$\begin{aligned}
E\left\{N_T\mathbf{1}_{\{N_T>n+1+m\}}\right\}&\leq\sqrt{E\{N_T^2\}\,E\left\{\mathbf{1}_{\{N_T>n+1+m\}}\right\}}\\
&\leq\sqrt{E\{N_T^2\}\,P\{N_T>m\}}\leq\frac{1}{\sqrt{m}}\sqrt{E\{N_T^2\}\,E\{N_T\}}
\end{aligned}$$

which tends to zero with $m\to\infty$ independently of the choice of $(t,x;k)\in E_R$. There exists thus m_ε, independent of $(t,x;k)\in E_R$ such that, for $m>m_\varepsilon$ one has $R\,E\left\{N_T\mathbf{1}_{\{N_T\geq n+1+m\}}\right\}<\varepsilon$ which concludes the proof of the Proposition. □

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