Expected log-utility maximization under incomplete information and with Cox-process observations

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Abstract

We consider the portfolio optimization problem for the criterion of maximization of expected terminal log-utility. The underlying market model is a regime-switching diffusion model where the regime is determined by an unobservable factor process forming a finite state Markov process. The main novelty is due to the fact that prices are observed and the portfolio is rebalanced only at random times corresponding to a Cox process where the intensity is driven by the unobserved Markovian factor process as well. This leads to a more realistic modeling for many practical situations, like in markets with liquidity restrictions; on the other hand it considerably complicates the problem to the point that traditional methodologies cannot be directly applied. The approach presented here is specific to the log-utility. For power utilities a different approach is presented in the companion paper [11].

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1 Introduction

Among the optimization problems in finance, portfolio optimization is one of the first and most important problems. A classical formulation for this problem is the maximization of expected utility from terminal wealth and here we shall consider the case of a log-utility. What is novel in our paper is the market model, where we assume that the dynamics of the prices, in which one invests, are of the usual diffusion type having however the following two peculiarities:

- the coefficients in the dynamics depend on an unobservable finite-state Markovian factor process θ_t (regime-switching model);
- the prices S_t^i of the risky assets, or equivalently their log-values, are observed only at doubly stochastic random times $\tau_0, \tau_1, \cdots$, for which the associated counting process forms a Cox process (see e.g. [4], [15]) with an intensity $n(\theta_t)$ that depends on the same unobservable factor process θ_t.

Such models are relevant in financial applications for various reasons: regime switching models, which are relevant also in various other applied areas, have been extensively used in the financial literature, because they account for various stylized facts, such as e.g. the volatility clustering. On the other hand, random discrete time observations are more realistic in comparison to diffusion-type models since, especially on small time scales, prices do not vary continuously, but rather change and are observed only at random times in reaction to trading or the arrival of significant new information and it is reasonable to assume that the intensity of the price changes depends on the same factors that specify the regime for the price evolution (see e.g. [10], [7]).

The partial observation setup, due to the non direct observability of the Markovian factors and their consequent updating on the basis of the actual observations, allows for a continuous updating of the underlying model and there is a huge literature on hidden Markov factor/regime-switching models (for a monograph see e.g. [9]). Concerning hidden Markov factors in portfolio optimization we limit ourselves to mention only some of the most recent ones that also summarize previous work in the area, more precisely [3], [20], [25] as well as the recent monograph [2] and, for the case when also defaultable securities are included, [5].

Due to the fact that the prices of the assets in which one invests are only observed at the random times τ_k, we shall restrict our investment strategies to be rebalanced only at those same time points. Although slightly less general from a theoretical point of view, restricting trading to discrete, in particular random times, is quite realistic in finance, where in practice one cannot rebalance a portfolio continuously: think of the case with transaction costs or with liquidity restrictions (in this latter context see e.g. [12], [13], [18], [22], [23], [24] where the authors consider illiquid markets, partly also with regime switching models as in this paper, but under complete information).

Our problem belongs thus to the class of stochastic control problems under incomplete information with the objective given by the maximization of expected log-utility, but it has the peculiarity that the observations are given by a Cox process with intensity depending on the unobserved factor process. Log-optimal investment problems have by now been studied extremely well in the literature and it is generally known that the log-optimal portfolio is myopic, i.e. it depends only on the local dynamics of the tradable assets. A recent rather complete account can be found in [14], where the asset prices are supposed to be general semimartingales, but there is complete information on the underlying market model and the asset price observations take place continuously in time. The conclusions of [14] cannot be extended directly to the setup of this paper (see more in the next paragraph) and one of the objectives here is to show that, also in our context, one can obtain similar results. Maximization of expected log-and power-utility of terminal wealth has in particular been studied also in [6], but for a simpler model than the present one, where the prices follow a pure jump process, the coefficients in the model are deterministically given, and only the jump intensity is unobserved. It is shown in [6] that, in the partial information case, the approach used for log-utility cannot be carried over straightforwardly to the power utility case, even if there are close analogies, and so for the latter case a different approach is presented. This happens also in our situation and, in fact, for the same model as in the present paper, we treat the power utility case in the companion paper [11] by a different approach.

The standard approach to incomplete observation stochastic control problems is to transform them into the so-called "separated problem", where the unobservable quantities are replaced by their conditional distributions. This requires to

- solve the associated filtering problem;
- formulate the separated problem so that its solution is indeed a solution of the original incomplete observation problem.

The filtering part of our problem has been studied in [7] (see also [8]), where it was found that (see Abstract in [7]) "the given problem does not fit into the standard diffusion or simple point process filtering framework and requires more technical tools". Indeed, for a given function $f(\theta)$ defined on the state space $E = \{e_1,\ e_2,\ \ldots, e_N\}$ of our hidden Markov process $\{\theta_t\}$, the filtering equation is given as in (3.6) according to [7]. However, since our observations take place only along a sequence of random times $\tau_0,\ \tau_1,\ \tau_2,\ \ldots$ up to a given time horizon T, useful information arrives in a discrete way, namely via $\pi_t^i \equiv \pi_t(\mathbf{1}_{e_i}(\theta_t))$ evaluated at the indicator functions $\mathbf{1}_{e_i}(\cdot)$, $i = 1, \ldots, N$ along the sequence $\tau_0,\ \tau_1,\ \tau_2,\ \ldots$. The corresponding dynamics are:

$$\pi^i_{\tau_{k+1}} = M^i(\tau_{k+1} - \tau_k, \tilde{X}_{\tau_{k+1}} - \tilde{X}_{\tau_k}, \pi_{\tau_k})$$

with the discounted log prices $\tilde{X}_{\tau_k}$, as it is seen in (3.13), where $M = (M^1, \ldots, M^N)$ is a function taking its values on the $N-1$- dimensional simplex (2.2) and defined by (3.10)-(3.11). This follows from the filtering results. Thus we obtain the Markov process $\{\tau_k, \pi_{\tau_k}, \tilde{X}_{\tau_k}\}_{k=1}^{\infty}$, with respect to the filtration $\mathcal{G}_k$ defined in (2.11), forming the state variable process of our reduced control problem with full information. Further, our portfolio strategies h_t^i on the interval $[\tau_k, \tau_{k+1})$ are determined by $\tilde{X}_t$, $\tilde{X}_{\tau_k}$ and the $\mathcal{G}_k$ measurable random variable h_k as is seen in (2.16). Note that once we choose a strategy h_k at a time instant τ_k, then the portfolio strategy on $[\tau_k, \tau_{k+1})$ is determined by the dynamics of the securities prices. Therefore we take as the set of admissible strategies the totality of the sequences of $\mathcal{G}_k$ measurable random variables taking their values in the m dimensional simplex $\bar{H}_m$ defined in (2.14), where m is the number of risky assets. Then, our original criterion defined by (4.4) and (4.5) can be reformulated by a function $\hat{C}$ of $(\tau_k, \pi_{\tau_k}, h_k)$ as is shown in (4.7) in Lemma 4.1. This part of our results has a crucial meaning since, even if we choose our strategy h_k only at the time instants τ_k, the portfolio proportion h_t depends on the evolution of the securities prices that, on each time interval (τ_k, τ_{k+1}) between the observation points τ_k and τ_{k+1} is unobservable and our original criterion depends on them as well as on the unobservable state process θ_t.

Next, we note that the sum appearing in the right hand side of (4.7) is infinite since, although the number of observation times τ_k up to T is finite a.s., it depends on ω. Therefore, the myopic strategy maximizing each $\hat{C}(\tau_k, \pi_{\tau_k}, h)$ on $\bar{H}_m$ cannot be shown directly to be the optimal one. We thus proceeded to obtain

- an approximation result leading to a "value iteration-type" algorithm;
- a general dynamic programming principle.

At this point one might observe that our problem setup is analogous to a discrete time, infinite horizon problem with discounting because, in fact, the trading times are discrete in nature and may be infinite in number as we mentioned above. Furthermore, as we shall show below, the operator that is implicit in the dynamic programming principle is a contraction operator. We want however to point out that our results are obtained by an approach that is specific to our problem as described above and they cannot just be obtained on the basis of the apparent analogy with the discrete time infinite horizon problem with discounting. On the other hand, concerning the optimal strategy we show that also in our setup it turns out to be of the myopic type.

Our results for the control part of the problem concern both the value function and the optimal control/strategy. Many studies in stochastic control concern only the value function and those deriving also the optimal control obtain it generally on the basis of the value function for which the latter has to be sufficiently regular. Since, as we show also for our setup, the optimal strategy can be derived directly on the basis of the local dynamics of the asset prices, the value function is derived here for its own interest (one may in fact want to know what is the best that one can achieve with a given problem formulation).

The paper is structured as follows. In Section 2 we give a more precise definition of the model and the investment strategy and specify the objective function. In Section 3 we recall the relevant filtering results from the literature and, on the basis of these results, we introduce an operator that is important for the control results. The control part is then studied in section 4 with the main result stated in Theorem 4.1. In view of proving this theorem, Section 4 contains various preliminary results with the more technical proofs deferred to the Appendix.

2 Market model and objective

2.1 Introductory remarks

As mentioned in the general Introduction, we consider here the problem of maximization of expected log-utility from terminal wealth, when the dynamics of the prices of the risky assets in which one invests are of the usual diffusion type but with the coefficients in the dynamics depending on an unobservable finite-state Markovian factor process (regime-switching model). In addition it is assumed that the risky asset prices S_t^i, or equivalently their logarithmic values $X_t^i := \log S_t^i$, are observed only at random times $\tau_0, \tau_1, \cdots$ for which the associated counting process forms a Cox process with an intensity $n(\theta_t)$ that also depends on the unobservable factor process θ_t.

2.2 The market model and preliminary notations

Let θ_t be the hidden finite state Markovian factor process. With Q denoting its transition intensity matrix ($Q-$matrix) its dynamics are given by

$$d\theta_t = Q^*\theta_t dt + dM_t, \quad \theta_0 = \xi, \tag{2.1}$$

where M_t is a jump-martingale on a given filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. If N is the number of possible values of θ_t, we may without loss of generality take as its state space the set $E = \{e_1, \ldots, e_N\}$, where e_i is a unit vector for each $i = 1, \ldots, N$ (see [9]).

The evolution of θ_t may also be characterized by the process π_t given by the state probability vector that takes values in the set

$$\mathcal{S}_N := \{\pi \in \mathbb{R}^N \mid \sum_{i=1}^N \pi^i = 1, 0 \le \pi^i \le 1, i = 1, 2, \ldots, N\} \tag{2.2}$$

namely the set of all probability measures on E and we have $\pi_0^i = P(\xi = e_i)$. Denoting by $\mathcal{M}(E)$ the set of all finite nonnegative measures on E, it follows that $\mathcal{S}_N \subset \mathcal{M}(E)$. In our study it will be convenient to consider on $\mathcal{M}(E)$ the Hilbert metric $d_H(\pi, \bar{\pi})$ defined (see [1] [16] [17]) by

$$d_H(\pi, \bar{\pi}) := \log(\sup_{\bar{\pi}(A)>0, A\subset E} \frac{\pi(A)}{\bar{\pi}(A)} \sup_{\pi(A)>0, A\subset E} \frac{\bar{\pi}(A)}{\pi(A)}). \tag{2.3}$$

Notice that, while d_H is only a pseudo-metric on $\mathcal{M}(E)$, it is a metric on $\mathcal{S}_N$ ([1]).

In our market we consider m risky assets, for which the price processes $S^i = (S_t^i)_{t\ge 0}, i = 1, \ldots, m$ are supposed to satisfy

$$dS_t^i = S_t^i \{r^i(\theta_t)dt + \sum_j \sigma_j^i(\theta_t) dB_t^j\}, \tag{2.4}$$

for given coefficients $r^i(\theta)$ and $\sigma_j^i(\theta)$ and with B_t^j $(j = 1, \cdots, m)$ independent $(\mathcal{F}_t, P)-$Wiener processes. Letting $X_t^i = \log S_t^i$, by Itô's formula we have, in vector notation,

$$X_t = X_0 + \int_0^t r(\theta_s) - d(\sigma\sigma^*(\theta_s))ds + \int_0^t \sigma(\theta_s) dB_s, \tag{2.5}$$

where by $d(\sigma\sigma^*(\theta))$ we denote the column vector $(\frac{1}{2}(\sigma\sigma^*)^{11}(\theta), \ldots, \frac{1}{2}(\sigma\sigma^*)^{mm}(\theta))$. As usual there is also a locally non-risky asset (bond) with price S_t^0 satisfying

$$dS_t^0 = r_0 S_t^0 dt \tag{2.6}$$

where r_0 stands for the short rate of interest. We shall also make use of discounted asset prices, namely

$$\tilde{S}_t^i := \frac{S_t^i}{S_t^0}, \quad \text{with} \quad \tilde{X}_t^i := \log \tilde{S}_t^i \tag{2.7}$$

for which, by Itô's formula

$$d\tilde{S}_t^i = \tilde{S}_t^i \{(r^i(\theta_t) - r_0)dt + \sum_j \sigma_j^i(\theta_t) dB_t^j\}, \tag{2.8}$$

$$d\tilde{X}_t^i = \{r^i(\theta_t) - r_0 - d(\sigma\sigma^*(\theta_t))^i\}dt + \sum_{j=1}^{m} \sigma_j^i(\theta_t) dB_t^j. \tag{2.9}$$

As already mentioned, the asset prices and thus also their logarithms are observed only at random times $\tau_0, \tau_1, \tau_2, \ldots$. The observations are thus given by the sequence $(\tau_k, \tilde{X}_{\tau_k})_{k\in\mathbb{N}}$ that forms a multivariate marked point process with counting measure

$$\mu(dt, dx) = \sum_k \mathbf{1}_{\{\tau_k<\infty\}} \delta_{\{\tau_k, \tilde{X}_{\tau_k}\}}(t, x) dt dx. \tag{2.10}$$

The corresponding counting process $\Lambda_t := \int_0^t \int_{\mathbb{R}^m} \mu(dt, dx)$ is supposed to be a Cox process with intensity $n(\theta_t)$, i.e. $\Lambda_t - \int_0^t n(\theta_s) ds$ is an $(\mathcal{F}_t, P)-$ martingale. We consider two sub-filtrations related to $(\tau_k, \tilde{X}_{\tau_k})_{k\in\mathbb{N}}$ namely

$$\begin{aligned} \mathcal{G}_t &:= \mathcal{F}_0 \vee \sigma\{\mu((0, s] \times B) : s \leq t, B \in \mathcal{B}(\mathbb{R}^m)\}, \\ \mathcal{G}_k &:= \mathcal{F}_0 \vee \sigma\{\tau_0, \tilde{X}_{\tau_0}, \tau_1, \tilde{X}_{\tau_1}, \tau_2, \tilde{X}_{\tau_2}, \ldots, \tau_k, \tilde{X}_{\tau_k}\}. \end{aligned} \tag{2.11}$$

where, again for simplicity, $\mathcal{G}_k$ stands for $\mathcal{G}_{\tau_k}$.

In our development below we shall often make use of the following notations. For the conditional (on $\mathcal{F}^\theta$) mean and variance of $\tilde{X}_t - \tilde{X}_{\tau_k}$ we set

$$\begin{aligned} m_k^\theta(t) &= \int_{\tau_k}^t [r(\theta_s) - r_0\mathbf{1} - d(\sigma\sigma^*(\theta_s))] ds, \\ \sigma_k^\theta(t) &= \int_{\tau_k}^t \sigma\sigma^*(\theta_s) ds \end{aligned} \tag{2.12}$$

and, for $z \in \mathbb{R}^m$, we set

$$\rho_{\tau_k,t}^\theta(z) \sim N(z; m_k^\theta(t), \sigma_k^\theta(t)) \tag{2.13}$$

namely the joint conditional (on $\mathcal{F}^\theta$) m-dimensional normal density function with mean vector $m_k^\theta(t)$ and covariance matrix $\sigma_k^\theta(t)$. In (2.13) the symbol $\sim$ stands for "distributed according to".

2.3 Investment strategies, portfolios, objective

As mentioned in the Introduction, since observations take place at random time points τ_k, we shall consider investment strategies that are rebalanced only at those same time points τ_k.

Let N_t^i be the number of assets of type i held in the portfolio at time t, $N_t^i = \sum_k \mathbf{1}_{[\tau_k, \tau_{k+1})}(t) N_k^i$. The wealth process is defined by

$$V_t := \sum_{i=0}^{m} N_t^i S_t^i.$$

Consider then the investment ratios

$$h_t^i := \frac{N_t^i S_t^i}{V_t},$$

and set, for simplicity of notation, $h_k^i := h_{\tau_k}^i$. The set of admissible investment ratios is given by

$$\bar{H}_m := \{(h^1, \ldots, h^m); h^1 + h^2 + \ldots + h^m \leq 1, 0 \leq h^i, i = 1, 2, \ldots, m\}, \tag{2.14}$$

i.e. no shortselling is allowed and notice that $\bar{H}_m$ is bounded and closed. Put $h = (h^1, \cdots, h^m)$. Analogously to [19] define next a function $\gamma : \mathbb{R}^m \times \bar{H}_m \to \bar{H}_m$ by

$$\gamma^i(z,h) := \frac{h^i \exp(z^i)}{1 + \sum_{i=1}^{m} h^i(\exp(z^i) - 1)}, \qquad i = 1,, \ldots, m. \tag{2.15}$$

Noticing that N_t is constant on $[\tau_k, \tau_{k+1})$, for $i = 1, \ldots, m$, and $t \in [\tau_k, \tau_{k+1})$ let

$$\begin{aligned} h_t^i &= \frac{N_t^i S_t^i}{\sum_{i=0}^m N_t^i S_t^i} = \frac{N_k^i S_t^i}{\sum_{i=0}^m N_k^i S_t^i} \\ &= \frac{N_k^i S_{\tau_k}^i S_t^i / S_{\tau_k}^i}{\sum_{i=0}^m N_k^i S_{\tau_k}^i S_t^i / S_{\tau_k}^i} = \frac{h_k^i S_t^i / S_{\tau_k}^i}{\sum_{i=0}^m h_k^i S_t^i / S_{\tau_k}^i} = \frac{h_k^i S_{\tau_k}^0 / S_t^0 S_t^i / S_{\tau_k}^i}{\sum_{i=0}^m h_k^i S_{\tau_k}^0 / S_t^0 S_t^i / S_{\tau_k}^i} \\ &= \frac{h_k^i \exp(\tilde{X}_t^i - \tilde{X}_{\tau_k}^i)}{h_k^0 + \sum_{i=1}^m h_k^i \exp(\tilde{X}_t^i - \tilde{X}_{\tau_k}^i)} = \frac{h_k^i \exp(\tilde{X}_t^i - \tilde{X}_{\tau_k}^i)}{1 + \sum_{i=1}^m h_k^i (\exp(\tilde{X}_t^i - \tilde{X}_{\tau_k}^i) - 1)} \\ &= \gamma^i(\tilde{X}_t - \tilde{X}_{\tau_k}, h_k). \end{aligned} \tag{2.16}$$

The set of admissible strategies $\mathcal{A}$ is defined by

$$\mathcal{A} := \{\{h_k\}_{k=0}^\infty | h_k \in \bar{H}_m,\ \mathcal{G}_k \text{ measurable for all } k \geq 0\}. \tag{2.17}$$

Furthermore, for $n > 0$, we let

$$\mathcal{A}^n := \{h \in \mathcal{A} | h_{n+i} = h_{\tau_{n+i}-} \text{ for all } i \geq 1\}. \tag{2.18}$$

Notice that, by the definition of $\mathcal{A}^n$, for all $k \geq 1,\ h \in \mathcal{A}^n$ we have

$$h_{n+k}^i = h_{\tau_{n+k}-}^i$$

$$\Leftrightarrow \frac{N_{n+k}^i S_{\tau_{n+k}}^i}{\sum_{i=0}^m N_{n+k}^i S_{\tau_{n+k}}^i} = \frac{N_{n+k-1}^i S_{\tau_{n+k}}^i}{\sum_{i=0}^m N_{n+k}^i S_{\tau_{n+k}}^i}$$

$$\Leftrightarrow N_{n+k} = N_{n+k-1}.$$

Therefore, for $k \geq 1$

$$N_{n+k} = N_n,$$

and

$$\mathcal{A}^0 \subset \mathcal{A}^1 \subset \cdots \mathcal{A}^n \subset \mathcal{A}^{n+1} \cdots \subset \mathcal{A}. \tag{2.19}$$

Remark 2.1. *Notice that, for a given finite sequence of investment ratios $h_0, h_1, \cdots, h_n$ such that h_k is an $\mathcal{G}_k-$measurable, $\bar{H}_m-$valued random variable for $k \leq n$, there exists $h^{(n)} \in \mathcal{A}^n$ such that $h_k^{(n)} = h_k$, $k = 0, \cdots, n$. Indeed, if N_t is constant on $[\tau_n, T)$, then for h_t we have $h_t = \gamma(\tilde{X}_t - \tilde{X}_{\tau_n}, h_n)$, $\forall t \geq \tau_n$. Therefore, by setting $h_\ell^{(n)} = h_\ell$, $\ell = 0, \cdots, n$, and $h_{n+k}^{(n)} = h_{\tau_{n+k}}$, $k = 1, 2, \cdots$, since the vector process S_t and the vector function $\gamma(\cdot, h_n)$ are continuous, we see that $h_{n+k}^{(n)} = h_{\tau_{n+k}-}$, $k = 1, 2, \cdots$.*

Finally, considering only self-financing portfolios, for their value process we have the dynamics

$$\frac{dV_t}{V_t} = [r_0 + h_t^* \{r(\theta_t) - r_0 \mathbf{1}\}]dt + h_t^* \sigma(\theta_t) dB_t. \tag{2.20}$$

Problem: Given a finite planning horizon $T > 0$, our problem of maximization of expected terminal log-utility consists in determining

$$\sup_{h \in \mathcal{A}} E[\log V_T | \tau_0 = 0, \pi_{\tau_0} = \pi]$$

as well as an optimal maximizing strategy $\hat{h} \in \mathcal{A}$.

3 Filtering

As mentioned in the Introduction, the standard approach to stochastic control problems under incomplete information is to first transform them into a so-called separated problem, where the unobservable part of the state is replaced by its conditional (filter) distribution. This implies that we first have to study this conditional distribution and its (Markovian) dynamics, i.e. we have to study the associated filtering problem.

The filtering problem for our specific case, where the observations are given by a Cox process with intensity expressed as a function of the unobserved state, has been studied in [7] (see also [8]). In the first subsection 3.1 we therefore summarize the main results from [7] in view of their use in our control problem in section 4. Related to the filter, in subsection 3.2 we shall then introduce a contraction operator that will play a major role in obtaining the results in section 4, in particular for the approximation result in the main Theorem 4.1.

3.1 General Filtering Equation

Recalling the definition of $\rho^\theta(z)$ in (2.13) and putting

$$\phi^\theta(\tau_k, t) = n(\theta_t) \exp(-\int_{\tau_k}^t n(\theta_s) ds), \tag{3.1}$$

for a given function $f(\theta)$ we let

$$\psi_k(f; t, x) := E[f(\theta_t) \rho^\theta_{\tau_k, t}(x - \tilde{X}_k) \phi^\theta(\tau_k, t) | \sigma\{\theta_{\tau_k}\} \vee \mathcal{G}_k] \tag{3.2}$$

$$\bar{\psi}_k(f; t) := \int \psi_k(f; t, x) dx = E[f(\theta_t) \phi^\theta(\tau_k, t) | \sigma\{\theta_{\tau_k}\} \vee \mathcal{G}_k] \tag{3.3}$$

$$\pi_t(f) = E[f(\theta_t) | \mathcal{G}_t] \tag{3.4}$$

with ensuing obvious meanings of $\pi_{\tau_k}(\psi_k(f; t, x))$ and $\pi_{\tau_k}(\bar{\psi}_k(f; t))$ where we consider $\psi_k(f; t, x)$ and $\bar{\psi}_k(f; t)$ as functions of θ_{τ_k}. The process $\pi_t(f)$ is called the *filter process* for $f(\theta_t)$.

We have the following lemma (see Lemma 4.1 in [7]), where by $\mathcal{P}(\mathcal{G})$ we denote the predictable σ -algebra on $\Omega \times [0, \infty)$ with respect to $\mathcal{G}$ and set $\tilde{\mathcal{P}}(\mathcal{G}) = \mathcal{P}(\mathcal{G}) \otimes \mathcal{B}(R^m)$.

Lemma 3.1. *The compensator of the random measure $\mu(dt, dx)$ in (2.10) with respect to $\tilde{\mathcal{P}}(\mathcal{G})$ is given by the following nonnegative random measure*

$$\nu(dt, dx) = \sum_k \mathbf{1}_{(\tau_k, \tau_{k+1}]}(t) \frac{\pi_{\tau_k}(\psi_k(1, t, x))}{\int_t^\infty \pi_{\tau_k}(\bar{\psi}_k(1, s))ds} dtdx. \tag{3.5}$$

The main filtering result is the following (see Theorem 4.1 in [7]).

Theorem 3.1. *For any bounded function $f(\theta)$, the differential of the filter $\pi_t(f)$ is given by*

$$\begin{aligned} d\pi_t(f) &= \pi_t(Lf)dt \\ &+ \int \sum_k \mathbf{1}_{(\tau_k, \tau_{k+1}]}(t) [\tfrac{\pi_{\tau_k}(\psi_k(f;t,x))}{\pi_{\tau_k}(\psi_k(1;t,x))} - \pi_{t-}(f)](\mu - \nu)(dt, dx), \end{aligned} \tag{3.6}$$

where L is the generator of the Markov process θ_t(namely $L = Q$).

Corollary 3.1. *We have*

$$\pi_{\tau_{k+1}}(f) = \left. \frac{\pi_{\tau_k}(\psi_k(f; t, x))}{\pi_{\tau_k}(\psi_k(1; t, x))} \right|_{t=\tau_{k+1}, x=\tilde{X}_{\tau_{k+1}}}. \tag{3.7}$$

Recall that in our setting θ_t is an N-state Markov chain with state space $E = \{e_1, \ldots, e_N\}$, where e_i is a unit vector for each $i = 1, \ldots, N$. One may then write $f(\theta_t) = \sum_{i=1}^N f(e_i)\mathbf{1}_{e_i}(\theta_t)$. For $i = 1, \ldots, N$ let $\pi_t^i = \pi_t(\mathbf{1}_{e_i}(\theta_t))$ and

$$r_{ji}(t, z) := E[\exp(\int_0^t -n(\theta_s)ds)\rho^\theta_{0,t}(z)|\theta_0 = e_j, \theta_t = e_i], \tag{3.8}$$

$$p_{ji}(t) := P(\theta_t = e_i|\theta_0 = e_j) \tag{3.9}$$

and, noticing that $\pi_t \in \mathcal{S}_N$, define the function $M : [0, \infty) \times \mathbb{R}^m \times \mathcal{S}_N \to \mathcal{S}_N$ by

$$M^i(t, x, \pi) \ := \ \frac{\sum_j n(e_i)r_{ji}(t,x)p_{ji}(t)\pi^j}{\sum_{ij} n(e_i)r_{ji}(t,x)p_{ji}(t)\pi^j}, \tag{3.10}$$

$$M(t, x, \pi) := (M^1(t, x, \pi), M^2(t, x, \pi), \ldots, M^N(t, x, \pi)). \tag{3.11}$$

For $A \subset E$

$$M(t, x, \pi)(A) := \sum_{i=1}^N M^i(t, x, \pi)1_{\{e_i \in A\}}. \tag{3.12}$$

The following corollary will be useful

Corollary 3.2. *For the generic i-th state one has*

$$\pi^i_{\tau_{k+1}} = M^i(\tau_{k+1} - \tau_k, \tilde{X}_{\tau_{k+1}} - \tilde{X}_{\tau_k}, \pi_{\tau_k}) \tag{3.13}$$

and the process $\{\tau_k, \pi_{\tau_k}, \tilde{X}_{\tau_k}\}_{k=1}^\infty$ is a Markov process with respect to $\mathcal{G}_k$.

Proof. The representation (3.13) and the fact that $\{\tau_k, \pi_{\tau_k}, \tilde{X}_{\tau_k}\}$ is a $\mathcal{G}_k-$adapted discrete stochastic processes on $[0,\infty)\times\mathcal{S}_N\times\mathbb{R}^m$ follow immediately from Corollary 3.1 and the preceding definitions. For the Markov property we calculate

$$
\begin{aligned}
& P(\tau_{k+1} < t, \tilde{X}^1_{\tau_{k+1}} < x_1, \ldots, \tilde{X}^m_{\tau_{k+1}} < x_m | \mathcal{G}_k) \\
= \; & E[P(\tau_{k+1} < t, \tilde{X}^1_{\tau_{k+1}} < x_1, \ldots, \tilde{X}^m_{\tau_{k+1}} < x_m | \mathcal{G}_k \vee \mathcal{F}^\theta) | \mathcal{G}_k] \\
= \; & E[\textstyle\int_{\tau_k}^t P(\tilde{X}^1_{\tau_{k+1}} < x_1, \ldots, \tilde{X}^m_{\tau_{k+1}} < x_m | \mathcal{G}_k \vee \mathcal{F}^\theta) n(\theta_s) \exp(-\int_{\tau_k}^s n(\theta_u) du) ds | \mathcal{G}_k] \\
= \; & E[\textstyle\int_{\tau_k}^t \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_m} \rho_{\tau_k, s}(z - \tilde{X}_{\tau_k}) n(\theta_s) \exp(-\int_{\tau_k}^s n(\theta_u) du) ds dz | \mathcal{G}_k] \\
= \; & \textstyle\int_{\tau_k}^t \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_m} \sum\limits_{ij} n(e_i) r_{ji}(s - \tau_k, z - \tilde{X}_{\tau_k}) p_{ji}(s - \tau_k) \pi^j_{\tau_k} ds dz,
\end{aligned}
$$

and for any bounded measurable function g on $[0,\infty)\times\mathcal{S}_N\times\mathbb{R}^m$ it then follows that

$$
\begin{aligned}
& E[g(\tau_{k+1}, \pi_{\tau_{k+1}}, \tilde{X}_{\tau_{k+1}}) | \mathcal{G}_k] \\
= \; & E[g(\tau_{k+1}, M(\tau_{k+1} - \tau_k, \tilde{X}_{\tau_{k+1}} - \tilde{X}_{\tau_k}, \pi_{\tau_k}), \tilde{X}_{\tau_{k+1}}) | \mathcal{G}_k] \\
= \; & E[E[g(\tau_{k+1}, M(\tau_{k+1} - \tau_k, \tilde{X}_{\tau_{k+1}} - \tilde{X}_{\tau_k}, \pi_{\tau_k}), \tilde{X}_{\tau_{k+1}}) | \mathcal{G}(k) \vee \mathcal{F}^\theta] | \mathcal{G}_k] \\
= \; & E[\textstyle\int_{\tau_k}^\infty E[g(t, M(t - \tau_k, \tilde{X}_t - \tilde{X}_{\tau_k}, \pi_{\tau_k}), \tilde{X}_t) n(\theta_t) \exp(-\int_{\tau_k}^t n(\theta_s) ds) | \mathcal{G}_k \vee \mathcal{F}^\theta] dt | \mathcal{G}_k] \\
= \; & \textstyle\int_{\tau_k}^\infty \int_{\mathbb{R}^m} g(t, M(t - \tau_k, x - \tilde{X}_{\tau_k}, \pi_{\tau_k}), x) \sum_{ij} n(e_i) r_{ji}(t - \tau_k, x - \tilde{X}_{\tau_k}) p_{ji}(t - \tau_k) \pi^j_{\tau_k} dx dt,
\end{aligned}
$$

where the last equation depends only on $\{\tau_k, \pi_{\tau_k}, \tilde{X}_{\tau_k}\}$ thus implying the Markov property. □

3.2 A contraction operator

In this subsection we define a contraction operator (see Definition 3.1 below) that will be relevant for deriving the results on the value function. In view of its definition and in order to derive its properties, we need first to introduce some additional notions.

We start by defining an operator on $\mathcal{M}(E)$ as follows

$$K^i(t,x)\pi := \sum_j n(e_i) r_{ji}(t,x) p_{ji}(t) \pi^j, \tag{3.14}$$

$$K(t,x)\pi := (K^1(t,x)\pi, K^2(t,x)\pi, \ldots, K^N(t,x)\pi). \tag{3.15}$$

For $t \in [0,\infty),\ x \in \mathbb{R}^m,\ K^i(t,x)$ is a positive linear operator on $\mathcal{M}(E)$. For $A \subset E$ set

$$K(t,x)\pi(A) := \sum_{i=1}^N K^i(t,x)\pi 1_{\{e_i \in A\}}. \tag{3.16}$$

By the definition of $M^i(t,x,\pi)$ and $K^i(t,x)\pi$, setting $\kappa(t,x,\pi) := \sum_i K^i(t,x)\pi$, for $t \in [0,\infty),\ x \in \mathbb{R}^m,\ \pi \in \mathcal{M}(E)$ we have

$$M^i(t,x,\pi) = \frac{1}{\kappa(t,x,\pi)} K^i(t,x)\pi \ . \tag{3.17}$$

By the definition of the Hilbert metric $d_H(\cdot,\cdot)$, for $t \in [0,\infty),\ x \in \mathbb{R}^m,\ \pi, \bar{\pi} \in \mathcal{M}(E)$ we then have

$$\begin{aligned}
d_H(M(t,x,\pi), M(t,x,\bar{\pi})) &= \log(\sup \tfrac{M(t,x,\pi)(A)}{M(t,x,\bar{\pi})(A)} \sup \tfrac{M(t,x,\bar{\pi})(A)}{M(t,x,\pi)(A)}) \\
&= \log(\sup \tfrac{\frac{1}{\kappa(t,x,\pi)}K(t,x)\pi(A)}{\frac{1}{\kappa(t,x,\bar{\pi})}K(t,x)\bar{\pi}(A)} \sup \tfrac{\frac{1}{\kappa(t,x,\bar{\pi})}K(t,x)\bar{\pi}(A)}{\frac{1}{\kappa(t,x,\pi)}K(t,x)\pi(A)}) \\
&= \log(\sup \tfrac{K(t,x)\pi(A)}{K(t,x)\bar{\pi}(A)} \sup \tfrac{K(t,x)\bar{\pi}(A)}{K(t,x)\pi(A)}) \\
&= d_H(K(t,x)\pi, K(t,x)\bar{\pi}).
\end{aligned} \tag{3.18}$$

Applying [1], Lemma 3.4 in [16] and Theorem 1.1 in [17] , for the positive linear operator K on $\mathcal{M}(E)$ it then follows that

$$d_H(M(t,x,\pi), M(t,x,\bar{\pi})) = d_H(K(t,x)\pi, K(t,x)\bar{\pi}) \leq d_H(\pi,\bar{\pi}) \tag{3.19}$$

for $t \in [0,\infty),\ x \in \mathbb{R}^m,\ \pi, \bar{\pi} \in \mathcal{S}_N$. By Lemma 3.4 in [16] , for $\forall \pi, \bar{\pi} \in \mathcal{S}_N$ we also have

$$\|\pi - \bar{\pi}\|_{TV} \leq \frac{2}{\log 3} d_H(\pi,\bar{\pi}), \tag{3.20}$$

where $\|\cdot\|_{TV}$ is the total variation on $\mathcal{S}_N$.

We finally introduce a metric on $[0,\infty) \times \mathcal{S}_N \times \bar{H}_m$ by

$$|t - \bar{t}| + d_H(\pi,\bar{\pi}) + \sum_{i=1}^{m} |h^i - \bar{h}^i| \tag{3.21}$$

for $(t,\ \pi,\ h), (\bar{t},\ \bar{\pi},\ \bar{h}) \in [0,\infty) \times \mathcal{S}_N \times \bar{H}_m$ and considering the state space

$$\Sigma := [0,\infty) \times \mathcal{S}_N, \tag{3.22}$$

let $C_b(\Sigma)$ be the set of bounded continuous functions $g : \Sigma \to \mathbb{R}$ with norm

$$\|g\| := \max_{x\in\Sigma} |\ g(x)\ | \ . \tag{3.23}$$

Definition 3.1. *Let the operator $J : C_b(\Sigma) \to C_b(\Sigma)$ be given as follows*

$$\begin{aligned}
&Jg(\tau,\pi) \\
&:= \int_\tau^T \int_{\mathbb{R}^m} g(t, M(t-\tau,z,\pi)) \sum_{ij} n(e_i) r_{ji}(t-\tau,z) p_{ji}(t-\tau)\pi^j dz dt \\
&= E[g(\tau_1, \pi_{\tau_1}) 1_{\{\tau_1 < T\}} | \tau_0 = \tau, \pi_{\tau_0} = \pi],
\end{aligned} \tag{3.24}$$

where M is defined in (3.10)-(3.11).

First we have

Lemma 3.2. *J is a contraction operator on $C_b(\Sigma)$ with contraction constant $c := 1 - e^{-\bar{n}T} < 1$, where $\bar{n} := \max n(\theta) = \max_i n(e_i)$.*

Proof. For $\forall g \in C_b(\Sigma)$

$$\begin{aligned} |\ Jg(t,\pi)\ | &= |\ E[g(\tau_1,\pi_1)1_{\{\tau_1<T\}}|\tau_0=t,\pi_{\tau_0}=\pi]\ | \\ &\le \|g\|P(\tau_1<T|\tau_0=t) \\ &= \|g\|E[(1-\exp(-\textstyle\int_t^T n(\theta_t)dt))] \\ &\le \|g\|(1-\exp(-\bar{n}(T-t))) \end{aligned}$$

and so

$$\|Jg\| \le c\|g\| \tag{3.25}$$

with c as specified in the statement. □

Let $C_{b,lip}(\Sigma)$ be the set of bounded and Lipschitz continuous functions $g:\ \Sigma\ \to\ \mathbb{R}$ and set for $g \in C_{b,lip}(\Sigma)$,[1]

$$N^{\lambda}(g) := \lambda\|g\| + [g]_{lip} \tag{3.26}$$

where,

$$[g]_{lip} := \sup_{\tau,\bar{\tau}\in[0,T]\ \pi,\bar{\pi}\in\mathcal{S}_N} \frac{|g(\tau,\pi)-g(\bar{\tau},\bar{\pi})|}{|\tau-\bar{\tau}|+d_H(\pi,\bar{\pi})}. \tag{3.27}$$

Note that $C_{b,lip}(\Sigma)$ is a Banach space with the norm $N^{\lambda}(g)$, for each $\lambda > 0$.
Take a sufficiently large constant λ such that

$$c' := (c + \max(\bar{n}, \frac{2}{\log 3})\frac{1}{\lambda}) < 1. \tag{3.28}$$

Proposition 3.1. *The operator J in Definition 3.1 is a contraction operator*

$$J : C_{b,lip}(\Sigma) \to C_{b,lip}(\Sigma)$$

with contraction constant c'.

Proof. Let us first prove that $Jg(t,\pi)$ is Lipschitz continuous with respect to t. By assumption, for all $g \in C_{b,lip}(\Sigma)$,

$$|g(\tau,\pi)-g(\bar{\tau},\pi)| \le [g]_{lip}|\tau-\bar{\tau}|, \tag{3.29}$$

$$|g(\tau,\pi)-g(\tau,\bar{\pi})| \le [g]_{lip}d_H(\pi,\bar{\pi}). \tag{3.30}$$

We change variables from t to $t+\tau$,

$$Jg(\tau,\pi) = \int_0^{T-\tau}\int_{\mathbb{R}^m} g(t+\tau, M(t,z,\pi))\sum_{ij} n(e_i)r_{ji}(t,z)p_{ji}(t)\pi^j dzdt. \tag{3.31}$$

[1] We are grateful for an anonymous suggestion of this useful norm

We then have

$$\begin{aligned}
&|Jg(\tau,\pi)-Jg(\bar\tau,\pi)| \\
&= \ |\textstyle\int_{T-\bar\tau}^{T-\tau}\int_{\mathbb{R}^m} g(t+\tau,M(t,z,\pi))\sum_{ij} n(e_i)r_{ji}(t,z)p_{ji}(t)\pi^j dzdt| \\
&\quad +|\textstyle\int_0^{T-\tau}\int_{\mathbb{R}^m}\{g(t+\tau,M(t,z,\pi))-g(t+\bar\tau,M(t,z,\pi))\} \\
&\qquad\qquad \cdot\textstyle\sum_{ij} n(e_i)r_{ji}(t,z)p_{ji}(t)\pi^j dzdt| \qquad (3.32)\\
&\le \ \bar n\|g\||\tau-\bar\tau| + [g]_{lip}|\tau-\bar\tau|\ |\textstyle\int_0^{T-\tau}\int_{\mathbb{R}^m}\sum_{ij} n(e_i)r_{ji}(t,z)p_{ji}(t)\pi^j dzdt| \\
&= \ \bar n\|g\||\tau-\bar\tau| + [g]_{lip}|\tau-\bar\tau|P(\tau_1<T|\tau_0=\tau,\pi_{\tau_0}=\pi) \\
&\le \ (\bar n\|g\|+c[g]_{lip})|\tau-\bar\tau|.
\end{aligned}$$

Next, let us prove that $Jg(t,\pi)$ is Lipschitz continuous with respect to π.

$$\begin{aligned}
&|Jg(\tau,\pi)-Jg(\tau,\bar\pi)| \\
&\le \ |\textstyle\int_0^{T-\tau}\int_{\mathbb{R}^m}\{g(t,M(t,z,\pi))-g(t,M(t,z,\bar\pi))\}\sum_{ij} n(e_i)r_{ji}(t,z)p_{ji}(t)\pi^j dzdt| \\
&\quad + |\textstyle\int_0^{T-\tau}\int_{\mathbb{R}^m} g(t,M(t,z,\bar\pi))\sum_{ij} n(e_i)r_{ji}(t,z)p_{ji}(t)(\pi^j-\bar\pi^j)dzdt| \\
&\le \ |\textstyle\int_0^{T-\tau}\int_{\mathbb{R}^m}[g]_{lip}d_H(M(t,z,\pi),M(t,z,\bar\pi))\sum_{ij} n(e_i)r_{ji}(t,z)p_{ji}(t)\pi^j dzdt| \qquad (3.33)\\
&\quad + \|g\|\tfrac{2}{\log 3}d_H(\pi,\bar\pi)P(\tau_1<T|\tau_0=\tau) \\
&\le \ (\tfrac{2}{\log 3}\|g\|+c[g]_{lip})d_H(\pi,\bar\pi).
\end{aligned}$$

Therefore,

$$\begin{aligned}
[Jg]_{lip} &= \sup_{\tau,\bar\tau\in[0,T]\ \pi,\bar\pi\in\mathcal{S}_N}\frac{|Jg(\tau,\pi)-Jg(\bar\tau,\bar\pi)|}{|\tau-\bar\tau|+d_H(\pi,\bar\pi)} \\
&\le \sup_{\tau,\bar\tau\in[0,T]\ \pi,\bar\pi\in\mathcal{S}_N}\frac{|Jg(\tau,\pi)-Jg(\bar\tau,\pi)|+|Jg(\bar\tau,\pi)-Jg(\bar\tau,\bar\pi)|}{|\tau-\bar\tau|+d_H(\pi,\bar\pi)} \qquad (3.34)\\
&\le \sup_{\tau,\bar\tau\in[0,T]\ \pi,\bar\pi\in\mathcal{S}_N}\frac{(\bar n\|g\|+c[g]_{lip})|\tau-\bar\tau|+(\frac{2}{\log 3}\|g\|+c[g]_{lip})d_H(\pi,\bar\pi)}{|\tau-\bar\tau|+d_H(\pi,\bar\pi)} \\
&\le \max(\bar n,\tfrac{2}{\log 3})\|g\|+c[g]_{lip}.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
N^\lambda(Jg) &= \lambda\|Jg\|+[Jg]_{lip} \\
&\le c\lambda\|g\|+\max(\bar n,\tfrac{2}{\log 3})\|g\|+c[g]_{lip} \qquad (3.35)\\
&\le c'\lambda\|g\|+c[g]_{lip} \\
&\le c'N^\lambda(g).
\end{aligned}$$

□

4 The Control Problem/Log-utility

Recall from (2.20) that the value process of a self financing portfolio satisfies

$$\frac{dV_t}{V_t} = [r_0 + h_t^*\{r(\theta_t) - r_0\mathbf{1}\}]dt + h_t^*\sigma(\theta_t)dB_t. \tag{4.1}$$

We have by Itô's formula

$$\begin{aligned}\log V_T = \log v_0 \quad &+ \quad \int_0^T h_t^*\sigma(\theta_t)dB_t \\ &+ \quad \int_0^T [r_0 + h_t^*\{r(\theta_t) - r_0\mathbf{1}\} - \tfrac{1}{2}h_t^*\sigma\sigma^*(\theta_t)h_t]dt.\end{aligned} \tag{4.2}$$

Put

$$f(\theta, h) := r_0 + h^*\{r(\theta) - r_0\mathbf{1}\} - \frac{1}{2}h^*\sigma\sigma^*(\theta)h \tag{4.3}$$

and notice that this function $f(\cdot)$ is bounded under our assumptions. The expected log-utility of terminal wealth then becomes

$$E[\log V_T|\tau_0 = 0, \pi_{\tau_0} = \pi] = \log v_0 + E[\int_0^T f(\theta_t, h_t)dt|\tau_0 = 0, \pi_{\tau_0} = \pi] \tag{4.4}$$

and, as mentioned in section 2.3, we want to consider the problem of maximization of expected terminal log-utility, namely

$$\sup_{h\in\mathcal{A}} E[\log V_T|\tau_0 = 0, \pi_{\tau_0} = \pi]. \tag{4.5}$$

The results that we shall derive for the control part of the problem, and that we synthesize in Theorem 4.1, concern both the optimal control that we shall show to be also here of the myopic type, as well as the optimal value function, for which we shall derive an approximation result (value iteration) as well as a Dynamic Programming principle. In subsection 4.1 we shall present preliminary results, mainly in view of the optimal strategy, while in subsection 4.2 we shall introduce the value function $W(\cdot)$ in the standard way and show some first properties related to $W(\cdot)$. Using only the standard value function $W(\cdot)$ it turns out to be very difficult to obtain the results that we are after and so in the further subsection 4.3 we introduce an auxiliary value function $\bar{W}(\cdot)$ that not only will be instrumental in obtaining our results, but is also the value function that enters explicitly into the approximation result in Theorem 4.1 below (it is in fact the value function that can be computed by value iteration).

4.1 Preliminary results in view of the optimal strategy

Definition 4.1. *Let $\hat{C}(\tau, \pi, h)$ be defined by*

$$\begin{aligned}\hat{C}(\tau, \pi, h) \quad &:= \quad E[\int_\tau^{T\wedge\tau_1} f(\theta_s, h_s)ds|\tau_0 = \tau, \pi_{\tau_0} = \pi] \\ &= \quad \int_\tau^T \int_{\mathbb{R}^m} \sum_{i,j} f(e_i, \gamma(x, h))r_{ji}(t-\tau, x)p_{ji}(t-\tau)\pi^j dxdt,\end{aligned} \tag{4.6}$$

where $\gamma(x, h) = [\gamma^1(x^1, h), \cdots, \gamma^m(x^m, h)]$.

Lemma 4.1.

(i) For the function defined by (4.3), we have the following equation

$$E[\int_t^T f(\theta_s, h_s)ds|\tau_0 = t, \pi_{\tau_0} = \pi] = E[\sum_k \hat{C}(\tau_k, \pi_{\tau_k}, h_k)1_{\{\tau_k<T\}}|\tau_0 = t, \pi_{\tau_0} = \pi]. \quad (4.7)$$

(ii) $\hat{C}$ is a bounded and continuous function on $[0,T] \times \mathcal{S}_N \times \bar{H}_m$.

For the proof see the Appendix.

Corollary 4.1.

(i) There exists a Borel function $\hat{h}(\tau, \pi)$ such that $\sup_{h\in\bar{H}_m} \hat{C}(\tau, \pi, h) = \hat{C}(\tau, \pi, \hat{h}(\tau, \pi))$.

(ii) The function

$$C(t, \pi) := \sup_{h\in\bar{H}_m} \hat{C}(t, \pi, h). \quad (4.8)$$

is Lipschitz continuous with respect to t, π in the metric introduced in (3.21).

Proof. $\bar{H}_m$ is compact and $\hat{C}(\tau, \pi, h)$ is a bounded continuous function on $[0,T]\times\mathcal{S}_N\times\bar{H}_m$; there exists then a Borel function $\hat{h}(\tau, \pi)$ such that (4.8) holds. Furthermore, $\hat{C}(t, \pi, h)$ is uniformly Lipschitz continuous with respect to t, π.

□

4.2 Value function and first properties

We start with the following basic definition

Definition 4.2. *For given initial data ($\tau_0 = t, \pi_{\tau_0} = \pi$), where we now start at a generic time t, consider the following value function for $h \in \mathcal{A}$*

$$\begin{aligned} W(t, \pi, h.) &:= E[\textstyle\int_t^T f(\theta_s, h_s)ds|\tau_0 = t,\ \pi_{\tau_0} = \pi] \\ &= E[\sum_{k=0}^{\infty} \hat{C}(\tau_k, \pi_{\tau_k}, h_k)1_{\{\tau_k<T\}}|\tau_0 = t,\ \pi_{\tau_0} = \pi], \end{aligned} \quad (4.9)$$

and define

$$\begin{aligned} W(t, \pi) &:= \sup_{h\in\mathcal{A}} W(t, \pi, h.) \\ &= \sup_{h\in\mathcal{A}} E[\int_t^T f(\theta_s, h_s)ds|\tau_0 = t,\ \pi_{\tau_0} = \pi] \\ &= \sup_{h\in\mathcal{A}} E[\sum_{k=0}^{\infty} \hat{C}(\tau_k, \pi_{\tau_k}, h_k)1_{\{\tau_k<T\}}|\tau_0 = t,\ \pi_{\tau_0} = \pi], \end{aligned} \quad (4.10)$$

$$
\begin{aligned}
W^n(t,\pi) &:= \sup_{h\in\mathcal{A}^n} W(t,\pi,h.) \\
&= \sup_{h\in\mathcal{A}^n} E[\int_t^T f(\theta_s,h_s)ds|\tau_0=t,\ \pi_{\tau_0}=\pi] \\
&= \sup_{h\in\mathcal{A}^n} E[\sum_{k=0}^{\infty}\hat{C}(\tau_k,\pi_{\tau_k},h_k)1_{\{\tau_k<T\}}|\tau_0=t,\ \pi_{\tau_0}=\pi],
\end{aligned}
\tag{4.11}
$$

where $\mathcal{A}^n$ was defined in (2.18).

Lemma 4.2. *For all $n\geq 0$ and $h\in\mathcal{A}^n$, we have the following equation*

$$
\begin{aligned}
W(t,\pi,h.) &= E[\sum_{k=0}^{n-1}\hat{C}(\tau_k,\pi_{\tau_k},h_k)1_{\{\tau_k<T\}} \\
&+ \textstyle\int_{\tau_n}^T f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_{\tau_n},h_n))ds1_{\{\tau_n<T\}}|\tau_0=t,\ \pi_{\tau_0}=\pi].
\end{aligned}
\tag{4.12}
$$

For the proof see the Appendix.

Corollary 4.2. *For $n\geq 0$, $t\in[0,T]$, $\pi\in\mathcal{S}_N$ we have the following equation*

$$
\begin{aligned}
W^n(t,\pi) &= \sup_{h\in\mathcal{A}^n} E[\sum_{k=0}^{n-1}\hat{C}(\tau_k,\pi_{\tau_k},h_k)1_{\{\tau_k<T\}} \\
&+ \textstyle\int_{\tau_n}^T f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_{\tau_n},h_n))ds1_{\{\tau_n<T\}}|\tau_0=t,\ \pi_{\tau_0}=\pi].
\end{aligned}
\tag{4.13}
$$

4.3 An auxiliary value function

Recall the function $C(t,\pi)$ defined in Corollary 4.1 as well as the operator J from Definition 3.1. By Proposition 3.1 we have that J is a contraction operator on the Banach space $C_{b,lip}$ with its norm $N^\lambda(\cdot)$. Therefore, $\lim_{n\to\infty}\sum_{k=0}^n J^kC$ exists and so we introduce the

Definition 4.3. *Define the auxiliary value function $\bar{W}(t,\pi)$ as*

$$\bar{W} := \sum_{k=0}^{\infty} J^kC$$

The following lemma then holds

Lemma 4.3. *We have $\bar{W}\in C_{b,lip}$ and it satisfies*

$$\bar{W}(t,\pi) = C(t,\pi) + J\bar{W}(t,\pi). \tag{4.14}$$

Proof. Due always to the fact that (see Proposition 3.1) J is a contraction operator on the Banach space $C_{b,lip}$ with its norm $N^\lambda(\cdot)$, in addition to the existence of $\lim_{n\to\infty}\sum_{k=0}^n J^kC$ we also have

$$(I-J)^{-1}C = \sum_{k=0}^{\infty} J^kC,$$

from which the result follows. ☐

In view of deriving a recursion related to $\bar{W}(t,x)$ (value iteration), we start with the

Definition 4.4. *Define, for $h \in \bar{H}_m$,*

$$\bar{W}^0(t,\pi,h) := E[\int_t^T f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_t, h))ds | \tau_0 = t,\ \pi_{\tau_0} = \pi]. \tag{4.15}$$

Furthermore, let

$$\bar{W}^0(t,\pi) := \max_{h \in \bar{H}_m} \bar{W}^0(t,\pi,h), \tag{4.16}$$

and, for $n \geq 1$

$$\begin{aligned} \bar{W}^n(t,\pi) &:= C(t,\pi) + J\bar{W}^{n-1}(t,\pi) \\ &= \sum_{k=0}^{n-1} J^k C(t,\pi) + J^n \bar{W}^0(t,\pi). \end{aligned} \tag{4.17}$$

Remark 4.1. *The function $\bar{W}^0(t,\pi,h)$ in (4.15) is bounded and continuous with respect to t,π,h. This follows by an analogous proof as in Lemma 4.1(ii).*

We first state and prove the following lemma (later we need a relation from the proof)

Lemma 4.4.

(i) We have the equation

$$\bar{W}^n(t,\pi) = E[\sum_{k=0}^{n-1} C(\tau_k, \pi_{\tau_k}) 1_{\{\tau_k < T\}} + \bar{W}^0(\tau_n, \pi_{\tau_n}) 1_{\{\tau_n < T\}} | \tau_0 = t,\ \pi_{\tau_0} = \pi]. \tag{4.18}$$

(ii) For any $\epsilon > 0$, we set $n_\epsilon := (\log(1-c') + \log \varepsilon - \log N^\lambda(\bar{W}^1 - \bar{W}^0))/\log c'$, where c' is the contraction constant defined in (3.28). For all $n > n_\epsilon$,

$$N^\lambda(\bar{W} - \bar{W}^n) < \epsilon. \tag{4.19}$$

Proof. We prove (i). For $n \geq 1$

$$\{\tau_{n-1} < T\} \supset \{\tau_n < T\}. \tag{4.20}$$

Therefore,

$$1_{\{\tau_{n-1} < T\}} 1_{\{\tau_n < T\}} = 1_{\{\tau_n < T\}}. \tag{4.21}$$

For all $g \in C_b([0,T] \times \mathcal{S}_N)$ and $n \geq 0$, we have

$$\begin{aligned} &E[g(\tau_n, \pi_{\tau_n}) 1_{\{\tau_n < T\}} | \tau_0 = t, \pi_{\tau_0} = \pi] \\ &= E[E[g(\tau_n, \pi_{\tau_n}) 1_{\{\tau_n < T\}} | \mathcal{G}_{n-1}] 1_{\{\tau_{n-1} < T\}} | \tau_0 = t, \pi_{\tau_0} = \pi]. \end{aligned} \tag{4.22}$$

because $1_{\{\tau_{n-1} < T\}} E[1_{\{\tau_n < T\}} | \mathcal{G}_{n-1}] = E[1_{\{\tau_n < T\}} | \mathcal{G}_{n-1}]$. Then, since (see (3.24))

$$\begin{aligned} &E[g(\tau_n, \pi_{\tau_n}) 1_{\{\tau_n < T\}} | \mathcal{G}_{n-1}] \\ &= \int_{\tau_{n-1}}^T \int_{\mathbb{R}^m} g(t, M(t-\tau_{n-1}, z, \pi_{\tau_{n-1}})) \sum_{ij} n(e_i) r_{ji}(t-\tau_{n-1}, z) p_{ji}(t-\tau_{n-1}) \pi^j_{\tau_{n-1}} dz dt \\ &= Jg(\tau_{n-1}, \pi_{\tau_{n-1}}), \end{aligned} \tag{4.23}$$

we have (see always (3.24))

$$\begin{aligned} E[g(\tau_n, \pi_{\tau_n}) 1_{\{\tau_n<T\}} | \tau_0 = t, \pi_{\tau_0} = \pi] &= E[Jg(\tau_{n-1}, \pi_{\tau_{n-1}}) 1_{\{\tau_{n-1}<T\}} | \tau_0 = t, \pi_{\tau_0} = \pi] \\ &= J^n g(t, \pi). \end{aligned} \tag{4.24}$$

We thus obtain

$$\begin{aligned} \bar{W}^n(t, \pi) &= \sum_{k=0}^{n-1} J^k C(t, \pi) + J^n \bar{W}^0(t, \pi) \\ &= E[\textstyle\sum_{k=0}^{n-1} C(\tau_k, \pi_{\tau_k}) 1_{\{\tau_k<T\}} + \bar{W}^0(\tau_n, \pi_{\tau_n}) 1_{\{\tau_n<T\}} | \tau_0 = t, \ \pi_{\tau_0} = \pi]. \end{aligned} \tag{4.25}$$

Next, we prove (ii). For any n,

$$\begin{aligned} N^\lambda(\bar{W} - \bar{W}^n) &= N^\lambda(\lim_{k\to\infty} \bar{W}^{n+k} - \bar{W}^n) = \lim_{k\to\infty} N^\lambda(\bar{W}^{n+k} - \bar{W}^n) \\ &\leq \lim_{k\to\infty} \sum_{i=0}^{k-1} N^\lambda(\bar{W}^{n+i+1} - \bar{W}^{n+i}) \leq N^\lambda(\bar{W}^{n+1} - \bar{W}^n) \sum_{i=0}^{\infty} (c')^i \\ &\leq N^\lambda(\bar{W}^1 - \bar{W}^0)(c')^n \sum_{i=0}^{\infty} (c')^i = \frac{(c')^n}{1-c'} N^\lambda(\bar{W}^1 - \bar{W}^0). \end{aligned} \tag{4.26}$$

□

We close this subsection with three crucial lemmas. The first one, Lemma 4.5, establishes the relationship between the actual and the auxiliary value functions. It is preliminary to the following two lemmas 4.6 and 4.7 that are the main ingredients for the approximation result in Theorem 4.1. Furthermore, Lemma 4.5 and its proof are also relevant in order to obtain the optimal strategy (point iii) in Theorem 4.1).

Lemma 4.5. *For all* $n \geq 0$*, we have the equality*

$$W^n(t, \pi) = \bar{W}^n(t, \pi). \tag{4.27}$$

Proof. By Corollary 4.2, for all $n \geq 0$

$$\begin{aligned} W^n(t, \pi) &= \sup_{h\in\mathcal{A}^n} E[\sum_{k=0}^{n-1} \hat{C}(\tau_k, \pi_{\tau_k}, h_k) 1_{\{\tau_k<T\}} \\ &+ \textstyle\int_{\tau_n}^T f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_{\tau_n}, h_n)) ds 1_{\{\tau_n<T\}} | \tau_0 = t, \pi_{\tau_0} = \pi]. \end{aligned} \tag{4.28}$$

Since $\bar{H}_m$ is compact and $\bar{W}^0(\tau, \pi, h)$ is a bounded continuous function on $[0, T] \times \mathcal{S}_N \times \bar{H}_m$, there exists a Borel function $w(\tau, \pi)$ such that $\sup_{h\in\bar{H}_m} \bar{W}^0(\tau, \pi, h) = \bar{W}^0(\tau, \pi, w(\tau, \pi))$. Furthermore, by Corollary 4.1(i) there exists a Borel function $\hat{h}(\tau, \pi)$ such that $\sup_{h\in\bar{H}_m} \hat{C}(\tau, \pi, h) = \hat{C}(\tau, \pi, \hat{h}(\tau, \pi))$ holds. For $n \geq 0$, we define the strategy

$$\begin{aligned} \tilde{h}_k &:= \hat{h}(\tau_k, \pi_{\tau_k}), && 0 \leq k \leq n-1 \\ \tilde{h}_k &:= w(\tau_n, \pi_{\tau_n}), && k = n \\ \tilde{h}_k &:= \gamma(\tilde{X}_{\tau_k} - \tilde{X}_{\tau_n}, \tilde{h}_n), && k > n. \end{aligned} \tag{4.29}$$

By definition of $\{\tilde{h}_k\}_{k\in\mathbb{N}}$, we have $\{\tilde{h}_k\}_{k\in\mathbb{N}} \in \mathcal{A}^n$. Using Lemma 4.4(i) and Lemma 4.2, for $n \geq 0, t \in [0,T], \pi \in \mathcal{S}_N$

$$\begin{aligned}
\bar{W}^n(t,\pi) &= E[\sum_{k=0}^{n-1} \hat{C}(\tau_k, \pi_{\tau_k}, \tilde{h}_k) 1_{\{\tau_k<T\}} \\
&\quad + \textstyle\int_{\tau_n}^{T} f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_{\tau_n}, \tilde{h}_n)) ds 1_{\{\tau_n<T\}} | \tau_0 = t, \pi_{\tau_0} = \pi] \\
&\leq W^n(t,\pi).
\end{aligned} \tag{4.30}$$

Using again Lemma 4.2, (4.15) and Lemma 4.4(i), for all $n \geq 0, h \in \mathcal{A}^n, t \in [0,T], \pi \in \mathcal{S}_N$

$$\begin{aligned}
W(t,\pi,h.) = E[&\sum_{k=0}^{n-1} \hat{C}(\tau_k, \pi_{\tau_k}, h_k) 1_{\{\tau_k<T\}} \\
&\quad + \textstyle\int_{\tau_n}^{T} f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_{\tau_n}, h_n)) ds 1_{\{\tau_n<T\}} | \tau_0 = t, \pi_{\tau_0} = \pi] \\
&= E[\sum_{k=0}^{n-1} \hat{C}(\tau_k, \pi_{\tau_k}, h_k) 1_{\{\tau_k<T\}} + \bar{W}^0(\tau_n, \pi_{\tau_n}, h_n) 1_{\{\tau_n<T\}} | \tau_0 = t, \pi_{\tau_0} = \pi] \\
&\leq E[\sum_{k=0}^{n-1} C(\tau_k, \pi_{\tau_k}) 1_{\{\tau_k<T\}} + \bar{W}^0(\tau_n, \pi_{\tau_n}) 1_{\{\tau_n<T\}} | \tau_0 = t, \pi_{\tau_0} = \pi] \\
&= \bar{W}^n(t,\pi).
\end{aligned} \tag{4.31}$$

Therefore, we have

$$W^n(t,\pi) = \sup_{h\in\mathcal{A}^n} W(t,\pi,h.) \leq \bar{W}^n(t,\pi), \tag{4.32}$$

and so we obtain for all $n \geq 0$

$$W^n(t,\pi) = \bar{W}^n(t,\pi). \tag{4.33}$$

□

Lemma 4.6. *For $n \geq 0$, we have the estimate*

$$\bar{W}^n(t,\pi) \leq \bar{W}^{n+1}(t,\pi) \leq \bar{W}(t,\pi) \leq W(t,\pi). \tag{4.34}$$

For the proof see the Appendix.

Lemma 4.7. *The following estimate holds*

$$W(t,\pi) \leq \bar{W}(t,\pi) \tag{4.35}$$

for $t \in [0,T], \forall\pi \in \mathcal{S}_N$.

For the proof see the Appendix.

4.4 Main result

Based on the previous subsections we obtain now the main result of this section

Theorem 4.1.

(i) Approximation theorem :
For any $\epsilon > 0, n > n_\epsilon$,

$$N^\lambda(W - \bar{W}^n) < \epsilon, \tag{4.36}$$

where n_ϵ is the constant defined in Lemma 4.4(ii) and, modulo the additive term $\log v_0$, the function $W = W(t,\pi)$ is the optimal value function (see (4.4), (4.9), (4.10)), N^λ is the norm introduced in (3.26), and $\bar{W}^n$ are computed recursively according to (4.16) and (4.17).

(ii) Dynamic programming principle : for any $n > 0$

$$\begin{aligned} W(t,\pi) = \sup_{h\in\mathcal{A}^n} E[\sum_{k=0}^{n} \hat{C}(\tau_k, \pi_{\tau_k}, h_k) 1_{\{\tau_k<T\}} \\ + W(\tau_{n+1}, \pi_{\tau_{n+1}}) 1_{\{\tau_{n+1}<T\}} | \tau_0 = t, \pi_{\tau_0} = \pi]. \end{aligned} \tag{4.37}$$

.

(iii) Optimal value and optimal strategy for the Log Utility Maximization Problem : for the utility maximization under the initial conditions $V_0 = v_0, \tau_0 = 0, \pi_{\tau_0} = \pi$ we have

$$\begin{aligned} \sup_{h\in\mathcal{A}} E[\log V_T | \tau_0 = 0, \pi_{\tau_0} = \pi] = \log v_0 + \sup_{h\in\mathcal{A}} E[\int_0^T f(\theta_t, h_t) dt | \tau_0 = 0, \pi_{\tau_0} = \pi] \\ = \log v_0 + C(0,\pi) + \textstyle\sum_{k=1}^{\infty} E[\hat{C}(\tau_k, \pi_{\tau_k}, \hat{h}_k) 1_{\{\tau_k<T\}} | \tau_0 = 0, \pi_{\tau_0} = \pi], \end{aligned} \tag{4.38}$$

with $\hat{h}_k$ defined in Corollary 4.1, namely $\sup_{h\in\bar{H}_m} \hat{C}(\tau,\pi,h) = \hat{C}(\tau,\pi,\hat{h}(\tau,\pi))$ and $\hat{h}_k = \hat{h}(\tau_k, \pi_{\tau_k})$ and

$$\hat{h}^i_t = \gamma^i(\tilde{X}_t - \tilde{X}_{\tau_k}, \hat{h}_k), \quad \tau_k \le t < \tau_{k+1} \tag{4.39}$$

Proof. Let us first prove (i). By Lemma 4.6 and Lemma 4.7,

$$W(t,\pi) = \bar{W}(t,\pi). \tag{4.40}$$

Therefore, applying Lemma 4.4(ii) one obtains

$$N^\lambda(W - \bar{W}^n) < \epsilon. \tag{4.41}$$

Next, let us prove (ii). By (4.40), Lemma 4.3, (4.24) and by Corollary 4.1

$$\begin{aligned} W(t,\pi) &= \bar{W}(t,\pi) = \sum_{k=0}^{n} J^k C + J^{n+1} W(t,\pi) \\ &= E[\sum_{k=0}^{n} C(\tau_k, \pi_{\tau_k}) 1_{\{\tau_k<T\}} + W(\tau_{n+1}, \pi_{\tau_{n+1}}) 1_{\{\tau_{n+1}<T\}} | \tau_0 = t, \pi_{\tau_0} = \pi] \\ &= \sup_{h\in\mathcal{A}^n} E[\sum_{k=0}^{n} \hat{C}(\tau_k, \pi_{\tau_k}, h_k) 1_{\{\tau_k<T\}} + W(\tau_{n+1}, \pi_{\tau_{n+1}}) 1_{\{\tau_{n+1}<T\}} | \tau_0 = t, \pi_{\tau_0} = \pi]. \end{aligned} \tag{4.42}$$

Finally, (iii) is an immediate consequence of (4.4), Lemma 4.1 and Lemma 4.5 and its proof. □

5 Appendix

Proof of Lemma 4.1.

Proof of statement (i). It follows from the two lemmas shown below.

Lemma 5.1. *We have the following representation,*

$$E[f(\theta_t, h_t)|\mathcal{G}_t] = \sum_{k\geq 0} 1_{]\tau_k,\tau_{k+1}]}(t)\frac{E[f(\theta_t, \gamma(\tilde{X}_t - \tilde{X}_{\tau_k}, h_k))1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]}{E[1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]}. \tag{5.1}$$

Proof. It suffices to prove that for any $\mathcal{G}_t-$adapted process Z_t

$$E[E[f(\theta_t, h_t)|\mathcal{G}_t]Z_t] = E[\sum_{k\geq 0} 1_{]\tau_k,\tau_{k+1}]}(t)\frac{E[f(\theta_t, \gamma(\tilde{X}_t - \tilde{X}_{\tau_k}, h_k))1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]}{E[1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]}Z_t]. \tag{5.2}$$

First notice that any $\mathcal{G}_t-$adapted process Z_t has the representation (see [4])

$$Z_t = \sum_{k\geq 0} 1_{]\tau_k,\tau_{k+1}]}(t)Z_k(t) + Z_\infty 1_{]\tau_\infty,\infty[}(t), \tag{5.3}$$

with the process $Z_k(t)$ being $\mathcal{G}_k \otimes \mathcal{B}(\mathbb{R}_+)$-measurable. Furthermore, under our assumptions, for all $t > 0$, $\lim_{n\to\infty} 1_{\{\tau_n<t\}} = 0$ and thus

$$Z_t = \sum_{k\geq 0} 1_{]\tau_k,\tau_{k+1}]}(t)Z_k(t). \tag{5.4}$$

Note, finally, that $E[1_{\{\tau_k<t\leq\tau_{k+1}\}}|\mathcal{G}_k] = 1_{]\tau_k,\infty)}(t)E[1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]]$. We then have

$$\begin{aligned}
E[E[f(\theta_t, h_t)|\mathcal{G}_t]Z_t] &= E[f(\theta_t, h_t)\sum_{k\geq 0} 1_{]\tau_k,\tau_{k+1}]}(t)Z_k(t)] \\
&= \sum_{k\geq 0} E[E[f(\theta_t, h_t)1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]1_{\{\tau_k<t\}}Z_k(t)] \\
&= \sum_{k\geq 0} E[\frac{E[f(\theta_t, h_t)1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]}{E[1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]}E[1_{]\tau_k,\tau_{k+1}]}(t)Z_k(t)|\mathcal{G}_k]] \\
&= E[\sum_{k\geq 0} 1_{]\tau_k,\tau_{k+1}]}(t)\frac{E[f(\theta_t, h_t)1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]}{E[1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]}Z_t],
\end{aligned}$$

and thus we obtain (5.2) since

$$f(\theta_t, h_t) = \sum_{k=0}^{\infty} 1_{[\tau_k,\tau_{k+1})}(t)f(\theta_t, \gamma(\tilde{X}_t - \tilde{X}_{\tau_k}, h_k)),$$

which follows from (2.16). □

Lemma 5.2. *We have the following equation*

$$E[\int_t^T f(\theta_s, h_s)ds|\tau_0 = t, \pi_{\tau_0} = \pi] = E[\sum_{k\geq 0} \hat{C}(\tau_k, \pi_{\tau_k}, h_k)1_{\{\tau_k<T\}}|\tau_0 = t, \pi_{\tau_0} = \pi] \tag{5.5}$$

with $\hat{C}(t, \pi, h)$ *defined by (4.6) in Definition 4.1.*

Proof. For simplicity, in the following formula we shall use the notation

$$E^{t,\pi}[\cdot] \equiv E[\cdot\ |\tau_0 = t, \pi_{\tau_0} = \pi]$$

Using (5.1) we have similarly as above

$$\begin{aligned}
E^{t,\pi}[\int_t^T E[f(\theta_s, h_s)|\mathcal{G}_s]ds] &= E^{t,\pi}[\int_t^T \sum_{k\geq 0} 1_{]\tau_k,\tau_{k+1}]}(s) \frac{E[f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_{\tau_k}, h_k))1_{\{s<\tau_{k+1}\}}|\mathcal{G}_k]}{E[1_{\{s\leq\tau_{k+1}\}}|\mathcal{G}_k]}ds] \\
&= E^{t,\pi}[\sum_{k\geq 0}\int_t^T 1_{]\tau_k,\infty)}(s)E[f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_{\tau_k}, h_k))1_{\{s<\tau_{k+1}\}}|\mathcal{G}_k]ds] \\
&= E^{t,\pi}[\sum_{k\geq 0}\int_t^T 1_{]\tau_k,\infty)}(s)E[e^{-\int_{\tau_k}^s n(\theta_u)du} f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_{\tau_k}, h_k))|\mathcal{G}_k]ds] \\
&= E^{t,\pi}[\sum_{k\geq 0}\int_t^T 1_{]\tau_k,\infty)}(s)E[E[e^{-\int_{\tau_k}^s n(\theta_u)du} f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_{\tau_k}, h_k))|\mathcal{G}_k \vee \sigma\{\theta_{\tau_k}\}]|\mathcal{G}_k]ds]
\end{aligned} \tag{5.6}$$

Since $(\theta_t, \tilde{X}_t)$ is a time homogeneous Markov process,

$$\begin{aligned}
&E[e^{-\int_{\tau_k}^s n(\theta_u)du} f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_{\tau_k}, h_k))|\mathcal{G}_k \vee \sigma\{\theta_{\tau_k}\}] \\
&\quad = E[e^{-\int_0^t n(\theta_u)du} f(\theta_t, \gamma(\tilde{X}_t - x, h))|\theta_0 = \theta, \tilde{X}_0 = x]|_{t=s-\tau_k,\theta=\theta_k,x=\tilde{X}_{\tau_k},h=h_k}
\end{aligned} \tag{5.7}$$

We now have, recalling the definition of $r_{ji}(t, z)$ in (3.8),

$$\begin{aligned}
&E[e^{-\int_0^t n(\theta_s)ds)} f(\theta_t, \gamma(\tilde{X}_t - x, h))|\theta_0 = \theta, \tilde{X}_0 = x] \\
&\quad = E[e^{-\int_0^t n(\theta_s)ds)} E[f(\theta_t, \gamma(\tilde{X}_t - x, h))|\mathcal{F}_t^\theta \vee \{\tilde{X}_0 = x\}]|\theta_0 = \theta, \tilde{X}_0 = x] \\
&\quad = E[e^{-\int_0^t n(\theta_s)ds)} \int_{\mathbb{R}^m} f(\theta_t, \gamma(z, h))\rho_{0,t}^\theta(z)dz|\theta_0 = \theta, \tilde{X}_0 = x] \\
&\quad = E[\int_{\mathbb{R}^m} \sum_{ij} 1_{\{\theta_t=e_i,\theta_0=e_j\}} f(e_i, \gamma(z, h)) \\
&\qquad\qquad \times E[e^{-\int_0^t n(\theta_s)ds}\rho_{0,t}^\theta(z)|\theta_t = e_i, \theta_0 = e_j]dz|\theta_0 = \theta, \tilde{X}_0 = x] \\
&\quad = E[\int_{\mathbb{R}^m} \sum_{ij} 1_{\{\theta_t=e_i,\theta_0=e_j\}} f(e_i, \gamma(z, h))r_{ji}(t, z)dz|\theta_0 = \theta, \tilde{X}_0 = x] \\
&\quad = \int_{\mathbb{R}^m} \sum_{ij} f(e_i, \gamma(z, h))r_{ji}(t, z)p_{ji}(t)1_{\{\theta=e_j\}}dz.
\end{aligned} \tag{5.8}$$

We finally have

$$
\begin{aligned}
& E^{t,\pi}[\int_t^T f(\theta_s,h_s)ds] = E^{t,\pi}[\int_t^T E[f(\theta_s,h_s)|\mathcal{G}_s]ds] \\
= \ & E^{t,\pi}[\sum_{k\geq 0}\int_t^T 1_{]\tau_k,\infty)}(s)E[E[e^{-\int_{\tau_k}^s n(\theta_u)du}f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_{\tau_k},h_k))|\mathcal{G}_k\vee\sigma\{\theta_{\tau_k}\}]|\mathcal{G}_k]ds] \\
= \ & E^{t,\pi}[\sum_{k\geq 0}1_{\{\tau_k<T\}}\int_{\tau_k}^T\int_{\mathbb{R}^m}\sum_{ij}f(e_i,\gamma(z,h_k))r_{ji}(s-\tau_k,z)p_{ji}(s-\tau_k)\pi^j_{\tau_k}dzds] \\
= \ & E^{t,\pi}[\sum_{k\geq 0}\hat{C}(\tau_k,\pi_{\tau_k},h_k)1_{\{\tau_k<T\}}].
\end{aligned}
\tag{5.9}
$$

□

Proof of statement (ii) of Lemma 4.1.

We start by proving that $\hat{C}(t,\pi,h)$ is Lipschitz continuous with respect to t.

$$
\begin{aligned}
\hat{C}(t,\pi,h) &= \int_t^T\int_{\mathbb{R}^m}\sum_{i,j}f(e_i,\gamma(x,h))r_{ji}(s-t,x)p_{ji}(s-t)\pi^j dxds \\
&= \int_0^{T-t}\int_{\mathbb{R}^m}\sum_{i,j}f(e_i,\gamma(x,h))r_{ji}(s,x)p_{ji}(s)\pi^j dxds.
\end{aligned}
\tag{5.10}
$$

Thus

$$
\begin{aligned}
|\hat{C}(t,\pi,h)-\hat{C}(\bar{t},\pi,h)| &= |\int_{T-\bar{t}}^{T-t}\int_{\mathbb{R}^m}\sum_{i,j}f(e_i,\gamma(x,h))r_{ji}(s,x)p_{ji}(s)\pi^j dxds| \\
&\leq \|f\||t-\bar{t}|,
\end{aligned}
\tag{5.11}
$$

where $\|f\| := \sup_{e\in E,h\in\bar{H}_m}\|f(e,h)\|$. Next, let us prove that $C(t,\pi,h)$ is Lipschitz continuous with respect to π (in the metric introduced in (3.21)).

$$
\begin{aligned}
|\hat{C}(t,\pi,h)-\hat{C}(t,\bar{\pi},h)| &= |\int_0^{T-t}\int_{\mathbb{R}^m}\sum_{i,j}f(e_i,\gamma(x,h))r_{ji}(s,x)p_{ji}(s)(\pi^j-\bar{\pi}^j)dxds| \\
&\leq \|f\|T|\pi-\bar{\pi}| = \|f\|T\sum_{i=1}^N|\pi(e_i)-\bar{\pi}(e_i)| \\
&\leq \|f\|T\|\pi-\bar{\pi}\|_{TV}\leq\|f\|T\frac{2}{\log 3}d_H(\pi,\bar{\pi}),
\end{aligned}
\tag{5.12}
$$

where we have used (3.20).

Next, let us prove that $C(t,\pi,h)$ is continuous with respect to h (always in the metric introduced in (3.21)). The function $f(e_i,h)$ is bounded and continuous with respect to h for all i. Furthermore, $\gamma(x,h)$ is continuous with respect to h for all $x\in\mathbb{R}^m$. Applying the dominated convergence theorem, for $h_n\subset\bar{H}_m$, s.t. $\lim_{n\to\infty}h_n = h\in\bar{H}_m$

$$
\begin{aligned}
\lim_{n\to\infty}\hat{C}(t,\pi,h_n) &= \int_0^{T-t}\int_{\mathbb{R}^m}\sum_{i,j}\lim_{n\to\infty}f(e_i,\gamma(x,h_n))r_{ji}(s,x)p_{ji}(s)\pi^j dxds \\
&= \int_0^{T-t}\int_{\mathbb{R}^m}\sum_{i,j}f(e_i,\gamma(x,h))r_{ji}(s,x)p_{ji}(s)\pi^j dxds \\
&= \hat{C}(t,\pi,h).
\end{aligned}
\tag{5.13}
$$

$\hat{C}(t,\pi,h)$ is thus continuous with respect to each of the variables t,π,h. However, continuity in t,π is independent of the other variable. Hence, $\hat{C}(t,\pi,h)$ is a continuous function on $[0,T] \times \mathcal{S}_N \times \bar{H}_m$.

Proof of Lemma 4.2

Fix $n \geq 0$. Recall the definition of h^i_n given in section 2.3. Since S_t is continuous and V_t satisfies the self-financing condition, we obtain

$$h^i_{\tau_n-} = \frac{N^i_{n-1}S^i_{\tau_n-}}{V_{\tau_n-}} = \frac{N^i_{n-1}S^i_{\tau_n}}{V_{\tau_n}} = \frac{N^i_{n-1}S^i_{\tau_n}}{\sum_{i=0}^m N^i_n S^i_{\tau_n}}.$$

Using (2.16), (2.18), for *all* $k \geq 1, h \in \mathcal{A}^n, t \in [\tau_{n+k}, T]$, one furthermore has

$$h^i_t = \gamma^i(\tilde{X}_t - \tilde{X}_{\tau_{n+k}}, h_{n+k}) = \gamma^i(\tilde{X}_t - \tilde{X}_{\tau_n}, h_n).$$

Therefore, using lemma 4.1(i) for $h \in \mathcal{A}^n$

$$\begin{aligned} W(t,\pi,h.) &= E[\sum_{k=0}^{n-1}\int_{\tau_k}^{T\wedge\tau_{k+1}} f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_{\tau_k}, h_k))ds 1_{\{\tau_k<T\}} \\ &\quad + \sum_{k=n}^{\infty}\int_{\tau_k}^{T\wedge\tau_{k+1}} f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_{\tau_k}, h_k))ds 1_{\{\tau_k<T\}} | \tau_0 = t,\ \pi_{\tau_0} = \pi] \\ &= E[\sum_{k=0}^{n-1} \hat{C}(\tau_k, \pi_{\tau_k}, h_k)1_{\{\tau_k<T\}} + \int_{\tau_n}^{T} f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_{\tau_n}, h_n))ds 1_{\{\tau_n<T\}} | \tau_0 = t,\ \pi_{\tau_0} = \pi]. \end{aligned} \tag{5.14}$$

Proof of Lemma 4.6

By the definition of $\mathcal{A}^n$, for $n \geq 0, \mathcal{A}^n \subset \mathcal{A}^{n+1} \subset \mathcal{A}$, hence,

$$\sup_{h\in\mathcal{A}^n} W(t,\pi,h.) \leq \sup_{h\in\mathcal{A}^{n+1}} W(t,\pi,h.) \leq \sup_{h\in\mathcal{A}} W(t,\pi,h.). \tag{5.15}$$

By the definition of $W^n(t,\pi)$ and $W(t,\pi)$

$$W^n(t,\pi) \leq W^{n+1}(t,\pi) \leq W(t,\pi). \tag{5.16}$$

Using Lemma 4.5, for $n, m \geq 0$

$$\bar{W}^n(t,\pi) \leq \bar{W}^{n+m}(t,\pi) \leq W(t,\pi). \tag{5.17}$$

Letting $m \to \infty$

$$\bar{W}^n(t,\pi) \leq \bar{W}(t,\pi) \leq W(t,\pi). \tag{5.18}$$

Proof of Lemma 4.7

For $h \in \mathcal{A}$, $W(t,\pi,h)$ defined by (4.8) satisfies

$$\begin{aligned} W(t,\pi,h.) &= E[\sum_{k=0}^{n-1} \hat{C}(\tau_k,\pi_{\tau_k},h_k)1_{\{\tau_k<T\}}|\tau_0=t,\pi_{\tau_0}=\pi] \\ &\qquad\qquad + E[\sum_{k=n}^{\infty} \hat{C}(\tau_k,\pi_{\tau_k},h_k)1_{\{\tau_k<T\}}|\tau_0=t,\pi_{\tau_0}=\pi] \\ &= E[\sum_{k=0}^{n-1} \hat{C}(\tau_k,\pi_{\tau_k},h_k)1_{\{\tau_k<T\}} + \int_{\tau_n}^{T} f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_{\tau_n},h_n))ds1_{\{\tau_n<T\}} \\ &\qquad - \textstyle\int_{\tau_n}^{T} f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_{\tau_n},h_n))ds1_{\{\tau_n<T\}}|\tau_0=t,\pi_{\tau_0}=\pi] \\ &\qquad + E[W(\tau_n,\pi_{\tau_n},h.)1_{\{\tau_n<T\}}|\tau_0=t,\pi_{\tau_0}=\pi]. \\ &\le W^n(t,\pi) + |E[\textstyle\int_{\tau_n}^{T} f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_{\tau_n},h_n))ds1_{\{\tau_n<T\}}|\tau_0=t,\pi_{\tau_0}=\pi]| \\ &\qquad + E[W(\tau_n,\pi_{\tau_n},h.)1_{\{\tau_n<T\}}|\tau_0=t,\pi_{\tau_0}=\pi] \\ &\le \bar{W}^n(t,\pi) + 2\|f\|TP(\tau_n<T|\tau_0=t). \end{aligned} \tag{5.19}$$

because of the representation of $W^n(t,\pi)$ in Corollary 4.2 (equation (4.13)) and Lemma 4.5. Thus, by letting $n \to \infty$, we obtain

$$W(t,\pi,h.) \le \bar{W}(t,\pi) \tag{5.20}$$

for all $h \in \mathcal{A}$.

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