Portfolio Optimization in discrete time

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Abstract

The paper is intended as a survey of some of the main aspects of portfolio optimization in discrete time. We consider three of the major criteria and discuss the method of Dynamic programming as well as the so-called martingale method as possible solution methods. We work out explicit examples for a logarithmic utility function and for the case of the complete binomial as well as the incomplete trinomial market models.

1 Introduction

The paper is intended as a survey of some of the main aspects concerning portfolio optimization in discrete time. We consider the three major types of problems :

- *Maximization of expected utility from terminal wealth*
- *Maximization of expected utility from consumption*
- *Maximization of expected utility from consumption and terminal wealth*

and discuss in more detail the case of a logarithmic utility function. For what concerns the solution methodologies we discuss the method of Dynamic programming (DP) as well as the so-called "martingale method". The latter method varies according to whether the market is complete or not, i.e. whether there exists or not an unique equivalent martingale measure. As example of a complete market in discrete time we consider the classical binomial market model and as an instance of an incomplete model we consider the trinomial model when there is only one risky underlying asset.

In section 2 we describe the problem setup and introduce the main definitions. The two main market models, the complete binomial model and the incomplete trinomial model, are then discussed in section 3 where we discuss also the equivalent martingale measures. A general description of the two main solution methodologies, namely DP and the martingale method, are then given in section 4 and their application to the solution of the three types of problems for the specific case of a logarithmic utility function are described in section 5. To better illustrate the computations described in section 5, some specific examples are explicitly solved in section 6.

In the text we shall not make specific references to the literature, but we limit ourselves to mention here that the main source of reference is the book [1] with additional references being [2],[3],[4].

2 Setting and problem definition

2.1 Assets and their dynamics

We consider a financial market, in which the prices of the assets evolve in discrete time. i.e. for $t = 1, \cdots, T$ on an underlying probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. There is a (locally) non risky asset, the price of which evolves as

$$B_{t+1} = (1 + r_t)B_t$$

where r_t is the so-called short rate of interest, and a certain number K of risky assets with price vector

$$S_t = (S_t^1, \cdots, S_t^K).$$

The generic S_t^i evolves according to

$$S_{t+1}^i = S_t^i \xi_{t+1}^i \tag{1}$$

with (ξ_t^i) i.i.d. sequences of random variables so that, as processes, S_t^i are Markov.

Note that we could equivalently write

$$S_{t+1}^i = S_t^i(1 + \bar{\xi}_{t+1}^i)$$

where $(\bar{\xi}_t^i)$ can be interpreted as *random returns*; in this case one has in fact $\frac{S_{t+1}^i - S_t^i}{S_t^i} = \bar{\xi}_{t+1}^i$. Notice also that as filtration we may take $\mathcal{F}_t = \sigma\{S_u, \, u \leq t\}$.

2.2 Portfolio strategies and self-financing portfolios

A portfolio strategy is a predictable process

$$\alpha_t = (\alpha_t^0, \alpha_t^1, \cdots, \alpha_t^K) \,, \quad \text{i.e. with } \alpha_t^i \in \mathcal{F}_{t-1}$$

where α_t^i denotes the number of units of the $i-$th risky asset held in the portfolio in period t $(i = 1, \cdots, K)$ while α_t^0 denotes the number of units of the non-risky asset. The value of the portfolio corresponding to α is thus

$$V_t = \alpha_t^0 B_t + \sum_{i=1}^{K} \alpha_t^i S_t^i \tag{2}$$

and we shall use the symbol α to denote both the strategy and the corresponding portfolio. Ignoring for the moment the possibility of consumption, we shall give the following

Definition 1. The portfolio α is said to be *self-financing* if

$$\begin{aligned} &V_{t+1} - V_t = \alpha^0_{t+1}\Delta B_t + \sum_{i=1}^{K} \alpha^i_{t+1}\Delta S^i_t \\ &(\Delta S^i_t = S^i_{t+1} - S^i_t) \end{aligned} \tag{3}$$

and this can equivalently be expressed as

$$\alpha^0_t B_t + \sum_{i=1}^{K} \alpha^i_t S^i_t = \alpha^0_{t+1} B_t + \sum_{i=1}^{K} \alpha^i_{t+1} S^i_t \tag{4}$$

Furthermore, we give

Definition 2. We call *gains process* of a self-financing strategy α the process

$$G_t = \sum_{s=0}^{t-1} \alpha^0_{s+1}\Delta B_s + \sum_{s=0}^{t-1}\sum_{i=1}^{K} \alpha^i_{s+1}\Delta S^i_s$$

The value in t of a self-financing portfolio with initial wealth V_0 can then be expressed as

$$V_t = V_0 + G_t$$

Later on it will be convenient to consider discounted values expressed in units of the non-risky asset B_t. In this case $\Delta B_t = 0$ and, denoting discounted values with a tilde, the discounted gains process $\tilde{G}_t$ for a self-financing strategy α becomes

$$\tilde{G}_t = \sum_{s=0}^{t-1}\sum_{i=1}^{K} \alpha^i_{s+1}\Delta \tilde{S}^i_s \tag{5}$$

with $\Delta \tilde{S}^i_t = \tilde{S}_{t+1} - \tilde{S}_t$ and we have

$$\tilde{V}_t = V_0 + \tilde{G}_t \tag{6}$$

(it is customary to normalize B_0 to 1 so that $\tilde{V}_0 = V_0$).

If we allow for the possibility of consumption, we have

Definition 3. A consumption process C is a non-negative, adapted process $C = (C_t)$ where C_t denotes the amount of funds consumed in period t.

Definition 4. A *Consumption-investment plan/strategy* is a pair (C, α) with α a portfolio strategy.

On the basis of (2) and (4) the self-financing condition then becomes

$$V_t = C_t + \alpha^0_{t+1} B_t + \sum_{i=1}^{K} \alpha^i_{t+1} S^i_t$$

implying

$$V_t = V_0 + G_t - \sum_{s=0}^{t-1} C_s \tag{7}$$

and

$$\tilde{V}_t = V_0 + \tilde{G}_t - \sum_{s=0}^{t-1} \tilde{C}_s \tag{8}$$

respectively.

Definition 5. The plan (C, α) is said to be *admissible* if $C_T \leq V_T$.

2.3 Investment criteria

We first recall the standard notion of a *utility function* that, as a function of the wealth process $V = (V_t)$ is such that

$$u(V) = \begin{cases} \text{strictly increasing} \\ \text{strictly concave, differentiable} \\ u'(\infty) := \lim_{V \to +\infty} u'(V) = 0 \\ \lim_{V \to 0^+} u'(V) = +\infty \end{cases}$$

It will be convenient to extend this definition also to $V < 0$ by putting $u(V) = -\infty$ there.

Examples of utility functions (in the range $V > 0$) are

$u(V) = \log V$ the *log-utility function* and
$u(V) = \frac{V^\alpha}{\alpha}$ $(0 < \alpha < 1)$the *power utility function* for a risk-averse investor

We may now consider three investment criteria :

- *Maximization of expected utility from terminal wealth*

$$\begin{cases} \max_{\alpha} \; E\{u(V_T^\alpha)\} \\ \text{with } \; V_0 = v \\ \alpha \; : \; \text{self-financing (predictable)} \end{cases}$$

- *Maximization of expected utility from consumption*

$$\begin{cases} \max_{(C,\alpha)} \; E\left\{ \sum_{t=0}^{T} \beta^t u(C_t) \right\} \\ \text{with } \; V_0 = v \; \text{ and } \; \beta \in (0,1) \; \text{ a discount factor} \\ (C, \alpha) : \text{self-financing and admissible} \end{cases}$$

- *Maximization of expected utility from consumption and terminal wealth*

$$\max_{(C,\alpha)\in\mathcal{A}_v} E\left\{\sum_{t=0}^{T}\beta^t u_c(C_t)+\beta^T u_p(V_T^\alpha - C_T)\right\}$$

$$\text{with } \mathcal{A}_v : \begin{cases} \alpha & : \text{ self-financing and predictable} \\ C & : \text{ nonnegative, adapted}, C_T \le V_T \end{cases}$$

3 Market models and equivalent martingale measures

Before coming to possible market models, let us recall the notion of an *equivalent martingale measure*. Given the real world probability measure P on $(\Omega, \mathcal{F})$, a probability measure $Q \sim P$, i.e. a measure Q that has the same null-sets as P, is called equivalent martingale measure if all the assets in the market, expressed in units of a same asset, for which one usually takes the locally non-risky asset B_t, are $(Q, \mathcal{F}_t)-$martingales. More precisely if, for $i = 1, \cdots, K$, we have

$$E^Q\left\{\frac{S^i_{t+1}}{B_{t+1}} \mid \mathcal{F}_t\right\} = \frac{S^i_t}{B_t} \tag{9}$$

which is equivalent to requiring

$$E^Q\{S^i_{t+1} \mid \mathcal{F}_t\} = \frac{B_{t+1}}{B_t} S^i_t = (1+r_t)\, S^i_t \tag{10}$$

Recalling the definition of $\tilde{G}_t$ in (5), from (9) it is easily seen that $E^Q\{\tilde{G}_t\} = E^Q\{E^Q\{\tilde{G}_t \,|\, \mathcal{F}_t\}\} = 0$. From (6) it then follows that, under Q, not only the discounted price processes, but also the discounted value processes of a self financing strategy are all martingales (In the case when there is also consumption, from the positivity of $\tilde{C}_t$ and (8) it follows that $\tilde{V}_t$ is a supermartingale).

In what follows we shall distinguish the case when there is a unique equivalent martingale measure and when there are more. The first case corresponds to what is called a *complete market*, a notion that comes from the problem of hedging a contingent claim that we do not consider here and so we shall simply identify a complete market as one where there is a unique equivalent martingale measure and incomplete market as one where there are more. In our incomplete markets below the set of equivalent martingale measures will be a convex polyhedron with a finite number of vertices (extreme points) that we shall call the *extremal martingale measures*.

3.1 A complete market model : the binomial or Cox-Ingersoll-Ross model

We shall begin by recalling a Bernoulli process that is a process $\{X_t\}$, $t \in [0,T] \cap \mathbb{N}$ having the property that

$$X_t \;\text{ i.i.d.} \sim P\{X_t = 1\} = 1 - P\{X_t = 0\} = p \in (0,1)$$

We may consider it as defined on the following sample space (*space of elementary events*)

$$\Omega = \{\omega\} \text{ with } \omega \text{ of the form } \omega = (\underbrace{0,1,0,0,1,1,\cdots}_{T \text{ times}})$$

The number of elements in Ω is finite and given by 2^T and the probabilities in the measure P are given by

$$P\{\omega\} = p^n(1-p)^{T-n} \quad (\text{if } \sum_{t=1}^{T} \omega_t = n)$$

Letting $N_t := X_1 + \cdots + X_t$, $t \in [0,T] \cap \mathbb{N}$ then

$$P\{N_t = n\} = \begin{pmatrix} t \\ n \end{pmatrix} p^n(1-p)^{t-n}$$

3.1.1 Binomial market model

We shall consider here a market with one locally riskless asset and a single risky asset ($K = 1$) evolving according to (1), i.e. as

$$S_{t+1} = S_t \cdot \xi_{t+1}$$

where ξ_t are i.i.d. and such that

$$\xi_t = \begin{cases} u & \text{with probab.} \quad p \\ d & \text{with probab.} \quad 1-p \end{cases}$$

It follows that we have the representation

$$\xi_t = uX_t + d(1 - X_t)$$

with X_t a Bernoulli process with parameter p. If N_t corresponds to this Bernoulli process, then

$$S_t = S_0 u^{N_t} d^{t-N_t}$$

and

$$P\{S_t = S_0 u^n d^{t-n}\} = \begin{pmatrix} t \\ n \end{pmatrix} p^n(1-p)^{t-n}$$

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3.1.2 Martingale measure in the binomial market model

We had seen at the beginning of this section 3 that an equivalent martingale measure is a measure $Q \sim P$ on Ω such that (9) holds. Putting

$$q := Q\{X_{t+1} = 1 \,|\, \mathcal{F}_t\}$$

and letting for simplicity $r_t \equiv r$, relation (9) becomes in the present case

$$(qu + (1-q)d)\, S_t = (1+r)\, S_t$$

namely

$$q = \frac{1+r-d}{u-d}$$

This q is unique and $\in (0,1)$ provided $d < 1+r < u$.

Recalling the sample space Ω of the binomial model, given $\omega \in \Omega$ with $\sum_i^T \omega_i = n$ we can write

$$P(\omega) = p^n (1-p)^{T-n} \;\; ; \;\; Q(\omega) = q^n (1-q)^{T-n}$$

so that for the Radon-Nikodym derivative on Ω we obtain

$$L(\omega) = \frac{Q(\omega)}{P(\omega)} = \left(\frac{q}{p}\right)^n \left(\frac{1-q}{1-p}\right)^{T-n} \tag{11}$$

3.2 An incomplete market model : the trinomial model

Corresponding to the Bernoulli process for the binomial model here we consider a process $\{X_t\}$, $t \in [0,T] \cap \mathbb{N}$ characterized by

$$X_t \text{ i.i.d.} \begin{cases} P\{X_t = 1\} &= p_1 \\ P\{X_t = 2\} &= p_2 \\ P\{X_t = 3\} &= p_3 \end{cases} \quad \forall t$$

$$p_1 + p_2 + p_3 = 1$$

The sample space can here be represented as

$$\Omega = \{\omega\} \text{ with } \omega \text{ of the form } \omega = (\underbrace{1,3,2,2,1,3,\cdots}_{T \text{ times}})$$

and $\#\Omega = 3^T$. The probabilities in the measure P are given by

$$P(\omega) = p_1^{n_1} p_2^{n_2} p_3^{n_3} \quad \text{if} \quad \begin{cases} \sum_{t=1}^T 1_{\{\omega_t = 1\}} &= n_1 \\ \sum_{t=1}^T 1_{\{\omega_t = 2\}} &= n_2 \\ \sum_{t=1}^T 1_{\{\omega_t = 3\}} &= n_3 \end{cases}$$

Letting this time

$$N_t^i := \sum_{s=1}^t 1_{\{X_s = i\}} \; , \; i = 1,2,3 \; ; \; t \in [0,T] \cap \mathbb{N} \tag{12}$$

one has

$$P\{N_t^1 = n_1, N_t^2 = n_2, N_t^3 = n_3\} = \tfrac{t!}{n_1! n_2! n_3!}\, p_1^{n_1} p_2^{n_2} p_3^{n_3}$$
$$(n_1 + n_2 + n_3 = t)$$

3.2.1 Trinomial market model

We shall consider here too a market with one locally riskless asset and a single risky asset ($K = 1$) evolving according to (1), i.e. as

$$S_{t+1} = S_t \cdot \xi_{t+1}$$

where ξ_t are i.i.d. and such that

$$\xi_t = \begin{cases} u & \text{with probab.} \quad p_1 \\ m & \text{with probab.} \quad p_2 \\ d & \text{with probab.} \quad p_3 \end{cases}$$

It follows that we have the representation

$$\xi_t = u 1_{\{X_t = 1\}} + m 1_{\{X_t = 2\}} + d 1_{\{X_t = 3\}}$$

and, if N_t^i, $i = 1, 2, 3,$ is as in (12), then we can write

$$S_t = S_0 \, u^{N_t^1} m^{N_t^2} d^{N_t^3}$$

and

$$P\{S_t = S_0 \, u^{n_1} m^{n_2} d^{n_3}\} = \tfrac{t!}{n_1! n_2! n_3!} \, p_1^{n_1} p_2^{n_2} p_3^{n_3}$$
$$(n_1 + n_2 + n_3 = t)$$

3.2.2 Martingale measures in the trinomial market model

As for the binomial market model, an equivalent martingale measure is also here a measure $Q \sim P$ on Ω such that (9) holds. Notice first that, if X_t are i.i.d. under P, this does not necessarily imply that they are independent also under Q. Put therefore

$$q_i(t+1) := Q\{X_{t+1} = i \,|\, \mathcal{F}_t\} \; ; \;\; i = 1, 2, 3 \; ; \;\; t = 0, 1$$

and let also here for simplicity $r_t \equiv r$. Relation (9) then becomes in the present case

$$\begin{cases} (q_1(t+1)u + q_2(t+1)m + q_3(t+1)d) \, S_t = (1 + r) \, S_t \\ q_1(t+1) + q_2(t+1) + q_3(t+1) = 1 \end{cases}$$

The latter, which is a condition on the conditional probabilities in the generic period t, admits infinitely many solutions $(q_1(t), q_2(t), q_3(t)) \in \Sigma^3$ (simplex in $\mathbb{R}^3$) and notice that, since the coefficients in the above system are independent of t and $\mathcal{F}_t$, the solutions are also independent of t and $\mathcal{F}_t$, i.e. $(q_1(t), q_2(t), q_3(t)) \equiv (q_1, q_2, q_3)$. These ∞^1 solutions form a convex set (segment) characterized by vertices (extremal points) that are independent of t and differ according to whether $m \geq 1 + r$ or $m < 1 + r$. The 1st vertex is given by

$$(q_1^0, q_2^0, q_3^0) = \begin{cases} \left(0, \frac{1+r-d}{m-d}, \frac{m-(1+r)}{m-d}\right) & \text{if} \quad m \geq 1 + r \\ \left(\frac{(1+r)-m}{u-m}, \frac{u-(1+r)}{u-m}, 0\right) & \text{if} \quad m < 1 + r \end{cases}$$

and the 2nd is characterized by

$$(q_1^1, q_2^1, q_3^1) = \left(\frac{(1+r)-d}{u-d}, \, 0, \, \frac{u-(1+r)}{u-d} \right)$$

These extremal probability measures are not equivalent to (p_1, p_2, p_3) (all p_i are > 0). The equivalent measures are obtained as the convex combinations

$$(q_1^\gamma, q_2^\gamma, q_3^\gamma) = \gamma \, (q_1^0, q_2^0, q_3^0) + (1-\gamma) \, (q_1^1, q_2^1, q_3^1)$$

for $\gamma \in (0,1)$ and notice that they concern the set of conditional measures in the generic period t which, as remarked previously, are here independent of the conditioning σ-algebra $\mathcal{F}_t$.

It turns out that also the set of martingale measures on all of Ω is a bounded convex set. However, determining its vertices is a more complex issue and we limit ourselves to illustrate it for a time horizon of $T = 2$. With S_0 fixed and recalling that ω_t corresponds to ξ_t ($t = 1, 2$), we have

$$\Omega = \{\omega\} \quad \text{with} \quad \omega = (\omega_1, \omega_2)$$

where, for the trinomial case,

$$\omega_1, \omega_2 \in \{u, m, d\}$$

and we put $\omega_t^1 = u$, $\omega_t^2 = m$, $\omega_t^3 = d$, $(t = 1, 2)$. For a generic martingale measure Q on Ω we now have

$$Q(\omega_1^i, \omega_2^j) = Q^1(\omega_1^i) \, Q^2(\omega_2^j | \, \omega_1^i) \,, \quad i, j \in \{1, 2, 3\}$$

with Q^1 the marginal measure in the 1st period and Q^2 the conditional measure for the 2nd period, given the 1st. Due to the previous remarks concerning the independence of $(q_1(t), q_2(t), q_3(t))$ from t and $\mathcal{F}_t$, this conditional measure Q^2 reduces here to a marginal measure as well, namely $Q^2(\omega_2^j | \, \omega_1^i) = Q^2(\omega_2^j)$.

For the sake of specificity assume now the case $m < 1 + r$ and let $Q^{1,0}$ be the extremal measure, among the measures Q^1, that corresponds to

$$(q_1^0, q_2^0, q_3^0) = \left(\frac{(1+r)-m}{u-m}, \, \frac{u-(1+r)}{u-m}, \, 0 \right)$$

and $Q^{1,1}$ the one corresponding to

$$(q_1^1, q_2^1, q_3^1) = \left(\frac{(1+r)-d}{u-d}, \, 0, \, \frac{u-(1+r)}{u-d} \right).$$

Analogously for $Q^{2,0}$ and $Q^{2,1}$.

We know now that Q^1 and Q^2 are convex combinations $\gamma_t Q^{t,0} + (1-\gamma_t) Q^{t,1}$ of $Q^{t,0}$ and $Q^{t,1}$ for $t = 1, 2$ respectively. In general, the coefficients γ_t in these convex combinations may depend not only on time, but in a predictive way also on ω itself (i.e. γ_t may depend on $(\omega_1, \cdots, \omega_{t-1})$) thereby inducing a dependence structure over time. Since in our case we have seen in the remarks above that $(q_1(t), q_2(t), q_3(t))$ are independent of t and $\mathcal{F}_t$, we shall let these coefficients depend only on time. We shall thus write

$$Q^t(\omega_t^i) = \gamma_t\, Q^{t,0}(\omega_t^i) + (1-\gamma_t)\, Q^{t,1}(\omega_t^i)$$

for suitable $\gamma_t \in (0,1)$. One thus obtains (recall that $\omega = (\omega_1^i, \omega_2^j)$)

$$\begin{array}{rcl} Q(\omega) & = & \gamma_1\gamma_2\, Q^{1,0}(\omega_1^i)Q^{2,0}(\omega_2^j) + (1-\gamma_1)\gamma_2\, Q^{1,1}(\omega_1^i)Q^{2,0}(\omega_2^j) \\ & + & \gamma_1(1-\gamma_2)\, Q^{1,0}(\omega_1^i)Q^{2,1}(\omega_2^j) + (1-\gamma_1)(1-\gamma_2)\, Q^{1,1}(\omega_1^i)Q^{2,1}(\omega_2^j) \end{array}$$

ending up with 4 extremal measures

$$\bar{Q}^1 = Q^{1,0}Q^{2,0},\ \bar{Q}^2 = Q^{1,1}Q^{2,0},\ \bar{Q}^3 = Q^{1,0}Q^{2,1},\ \bar{Q}^4 = Q^{1,1}Q^{2,1}$$

where, e.g.

$$\bar{Q}^1(u,u) = \left(\tfrac{(1+r)-m}{u-m}\right)^2,\ \bar{Q}^1(u,d) = 0,\ \bar{Q}^1(u,m) = \tfrac{((1+r)-m)(u-(1+r))}{(u-m)^2}$$

$$\bar{Q}^4(d,u) = \tfrac{((1+r)-d)(u-(1+r))}{(u-d)^2},\ \bar{Q}^4(d,m) = 0,\ \bar{Q}^4(d,d) = \left(\tfrac{u-(1+r)}{u-d}\right)^2$$

Finally, for the Radon-Nikodym derivatives we obtain

$$L^j(\omega) = \frac{\bar{Q}^j(\omega)}{P(\omega)} \quad , \quad j = 1, \cdots, 4$$

with $P(\omega_1^i, \omega_2^j) = p_i p_j$.

4 Solution methods (general description)

In this section we recall the basic aspects of the Dynamic Programming approach and of the so-called "martingale method".

4.1 Dynamic programming

4.1.1 General context

Let $V_t\,,\ t = 0, 1, \cdots, T\,,$ be a given process (as e.g. the portfolio value process), of which the evolution depends on the choice of a "control sequence" (equivalent to a strategy) $\pi_t\,,\ t = 0, \cdots, T$, adapted to a given observed filtration. A typical example for such a situation is

$$V_{t+1} = G_t(V_t, \pi_t, \xi_{t+1})\,, \quad \xi_t \quad \text{i.i.d.}$$

and notice that, if $\pi_t = \pi_t(V_t)$, (*Markov controls*), then V_t is a Markov process. The control sequence π_t is chosen so as to maximize an (additive over time) criterion, i.e.

$$\max_{\pi_0, \cdots, \pi_T} E\left\{\sum_{t=0}^{T} u(V_t, \pi_t)\right\} \tag{13}$$

with $u(\cdot)$ a utility function (in traditional stochastic control applications the last term for $t = T$, i.e. the terminal utility, does not depend on the control π).

4.1.2 The Dynamic programming principle

The method of Dynamic programming (DP) to obtain the control maximizing (13) is based on the so-called *Dynamic Programming Principle*, which states that : *if a process is optimal over an entire sequence of periods, then it has to be optimal over each single period.* In the general context described above, the DP principle allows to determine the optimal sequence $\pi_0, \cdots, \pi_T$ by a sequence of minimizations over the individual controls π_t (scalar minimizations). In fact, due to the Markovianity of π_t, V_t and the additivity over time of the criterion,

$$\max_{\pi_0,\cdots,\pi_T} E\left\{\sum_{t=0}^{T} u(V_t,\pi_t)\right\} = \max_{\pi_0}\left[u(V_0,\pi_0) + E\left\{\max_{\pi_1}\left[u(V_1,\pi_1)\right.\right.\right.$$

$$+E\left\{\cdots + E\left\{\max_{\pi_{T-1}}\left[u(V_{T-1},\pi_{T-1})\right.\right.\right.$$

$$\left.\left.\left. + E\left\{\max_{\pi_T} u(V_T,\pi_T) \mid V_{T-1},\pi_{T-1}\right\}\right] \mid V_{T-2},\pi_{T-2}\right\}\cdots \mid V_1,\pi_1\}] \mid V_0,\pi_0\}]$$

4.1.3 Implementation of the DP principle

The way the DP principle is used to determine the control maximizing (13) is as follows. Let

$$U_t(v) := \max_{\pi_t,\cdots,\pi_T} E\left\{\sum_{s=t}^{T} u(V_s,\pi_s) \mid V_t = v\right\} \tag{14}$$

then the DP principle leads to the following *DP-algorithm*

$$\begin{cases} U_T(v) = \max_{\pi_T}\ u(v,\pi_T) \quad \text{and, for } t<T, \\ U_t(v) = \max_{\pi_t}\left[u(v,\pi_t) + E\left\{U_{t+1}\left(G_t(V_t,\pi_t,\xi_{t+1})\right) \mid V_t = v\right\}\right] \end{cases} \tag{15}$$

which gives the optimal value and optimal control (obtained by backwards induction with scalar maximization). Notice that one automatically has $\pi_t^{\max}$ as a function of only V_t.

4.1.4 Specific context (expected utility from terminal wealth)

Here we mention how the general dynamic programming algorithm, described above, can be applied to the specific case of maximization of expected utility from terminal wealth.

We consider as V_t the portfolio value. As control/strategy we take $\pi_t = (\pi_t^0, \pi_t^1, \cdots, \pi_t^K)$ with

$$\pi_t^0 = \frac{\alpha_{t+1}^0 B_t}{V_t} \quad , \quad \pi_t^i = \frac{\alpha_{t+1}^i S_t^i}{V_t} \ (i = 1,\cdots,K)$$

i.e. the fractions of wealth invested in the individual assets so that

$$\pi_t^0 = 1 - \sum_{i=1}^{K} \pi_t^i \quad (\pi_t^i \in (0,1))$$

Recall also that, by the self-financing property,

$$V_t = \alpha_t^0 B_t + \sum_{i=1}^{K} \alpha_t^i S_t^i = \alpha_{t+1}^0 B_t + \sum_{i=1}^{K} \alpha_{t+1}^i S_t^i$$

and that we had supposed α to be predictable with

$$S_{t+1}^i = S_t^i \cdot \xi_{t+1}^i \quad (\xi_t^i \text{ i.i.d. sequences})$$

By the self-financing property and the previous definitions we obtain

$$\begin{aligned} V_{t+1} &= V_t + \alpha_{t+1}^0 \Delta B_t + \sum_{i=1}^{K} \alpha_{t+1}^i \Delta S_t^i \\ &= V_t + \alpha_{t+1}^0 B_t r_t + \sum_{i=1}^{K} \alpha_{t+1}^i S_t^i \left(\xi_{t+1}^i - 1\right) \\ &= V_t + V_t \left[\pi_t^0 r_t + \sum_{i=1}^{K} \pi_t^i \left(\xi_{t+1}^i - 1\right)\right] \end{aligned} \tag{16}$$

i.e. we have

$$V_{t+1} = G_t(V_t, \pi_t, \xi_{t+1})$$

with

$$G_t(V_t, \pi_t, \xi_{t+1}) = V_t \left[\pi_t^0 (1 + r_t) + \sum_{i=1}^{K} \pi_t^i \xi_{t+1}^i\right] \tag{17}$$

and where now $\pi_t = (\pi_t^1, \cdots, \pi_t^K)$.

In the present case we have only a terminal utility, i.e.

$$u(V_t, \pi_t) = 0 \quad \text{for } t < T, \text{ and} \quad u(V_T, \pi_T) = u(V_T)$$

which implies (there is no π_T over which to maximize)

$$U_t(v) := \max_{\pi_t, \cdots, \pi_{T-1}} E\left\{u(V_T) \mid V_t = v\right\}$$

and the *DP algorithm* becomes

$$\begin{cases} U_T(v) = u(v) \quad \text{and, for } t < T, \\ U_t(v) = \max_{\pi_t} E\left\{U_{t+1}\left(G_t(V_t, \pi_t, \xi_{t+1})\right) \mid V_t = v\right\} \end{cases} \tag{18}$$

4.2 The "martingale" method

4.2.1 Introductory remarks

To introduce this method, recall from the beginning of section 3 that, in the case when there is no consumption, the discounted value process of any self financing strategy is a martingale under any martingale measure (we implicitly assume the required integrability conditions on the strategy).

The martingale method can be used for the solution of general stochastic optimization problems, but it draws its origin from the financial problem of hedging a contingent claim. Although, as already mentioned in the beginning of section 3, the hedging problem as such is not in the scope of this survey, we recall here just what is useful for the understanding of the martingale method.

Given is a horizon T and a "claim" H_T, namely a random variable, which is $\mathcal{F}_T-$measurable (recall from subsection 2.1 that we may assume $\mathcal{F}_t = \sigma\{S_u,\, u \leq t\}$). The hedging problem consists in finding an initial wealth $V_0 = v$ and a self financing strategy α_t such that $V_T = H_T$, $P-$ and therefore also $Q-$a.s., which is equivalent to requiring $\tilde{V}_T = \tilde{H}_T$, $P(Q)-$a.s. (Recall that the quantities with a "tilde" represent values that are discounted with respect to B_t). Since, as recalled above, the discounted value process of a self financing strategy is a martingale under any martingale measure, this will only be possible if the initial wealth v is such that $v = E^Q\{\tilde{H}_T\}$ for any martingale measure Q. The problem of finding the self financing hedging strategy α can then be seen as equivalent to a "martingale representation" problem in the following sense. Fix for the moment a MM Q and define the martingale $\tilde{M}_t := E^Q\{\tilde{H}_T \mid \mathcal{F}_t\}$. If we now determine the self financing strategy $\alpha_t = (\alpha_t^0, \cdots, \alpha_t^K)$ in such a way that, for $\tilde{V}_t$ from (6) with $V_0 = v$ and $\tilde{G}_t$ according to (5), we have $\tilde{V}_t = \tilde{M}_t$ then, starting from the initial wealth $V_0 = v$ and following this strategy α_t, we have $\tilde{V}_T = \tilde{H}_T$ and therefore also $V_T = H_T$. Finding the strategy α_t is therefore equivalent to finding the representation of the martingale $\tilde{V}_t = \tilde{M}_t$ in the form of (6) with $\tilde{G}_t$ as in (5) and where the unknown is the strategy α. Notice that, since for the same v we have $v = E^Q\{\tilde{H}_T\}$ for any MM Q, the strategy α_t will also be the same under any MM Q (to this effect see also subsection 5.2 below).

The martingale method itself consists now of three steps :

- *Determine the set of reachable values for the wealth V_T in period T;*
- *determine the optimal reachable wealth V_T^*;*
- *determine a self financing strategy α^* such that $V_T^{\alpha^*} = V_T^*$ where in V_T^α we make explicit the dependence of the terminal wealth on the strategy α.*

Notice that the last step corresponds indeed to a hedging problem, where the "claim" to be hedged is the optimal reachable wealth. Notice also that this method decomposes the originally dynamic problem of portfolio optimization problem (or any other dynamic optimization problem) into a static problem (determination of the optimal reachable wealth) and a hedging problem (that corresponds to a "martingale representation problem").

We proceed now with a description of the first two steps; the third step will be illustrated below when solving specific portfolio optimization problems. More specifically, we shall illustrate the first two steps for the problem of maximization of expected utility from terminal wealth; for the other two investment criteria these steps will be discussed in the next section 5.

4.2.2 First step : set of reachable portfolio values

On a more formal basis the problem consists in determining the set

$$\mathcal{V}_v = \{V \;:\; V = V_T^\alpha \;\textit{ for some self-financing and predictable } \alpha \text{ with } V_0 = v\}$$

of reachable values for the wealth in period T.

If the market is complete with unique equivalent martingale measure (EMM) Q, then

$$\mathcal{V}_v = \{V \;:\; E^Q\{B_T^{-1}V\} = v\}$$

If the market is not complete and the set of all martingale measures forms a bounded convex set (convex polyhedron) with a finite number of extremal values (vertices) Q^j $(j = 1, \cdots, J)$ as was the case for the trinomial market model that was described in section 3.2 as an example of an incomplete market model in discrete time, then

$$\mathcal{V}_v = \{V \;:\; E^{Q^j}\{B_T^{-1}V\} = v \text{ for } j = 1, \cdots, J\}.$$

4.2.3 Second step : optimal reachable portfolio value

In formal terms the problem is : determine V^* such that

$$E\{u(V^*)\} \geq E\{u(V)\}\,,\;\forall\, V \in \mathcal{V}_v$$

To solve this 2nd step problem we shall mention here essentially the method based on the Lagrange multiplier technique. We shall briefly recall also a technique based on convex duality to show that a certain V^* is indeed optimal.

Lagrange multiplier technique

a) *Case of a complete market - unique martingale measure Q*

Let $L := \frac{dQ}{dP}$ and denote by λ the Lagrange multiplier. Recalling that the requirement $V_0 = v$ can be translated into $E^Q\{B_T^{-1}V\} = v$ the problem in this second step becomes

$$\max_V \left[E\{u(V)\} - \lambda\, E^Q\{B_T^{-1}V\}\right] = \max_V \; E\{u(V) - \lambda\, B_T^{-1}LV\} \qquad (19)$$

By the properties of $u(\cdot)$ as utility function we have that the inverse of the derivative exists and we shall denote it by $I(\cdot) = (u'(\cdot))^{-1}$. On the other hand, maximizing the expectation on the right hand side of (19) is equivalent to maximizing its argument for each possible scenario/state of nature $\omega \in \Omega$. A necessary condition for this is that

$$u'(V) = \lambda B_T^{-1} L \quad \text{implying} \quad V = I(\lambda B_T^{-1} L)$$

To this we have to add that the Lagrange multiplier λ has to satisfy the following "budget equation"

$$E^Q[B_T^{-1} I(\lambda B_T^{-1} L)] = v \quad \Leftrightarrow \quad v = E\{L B_T^{-1} I(\lambda B_T^{-1} L)\} := V(\lambda)$$

This implies that, whenever $V(\cdot)$ is invertible, $\lambda = V^{-1}(v)$ and, consequently, we obtain

$$V^* = I(V^{-1}(v) B_T^{-1} L)$$

To illustrate the feasibility of this procedure we mention the case of a log-utility function, i.e. $u(V) = \log V$, for which the inverse of the derivative is $I(y) = \frac{1}{y}$. The "Budget Equation" then becomes

$$v = E\{L B_T^{-1} I(\lambda B_T^{-1} L)\} = E\left\{L B_T^{-1} \frac{B_T}{\lambda L}\right\} = \frac{1}{\lambda} = V(\lambda)$$

implying that $\lambda = \frac{1}{v}$ and so the optimal wealth becomes

$$V^* = I(\lambda B_T^{-1} L) = v L^{-1} B_T \tag{20}$$

b) *Case of an incomplete market*

We consider here the case when, as in the trinomial model, the set of all martingale measures is bounded and has a finite number J of vertices/extremal points Q^j. Let $L^j := \frac{dQ^j}{dP}$, $j = 1, \cdots, J$ and notice that

$$E^Q\{B_T^{-1} V\} = v \;\; \forall \;\; Q \quad \Leftrightarrow \quad E^{Q^j}\{B_T^{-1} V\} = v \;\; \forall \;\; j =, \cdots, J$$

Having now J constraints we shall consider a vector $\lambda = (\lambda_1, \cdots, \lambda_J)$ of Lagrange multipliers and the problem of the second step becomes then

$$\max_V \; E\left\{u(V) - \sum_{j=1}^{J} \lambda_j B_T^{-1} L^j V\right\}$$

As in the case of the complete market, we impose the necessary condition for V to maximize, for each scenario, the argument of the expectation namely

$$u'(V) = \sum_{j=1}^{J} \lambda_j B_T^{-1} L^j \;\text{ implying }\; V = I\left(\sum_{j=1}^{J} \lambda_j B_T^{-1} L^j\right)$$

This time the Lagrange multiplier vector λ has to satisfy the following system of "*budget equations*"

$$E\left\{L^j B_T^{-1} I\left(\sum_{j=1}^{J} \lambda_j B_T^{-1} L^j\right)\right\} = v\,; \quad j = 1, \cdots, J$$

Convex duality - only case of a complete market

We shall now additionally use also the convex dual to verify that the V^* determined above with the Lagrange multiplier technique is indeed optimal. For this purpose let us first recall the Legendre-Fenchel transform of a function $U(x)$ that leads to the so-called convex dual $\tilde{U}(y)$ of $U(x)$, which is indeed a convex function and is given by

$$\tilde{U}(y) := \max_x \{U(x) - xy\} = U(I(y)) - yI(y)$$

It follows that

$$U(I(y)) - yI(y) \geq U(x) - xy\ , \ \ \forall\, x \tag{21}$$

Recall now from the Lagrange multiplier approach that

$$\begin{cases} V^* = I(\lambda B_T^{-1} L) \ \text{ with } \lambda \text{ such that} \\ \\ E\left\{LB_T^{-1}V^*\right\} = E\{\underbrace{LB_T^{-1} I(\lambda B_T^{-1} L)}_{V(\lambda)}\} = v \end{cases} \tag{22}$$

Put now $x = V$, $y = \lambda B_T^{-1} L$ for some positive λ. Then, by (21) and (22),

$$U(V^*) - \lambda B_T^{-1} L V^* \geq U(V) - \lambda B_T^{-1} L V \tag{23}$$

Taking expectations in (23) and using (22), one has that, for all V satisfying the budget equation i.e. such that $E\{LB_T^{-1}V\} = v$,

$$E\{U(V^*)\} - \lambda v \geq E\{U(V)\} - \lambda v$$

namely

$$E\{U(V^*)\} \geq E\{U(V)\}$$

and this $\forall\, V$ satisfying the budget equation.

5 The case of logarithmic utility, explicit computations

In this section we shall describe more explicitly the individual steps required both for the method of Dynamic programming and the martingale method in the case of a log-utility function and for each of the three investment criteria. In particular we shall show how to perform the third step in the martingale approach, namely determining the optimal investment strategy. To simplify the presentation we shall assume without loss of generality that the price of the non-risky asset, which is used to discount the various other prices, is $B_t \equiv 1$; since we implicitly take $B_0 = 1$, this is equivalent to assuming that the short rate of interest is $r_t \equiv 0$. While the possibility of explicitly computing the various steps in the martingale method depends rather crucially on the choice of the utility function, the calculations by the method of

Dynamic programming (DP) are less dependent on the choice of the utility function. Still, in this section we shall make a bit more explicit also the steps required by DP as this is not immediately obvious for the second and third investment criteria. Numerical calculations for specific examples are then reported in the next section 6 for the maximization of utility from terminal wealth.

5.1 Maximizing expected utility from terminal wealth (binomial market model)

After briefly recalling the particular form that the Dynamic programming approach takes in this case (as mentioned above, for this investment criterion this is almost immediate), we shall describe the particular form that the martingale approach takes in this case and illustrate the computation of the optimal strategy.

5.1.1 Dynamic programming

Recall from (18) the Dynamic programming algorithm, namely

$$\begin{cases} U_T(v) = u(v) \quad \text{and, for } t < T, \\ U_t(v) = \max_{\pi_t} E\left\{U_{t+1}\left(G_t(V_t, \pi_t, \xi_{t+1})\right) \mid V_t = v\right\} \end{cases} \tag{24}$$

where the dynamics $G_t(V_t, \pi_t, \xi_{t+1})$ were specified in (17), namely

$$G_t(V_t, \pi_t, \xi_{t+1}) = V_t \left[\pi_t^0(1 + r_t) + \sum_{i=1}^{K} \pi_t^i \xi_{t+1}^i\right]$$

The only particular aspect here is that $K = 1$ and, letting $\pi_t = \pi_t^1 = \frac{\alpha_{t+1}^1 S_t}{V_t}$, one has $\pi_t^0 = (1 - \pi_t)$. With prohibition of short selling, i.e. requiring $\alpha_t^1 > 0$, we obtain $\pi_t \in (0, 1)$. Since $r_t = 0$, we may then write

$$G_t(V_t, \pi_t, \xi_{t+1}) = V_t \left[1 + \pi_t\left(\xi_{t+1} - 1\right)\right]$$

5.1.2 Martingale method

Recalling from section 3.1 the process N_t, we have that $N_T \sim b(T, p)$ and it represents the r.v. that counts the total number of "up-movements" of the price process S_t. With $u(x) = \log x$ one has by (20) and (11) and recalling that we take $B_t \equiv 1$,

$$V^* = v\left(\frac{p}{q}\right)^{N_T}\left(\frac{1-p}{1-q}\right)^{T-N_T} \tag{25}$$

and the optimal value of expected utility from terminal wealth is then

$$\begin{aligned} E\{u(V^*)\} &= \log v - \log\left(\tfrac{q}{p}\right) E(N_T) - \log\left(\tfrac{1-q}{1-p}\right)\left(T - E\{N_T\}\right) \\ &= \log v - pT\log\left(\tfrac{q}{p}\right) - T(1-p)\log\left(\tfrac{1-q}{1-p}\right) \end{aligned} \tag{26}$$

Computation of the optimal strategy - Backwards recursion

In line with section 2.2 an investment/portfolio strategy is here given by the two-dimensional vector $\alpha_t = (\alpha_t^0, \alpha_t^1)$ where α_t^0 and α_t^1 denote the number of units of the non-risky and the risky assets respectively held in the portfolio in period t. Recall also that the process α_t is supposed to be predictable, i.e. α_t is taken to be $\mathcal{F}_{t-1}-$measurable. We now describe the backwards recursion, from period $T-1$ to 0, to determine the optimal investment strategy $(\alpha_t^*)_{t=0,\cdots,T-1}$.

Take period $T-1$; we have to decide for α_T^0, α_T^1 . Let $N_{T-1} = n < T$ (and therefore $S_{T-1} = S_0 u^n d^{T-n-1}$). Recalling that we had assumed $r = 0$, we have to impose that for all "states of nature", i.e. independently on whether prices go up or down,

$$\alpha_T^1 S_T + \alpha_T^0 = V_T = V^* \tag{27}$$

This implies that the following system of equations in α_T^1, α_T^0 has to be satisfied, namely

$$\begin{cases} \alpha_T^1 S_{T-1} u + \alpha_T^0 = v \left(\frac{p}{q}\right)^{n+1} \left(\frac{1-p}{1-q}\right)^{T-n-1} & \text{(prices go up)} \\ \alpha_T^1 S_{T-1} d + \alpha_T^0 = v \left(\frac{p}{q}\right)^{n} \left(\frac{1-p}{1-q}\right)^{T-n} & \text{(prices go down)} \end{cases}$$

from which we obtain

$$\alpha_T^1 S_{T-1}(u-d) = v(1+r)^T \left(\frac{p}{q}\right)^n \left(\frac{1-p}{1-q}\right)^{T-n-1} \left[\frac{p}{q} - \frac{1-p}{1-q}\right]$$

and, consequently,

$$\begin{cases} \alpha_T^1 = \frac{v\left(\frac{p}{q}\right)^n \left(\frac{1-p}{1-q}\right)^{T-n-1}(p-q)}{S_{T-1}(u-d)q(1-q)} \\ \\ \alpha_T^0 = \frac{v\left(\frac{p}{q}\right)^n \left(\frac{1-p}{1-q}\right)^{T-n-1}[u(1-p)q-d(1-q)p]}{(u-d)q(1-q)} \end{cases}$$

Recalling the self-financing condition, the fact that $q = \frac{1+r-d}{u-d}$, and putting $C(p,q) := (p-q) + (uq-dp) + (dpq-upq)$, for the optimal value induced in period $T-1$ we thus obtain

$$\begin{aligned} V_{T-1}^* &= \alpha_{T-1}^1 S_{T-1} + \alpha_{T-1}^0 = \alpha_T^1 S_{T-1} + \alpha_T^0 \\ &= \frac{v\left(\frac{p}{q}\right)^n \left(\frac{1-p}{1-q}\right)^{T-n-1} C(p,q)}{(u-d)q(1-q)} = v \left(\frac{p}{q}\right)^n \left(\frac{1-p}{1-q}\right)^{T-n-1} \end{aligned}$$

Notice now that V_{T-1}^* has the same structure as V_T^* so that the calculations for the following period $T-2$ proceed in exactly the same way as for $T-1$ and so forth, which allows to straightforwardly complete the backwards recursion. In fact, in the generic period $t \leq T$, with $N_t = n \leq t$, the condition (27) becomes

$$\alpha_t^1 S_t + \alpha_t^0 = V_t^* \tag{28}$$

and one obtains

$$\begin{cases} \alpha_t^1 &= \frac{v\left(\frac{p}{q}\right)^n \left(\frac{1-p}{1-q}\right)^{t-n-1}(p-q)}{S_0 u^n d^{t-n-1}(u-d)q(1-q)} \\ \\ \alpha_t^0 &= \frac{v\left(\frac{p}{q}\right)^n \left(\frac{1-p}{1-q}\right)^{t-n-1}[u(1-p)q-d(1-q)p]}{(u-d)q(1-q)} \end{cases}$$

with the optimal wealth

$$V_t^* = v\left(\frac{p}{q}\right)^n \left(\frac{1-p}{1-q}\right)^{t-n}$$

Notice also that the ratio of the wealth invested in the risky asset is

$$\pi_t^1 = \frac{\alpha_{t+1}^1 S_t}{V_t^*} = \frac{(p-q)}{(u-d)q(1-q)}$$

and it is independent of t and S_t. It follows that the optimal strategy requires investing in the risky asset, in each period and in every state, the same fraction of wealth. Notice finally that the fact that the ratio π_t^1 does not depend neither on the period nor on the state of the underlying risky asset does not imply that the portfolio itself remains constant; in fact, since the price of the underlying is changing, to keep the ratio constant, one still has to continuously change the number of units invested in the underlying risky asset.

5.2 Maximizing expected utility from terminal wealth (trinomial market model)

Here we proceed analogously to the previous subsection for the binomial model noticing that nothing changes concerning the Dynamic programming approach. We shall thus consider here only the martingale method; in fact, while the Dynamic programming approach is independent on whether the market is complete or not (unique or more martingale measures), this is not the case with the martingale method.

5.2.1 Martingale method

Consider the situation described in section 3.2, in particular the subsubsection concerning martingale measures in the trinomial market model, where we had taken $T = 2$ and for which one has four extremal MM's, namely $\bar{Q}^1$, $\bar{Q}^2$, $\bar{Q}^3$, $\bar{Q}^4$. Recall also from section 4.2, in particular point b) in the subsubsection concerning the optimal reachable portfolio value that, for the log-utility function and $B_t \equiv 1$, the optimal terminal wealth is

$$V^* = \frac{1}{\lambda_1 L^1 + \lambda_2 L^2 + \lambda_3 L^3 + \lambda_4 L^4} \tag{29}$$

with $\lambda_1, \cdots, \lambda_4$ determined according to the system of equations ($j = 1, \cdots, 4$)

$$v = E\left\{\frac{L^j}{\lambda_1 L^1 + \cdots + \lambda_4 L^4}\right\} = E\left\{\frac{\bar{Q}^j}{\lambda_1 \bar{Q}^1 + \cdots + \lambda_4 \bar{Q}^4}\right\} \tag{30}$$

For the explicit computation of the values of the λ_i's we refer to the numerical calculations in the next section 6.2.

There remains to compute the optimal strategy. For this purpose notice that the market is incomplete and so not all future amounts can be replicated by a self-financing strategy. However, given the constraint $E^{Q^j}\{B_T^{-1}V^*\} = v$ for $j = 1, \cdots, J$, which implies that $E^Q\{B_T^{-1}V^*\} = v$ for all MM Q, one has that V^* is indeed replicable with a self-financing portfolio (consisting of the risky and non-risky assets) starting from an initial wealth of $V_0 = v$. To determine e.g. α_2^1, α_2^0 (in the case of $T = 2$) it thus suffices to impose that

$$\alpha_2^1 S_2 + \alpha_2^0 = V^*$$

holds in only two of the three states of nature; the condition is then automatically satisfied in the third one. At this point we are back to the same situation as for the binomial model and the calculations that lead to the optimal strategy are thus the same as they were there (an example showing the calculations is given in the next section 6.2).

We close this subsection 5.2 by mentioning the following alternative way to compute the optimal investment strategy in the trinomial model and, in general, in an incomplete market model. It consists in completing the market by adding fictitious assets and imposing on the optimal strategy that the investment in these assets is null.

5.3 Maximizing expected utility from consumption

As mentioned in the previous subsection, the method of Dynamic programming does not depend on whether the market is complete or not, only the martingale method does. For the present case of expected utility from consumption we therefore do not anymore consider two separate subsection, one for the complete binomial market model and one for the incomplete trinomial model as we did for expected utility from terminal wealth and where it was motivated by the desire to better illustrate the possible differences. Here we simply mention the differences that arise when we discuss the martingale method itself.

5.3.1 Dynamic programming

In the present context, considering again K underlying risky assets with prices

$$S_{t+1}^i = S_t^i \cdot \xi_{t+1}^i \quad (i = 1, \cdots, K)$$

we have, by the self-financing condition for the case with consumption as described in section 2.2 and in complete analogy with (16) and (17),

$$\begin{aligned} V_{t+1} &= V_t + \alpha^0_{t+1}\Delta B_t + \sum_{i=1}^K \alpha^i_{t+1}\Delta S^i_t - C_t \\ &= V_t + V_t\left[\pi^0_t r_t + \sum_{i=1}^K \pi^i_t\left(\xi^i_{t+1} - 1\right)\right] - C_t \\ &= V_t\left[(1+r_t) + \sum_{i=1}^K \pi^i_t\left(\xi^i_{t+1} - (1+r_t)\right)\right] - C_t(1+r_t) \end{aligned}$$

i.e. we have

$$V_{t+1} = G_t(V_t, \pi_t, \xi_{t+1})$$

with

$$G_t(V_t, \pi_t, \xi_{t+1}) = V_t\left[(1+r_t) + \sum_{i=1}^K \pi^i_t\left(\xi^i_{t+1} - (1+r_t)\right)\right] - C_t(1+r_t)$$

and with the control sequence/strategy

$$\pi_t = (\pi^1_t, \cdots, \pi^K_t, C_t)$$

where, as before, $\pi^i_t = \frac{\alpha^i_{t+1} S^i_t}{V_t}$.

With respect to the general description of the method of Dynamic programming in section 4.1 in this case we have

$$u(V_t, \pi_t) = u(C_t)\,, \quad t = 0, \cdots, T$$

with the constraint $C_T \leq V_T$.

To implement the Dynamic programming principle, corresponding to (14) here we put

$$U_t(v) := \max_{\substack{C_t,\cdots,C_T \\ (C_T \leq V_T)}} E\left\{\sum_{s=t}^T \beta^{s-t} u(C_s) \mid V_t = v\right\}$$

where the condition $C_T \leq V_T$ can be accounted for by setting

$$u(C_T) = -\infty \quad \text{for} \quad C_T > V_T$$

The *DP algorithm* (see (15) now becomes

$$\begin{cases} U_T(v) &= \begin{cases} u(v) & \text{if} \quad C_T = v \\ -\infty & \text{if} \quad C_T > v \end{cases} \\ \\ U_t(v) &= \max\limits_{\pi_t}\left[u(C_t) + \beta\, E\left\{U_{t+1}\left(G_t(V_t, \pi_t, \xi_{t+1})\right) \mid V_t = v\right\}\right] \end{cases}$$

where nonnegativity of C is guaranteed by $u(C) = -\infty$ for $C < 0$.

5.3.2 Martingale method

Recall that for the maximization of utility from terminal wealth we required :

- determining the set of reachable/attainable values of wealth in T;
- determining the optimal such wealth;
- determining a self-financing strategy that replicates the optimal wealth.

Here the terminal wealth is replaced by the adapted consumption process C_t and so we change the above requirements into the following steps :

- determine the set of "attainable" consumption processes;
- determine the optimal attainable consumption process;
- determine an investment strategy allowing to consume according to the optimal consumption process.

To implement these steps we start from the following lemma.

Lemma 1. *Given an initial wealth $v \geq 0$, a consumption process C_t, and a self-financing strategy α, one has*

$$\frac{V_t}{B_t} = v + \tilde{G}_t - \sum_{s=0}^{t-1} \frac{C_s}{B_s}\,, \quad t = 1, \cdots, T \tag{31}$$

where $\tilde{G}_t$ is the discounted gains process

$$\tilde{G}_t = \sum_{s=0}^{t-1} \sum_{i=1}^{K} \alpha_{s+1}^i \Delta \tilde{S}_s^i$$

Proof : By the definition (2) of the portfolio value and the self financing property (7) when there is consumption one has, respectively,

$$\alpha_t^0 = \tfrac{V_t}{B_t} - \textstyle\sum_{i=1}^K \alpha_t^i \tfrac{S_t^i}{B_t} \quad \text{and}$$

$$\alpha_t^0 = \tfrac{V_{t-1}}{B_{t-1}} - \tfrac{C_{t-1}}{B_{t-1}} - \textstyle\sum_{i=1}^K \alpha_t^i \tfrac{S_{t-1}^i}{B_{t-1}}$$

so that, since $V_0 = v$, $B_0 = 1$, relation (31) follows immediately for $t = 1$ and, for $t > 1$, by induction. $\diamond$

First step :

We start from the following

Definition 6. A consumption process C_t is said to be *attainable* if $\exists \alpha$ with (C, α) admissible (see Definition 5) s.t. $C_T = V_T$. *(α "replicates" or "generates" C)*

For any MM Q, we now have that the process $v + \tilde{G}_t$ is a $Q-$martingale with value v at $t = 0$. Putting $V_T = C_T$, from (31) one immediately has:

Proposition 2. *Given v, a consumption process C is attainable if*

a) *(complete market; unique MM Q)*

$$v = E^Q \left\{ \frac{C_0}{B_0} + \cdots , + \frac{C_T}{B_T} \right\}$$

b) *(incomplete market; extremal MM's Q^i for $j = 1, \cdots , J$)*

$$v = E^{Q^j} \left\{ \frac{C_0}{B_0} + \cdots , + \frac{C_T}{B_T} \right\} \; ; \quad j = 1, \cdots , J$$

Second step :

We shall discuss this step directly for the more general case of an incomplete market with a finite number J of extremal martingale measures. In the present case of maximizing expected utility from consumption, this second step consists in obtaining the optimal attainable consumption process by solving

$$\begin{cases} \max\limits_C \; E \left\{ \sum\limits_{t=0}^{T} \beta^t u(C_t) \right\} \\ \\ \text{subject to } \; E^{Q^j} \left\{ \sum_{t=0}^{T} \frac{C_t}{B_t} \right\} = v \; ; \; j = 1, \cdots , J \end{cases} \tag{32}$$

Notice that the only decision variable here is C_t $(t = 0, \cdots , T)$; α_t does not appear. Notice also that nonnegativity of C_t is guaranteed by $u(C) = -\infty$ for $C < 0$.

In order to solve problem (32) define $N_t^j := B_t^{-1} E\{L^j | \, \mathcal{F}_t\}$ $\left(L^j = \frac{dQ^j}{dP} \right)$ so that

$$E^{Q^j} \left\{ \sum_{t=0}^{T} \frac{C_t}{B_t} \right\} = E \left\{ L^j \sum_{t=0}^{T} \frac{C_t}{B_t} \right\}$$

$$= E \left\{ \sum_{t=0}^{T} E \left\{ B_t^{-1} C_t L^j | \, \mathcal{F}_t \right\} \right\} = E \left\{ \sum_{t=0}^{T} C_t N_t^j \right\}$$

then (32) becomes

$$\begin{cases} \max\limits_C \; E \left\{ \sum\limits_{t=0}^{T} \beta^t u(C_t) \right\} \\ \\ \text{subject to } E \left\{ \sum_{t=0}^{T} C_t N_t^j \right\} = v \; ; \; j = 1, \cdots , J \end{cases} \tag{33}$$

Using, as before, the Lagrange multiplier technique to solve (33), we have to compute

$$\max_{C} E\left\{\sum_{t=0}^{T}\beta^t u(C_t) - \sum_{j=1}^{J}\lambda_j \sum_{t=0}^{T} C_t N_t^j\right\}$$

A necessary condition for C_t to maximize, for each scenario, the argument of the expectation is

$$\beta^t u'(C_t) = \sum_{j=1}^{J}\lambda_j N_t^j \,; \quad t = 0, \cdots, T$$

which implies

$$C_t = I\left(\frac{\sum_{j=1}^{J}\lambda_j N_t^j}{\beta^t}\right) \;; \quad t = 0, \cdots, T$$

with λ_j satisfying the system of *"budget equations"*

$$E\left\{\sum_{t=0}^{T} N_t^j \, I\left(\frac{\sum_{j=1}^{J}\lambda_j N_t^j}{\beta^t}\right)\right\} = v \,; \quad j = 1, \cdots, J$$

and the optimal value becomes

$$J(v) = E\left\{\sum_{t=0}^{T}\beta^t u\left(I\left(\frac{\sum_{j=1}^{J}\lambda_j N_t^j}{\beta^t}\right)\right)\right\}$$

Third step :

In complete analogy with the previous case of expected utility from terminal wealth, it consists here in determining the investment strategy of a self-financing portfolio having terminal value $V_T = C_T$ and the procedure itself can be carried over directly from there.

5.4 Maximizing expected utility from consumption and terminal wealth

In the previous case of expected utility only from consumption, all of V_T was consumed at the terminal time $t = T$. Here we generalize this problem by leaving $V_T - C_T$ for future investment and maximizing expected utility from consumption of C_t for $t = 0, \cdots, T$, as well as from the terminal wealth $V_T - C_T$.

Letting $u_c(\cdot)$ and $u_p(\cdot)$ denote the utility functions for consumption and terminal wealth respectively, in accordance with section 2.3 our problem here is the following

Problem : (*max utility from consumption and terminal wealth*)

$$J(v) = \max_{(C,\alpha)\in\mathcal{A}_v} E\left\{\sum_{t=0}^{T}\beta^t u_c(C_t) + \beta^T u_p(V_T - C_T)\right\}$$

$$\text{with } \mathcal{A}_v : \begin{cases} \alpha & : \text{ self-financing and predictable} \\ C & : \text{ nonnegative, adapted}, C_T \le V_T \end{cases}$$

We can now easily adapt all previous solution methods.

5.4.1 Dynamic programming

As in the previous pure consumption-investment problem here too we have

$$V_{t+1} = G_t(V_t, \pi_t, \xi_{t+1})$$

with

$$G_t(V_t, \pi_t, \xi_{t+1}) = V_t \left[(1 + r_t) + \sum_{i=1}^{K} \pi_t^i \left(\xi_{t+1}^i - (1 + r_t) \right) \right] - C_t(1 + r_t)$$

and with the *control sequence/strategy*

$$\pi_t = (\pi_t^1, \cdots, \pi_t^K, C_t)$$

where, again as before, $\pi_t^i = \frac{\alpha_{t+1}^i S_t^i}{V_t}$. In this case, however, in addition to a *"running utility"* up to the terminal time, i.e.

$$u(V_t, \pi_t) = u_c(C_t), \quad t = 0, \cdots, T$$

we also have a *"terminal utility"*

$$u(V_T, \pi_T) = u_p(V_T - C_T)$$

The *DP algorithm* then becomes

$$\begin{cases} U_T(v) &= \max_{0 \le C \le v} \left[u_c(C) + u_p(v - C) \right] \\ U_t(v) &= \max_{\pi_t} \left[u_c(C_t) + \beta\, E\left\{ U_{t+1}\left(G_t(V_t, \pi_t, \xi_{t+1}) \right) \mid V_t = v \right\} \right] \end{cases}$$

and notice that the only difference with the simple consumption-investment problem is the form of $U_T(v)$ (in the previous case it is essentially given by $U_T(v) = u(v) = u_c(v)$).

5.4.2 Martingale method

The only difference with the previous case is that now we shall not require $C_T = V_T$ (see Definition 6 of attainability).

First step :

Instead of the notion of attainability for the previous pure consumption-investment problem given in Definition 6, in the present case we shall require for attainability that

$$v = E^{Q^j}\left\{\frac{C_0}{B_0} + \cdots + \frac{C_{T-1}}{B_{T-1}} + \frac{V_T}{B_T}\right\}, \quad j = 1, \cdots, J$$

(again we consider directly the more general case of an incomplete market with the finite number J of extremal martingale measures).

Second step :

Corresponding to (33) we now have

$$\begin{cases} \max\limits_{C,V_T} E\left\{\sum_{t=0}^{T}\beta^t u_c(C_t) + \beta^T u_p(V_T - C_T)\right\} \\ \text{subject to } E\left\{\sum_{t=0}^{T-1} C_t N_t^j + V_T N_T^j\right\} = v\,;\ j = 1, \cdots, J \\ \text{with } C \text{ adapted and } V_T \in \mathcal{F}_T \end{cases} \tag{34}$$

Notice that the fact that $u_c(C) = -\infty$ for $C < 0$, and the same for $u_p(\cdot)$, implies that the solution of (34) automatically satisfies $C_t > 0$ and $C_T < V_T$.

Using, as before, the Lagrange multiplier technique to solve (34), we have to compute

$$\max\ E\left\{\sum_{t=0}^{T}\beta^t u_c(C_t) + \beta^T u_p(V_T - C_T) - \sum_{j=1}^{J}\lambda_j\left[\sum_{t=0}^{T-1} C_t N_t^j + V_T N_T^j\right]\right\}$$

A necessary condition for C_t and V_T to maximize, for each scenario, the argument of the expectation is (differentiate w.r.to each C_t and V_T)

$$\begin{cases} \beta^t u_c'(C_t) = \sum_{j=1}^{J}\lambda_j N_t^j\,, \quad t = 0, \cdots, T-1 \\ \beta^T u_c'(C_T) = \beta^T u_p'(V_T - C_T) \\ \beta^T u_p'(V_T - C_T) = \sum_{j=1}^{J}\lambda_j N_T^j \end{cases}$$

Denoting by $I_c(\cdot)$ and $I_p(\cdot)$ the inverses of $u_c'(\cdot)$ and $u_p'(\cdot)$ respectively, one has

$$\begin{cases} C_t = I_c\left(\frac{\sum_{j=1}^{J}\lambda_j N_t^j}{\beta^t}\right)\ ; \quad t = 0, \cdots, T \\ V_T = I_p\left(\frac{\sum_{j=1}^{J}\lambda_j N_T^j}{\beta^T}\right) + I_c\left(\frac{\sum_{j=1}^{J}\lambda_j N_T^j}{\beta^T}\right) \end{cases}$$

with λ_j satisfying the system of *"budget equations"*

$$E\left\{\sum_{t=0}^{T} N_t^j\, I_c\left(\frac{\sum_{j=1}^{J}\lambda_j N_t^j}{\beta^t}\right) + N_T^j\, I_p\left(\frac{\sum_{j=1}^{J}\lambda_j N_T^j}{\beta^T}\right)\right\} = v\,; \quad j = 1, \cdots, J$$

and the optimal value becomes

$$J(v) = E\left\{\sum_{t=0}^{T} \beta^t u_c \left(I_c \left(\frac{\sum_{j=1}^{J} \lambda_j N_t^j}{\beta^t} \right) \right) + \beta^T u_p \left(I_p \left(\frac{\sum_{j=1}^{J} \lambda_j N_T^j}{\beta^T} \right) \right) \right\}$$

Third step :

In complete analogy to the previous cases, it consists in determining the investment strategy of a self-financing portfolio having terminal value V_T.

6 Numerical examples (logarithmic utility)

To illustrate the computations described in the previous section for the case of maximization of expected log-utility from terminal wealth (subsections 5.1 and 5.2), we present here numerical examples for the binomial and trinomial market models (complete and incomplete markets) using both the method of dynamic programming as well as the martingale method.

6.1 Binomial market model

We consider the following model : let the horizon be $T = 2$ and $S_0 = 1$. Furthermore, let $r = 0$ (equivalent to taking $B_t \equiv 1$), $u = 2$, $d = \frac{1}{2}$ which implies $q = \frac{1}{3}$ and let $p = \frac{4}{9}$.

$$\underline{\text{Objective}} \quad \begin{cases} \max E\{\log V_2\} \\ \\ E^Q\{V_2\} = v, \quad Q \ : \text{MM} \end{cases}$$

i.e. maximize the expected log-utility of the terminal value of a self financing portfolio, starting from an initial capital of $V_0 = v$.

6.1.1 Dynamic programming

Recalling from section 5.1 the Dynamic programming algorithm (24) and prohibiting here too short selling (which amounts to $\pi_t \in (0, 1)$), for the present example we obtain

$$\begin{array}{llcl}
i) & U_2(v) & = & \log v \\
ii) & U_1(v) & = & \log v + \max_{\pi_1 \in (0,1)} \left[\frac{4}{9} \log(1+\pi_1) + \frac{5}{9} \log\left(1 - \frac{\pi_1}{2}\right) \right] \\
 & & = & \log v + \max_{\pi_1 \in (0,1)} G(\pi_1) = \log v + G(\pi_1^*) \\
iii) & U_0(v) & = & \log v + G(\pi_1^*) + \max_{\pi_0 \in (0,1)} G(\pi_0) \\
 & & = & \log v + G(\pi_1^*) + G(\pi_0^*)
\end{array}$$

thereby obtaining the maximizing values $\pi_1^* = \pi_0^1 = \frac{1}{3}$, i.e. (see also end of section 5.1) the optimal strategy requires investing in the risky asset in each period and in every state simply the same fraction $\frac{1}{3}$ of the wealth. Furthermore,

$$G(\pi_1^*) = \frac{4}{9} \log \frac{4}{3} + \frac{5}{9} \log \frac{5}{6} = \frac{1}{3} \log 2 - \log 3 + \frac{5}{9} \log 5$$

and the optimal value of expected utility is

$$\log v + G(\pi_0^*) + G(\pi_1^*) = \log v + \frac{2}{3} \log 2 - 2 \log 3 + \frac{10}{9} \log 5$$

6.1.2 Martingale method

Approach according to section 5.1.

From (25) we have that the optimal terminal wealth is

$$V^* = v \left(\frac{4}{3}\right)^{N_2} \left(\frac{5}{6}\right)^{2-N_2}$$

$$\text{namely} \quad \begin{cases} \frac{16}{9} v & \text{if} \quad N_2 = 2 \\ \frac{10}{9} v & \text{if} \quad N_2 = 1 \\ \frac{25}{36} v & \text{if} \quad N_2 = 0 \end{cases} \tag{35}$$

and, according to (26), the optimal value of expected terminal utility is

$$E\{u(V^*)\} = \log v - 2p \log\left(\frac{q}{p}\right) - 2(1-p) \log\left(\frac{1-q}{1-p}\right)$$

$$= \log v - \frac{8}{9} \log\left(\frac{3}{4}\right) - \frac{10}{9} \log\left(\frac{6}{5}\right)$$

$$= \log v - \frac{8}{9}[\log 3 - 2\log 2] - \frac{10}{9}[\log 2 + \log 3 - \log 5]$$

$$= \log v - 2\log 3 + \frac{2}{3} \log 2 + \frac{10}{9} \log 5$$

which coincides with that obtained by the DP approach.

Alternative elementary approach

It may be instructive to consider also the following direct elementary approach (taken from [3]).

Let a, b, c, d be the four possible values of V_2. The problem then becomes (recall that the objective function is an expectation under the real-world probability measure, while the constraint is an expectation under the martingale measure)

$$\begin{cases} \max\left\{\frac{16}{81}\log a + \frac{20}{81}\log b + \frac{20}{81}\log c + \frac{25}{81}\log d\right\} \\ \frac{1}{9}a + \frac{2}{9}b + \frac{2}{9}c + \frac{4}{9}d = v \\ a, b, c, d \geq 0 \end{cases}$$

Using the technique of Lagrange multipliers to deal with the constraint, we have to maximize

$$\left(\frac{16}{81}\log a + \cdots + \frac{25}{81}\log c\right) - \lambda\left(\frac{1}{9}a + \cdots, \frac{4}{9}d\right)$$

and the necessary conditions lead to

$$\begin{cases} \frac{16}{81}\frac{1}{a} - \frac{\lambda}{9} = 0 \\ \frac{20}{81}\frac{1}{b} - \frac{2\lambda}{9} = 0 \\ \frac{20}{81}\frac{1}{c} - \frac{2\lambda}{9} = 0 \\ \frac{25}{81}\frac{1}{d} - \frac{4\lambda}{9} = 0 \end{cases} \Rightarrow \quad \begin{matrix} a = \frac{16}{9}\lambda \\ b = \frac{10}{9}\lambda \\ c = \frac{10}{9}\lambda \\ d = \frac{25}{36}\lambda \end{matrix}$$

The "budget equation" reads as

$$\left(\frac{16}{81} + \frac{20}{81} + \frac{20}{81} + \frac{25}{81}\right)\lambda = v$$

and implies

$$\lambda = v$$

The optimal expected value of terminal wealth is then

$$E\{\log V_2^*\} = \log v + \left(\frac{16}{81}\log\frac{16}{9} + 2\frac{20}{81}\log\frac{10}{9} + \frac{25}{81}\log\frac{25}{36}\right)$$

$$= \log v + \frac{16}{81}[4\log 2 - 2\log 3] + \frac{40}{81}[\log 2 + \log 5 - 2\log 3] + \frac{25}{81}[2\log 5 - 2\log 2 - 2\log 3]$$

$$= \log v + \frac{54}{81}\log 2 - \frac{162}{81}\log 3 + \frac{90}{81}\log 5$$

$$= \log v + \frac{2}{3}\log 2 - 2\log 3 + \frac{10}{9}\log 5$$

Optimal investment strategy

From section 5.1 we already know that the optimal investment strategy requires investing in the risky asset in each period and for every value of the underlying the constant fraction of wealth

$$\pi_t^1 = \frac{\alpha_{t+1}^1 S_t}{V_t^*} = \frac{(p-q)}{(u-d)q(1-q)}$$

i.e. $\pi_t^* = \frac{1}{3}$ in line with the results of the Dynamic programming approach.

Optimal strategy : independent direct approach

In view of the approach that we shall use in the next subsection concerning the trinomial model, we present here an independent direct approach to determine the optimal investment strategy. This will also allow for better insight and for possible comparison with the method of Dynamic programming.

Let (α_t^1, α_t^0) be as before and denote by (ϕ_t, ψ_t) the amounts invested in period t in the risky and non-risky assets respectively, i.e. $\phi_t = \alpha_t^1 S_{t-1}$, $\psi_t = \alpha_t^0$. Notice that, while the optimal investment ratio does not depend on the value of the underlying, the number of units and the amounts that are invested do and so below we make explicit this dependence. We now perform a backwards recursion for each period and for each possible value of the underlying risky asset. Recalling that we had supposed $S_0 = 1$ and $u = 2$, we start from

i) $t = 1$, $S_1 = 2$

Taking into account the requirement imposed on the strategy in the next-to-last period and expressed in (27), as well as the optimal portfolio values in the last period as expressed in (35), we obtain the system of equations (in (α_t^1, α_t^0) and (ϕ_t, ψ_t) respectively)

$$\begin{cases} 4\alpha_2^1(2) + \alpha_2^0(2) = \frac{16}{9}v \\ \\ \alpha_2^1(2) + \alpha_2^0(2) = \frac{10}{9}v \end{cases} \Leftrightarrow \begin{cases} 2\phi_2(2) + \psi_2(2) = \frac{16}{9}v \\ \\ \frac{1}{2}\phi_2(2) + \psi_2(2) = \frac{10}{9}v \end{cases}$$

having as solutions

$$\begin{cases} \alpha_2^1(2) = \frac{2}{9}v \\ \\ \alpha_2^0(2) = \frac{8}{9}v \end{cases} \qquad \begin{cases} \phi_2(2) = \frac{4}{9}v \\ \\ \psi_2(2) = \frac{8}{9}v \end{cases}$$

and implying that the optimal portfolio value in period $t = 1$ and with a price of the underlying $S_1 = 2$ is

$$V_1^*(2) = 2\alpha_2^1(2) + \alpha_2^0(2) = \phi_2(2) + \psi_2(2) = \frac{12}{9}v$$

Denoting again by π_t the fraction invested in the risky asset in period t, one has for its optimal value in $t = 2$

$$\pi_1^*(2) = \frac{\alpha_2^1(2)S_1}{V_1(2)} = \frac{\phi_2(2)}{V_1(2)} = \frac{1}{3}$$

independently of the choice of t and S and this coincides with the value that we had already determined previously.

Next, recalling that we had assumed $d = \frac{1}{2}$, we perform the computations for

ii) $t = 1,\ S_1 = \frac{1}{2}$

Proceeding analogously to the previous point i), we obtain the systems

$$\begin{cases} \alpha_2^1(\frac{1}{2}) + \alpha_2^0(\frac{1}{2}) = \frac{10}{9}v \\ \\ \frac{1}{4}\alpha_2^1(\frac{1}{2}) + \alpha_2^0(\frac{1}{2}) = \frac{25}{36}v \end{cases} \Leftrightarrow \begin{cases} 2\phi_2(\frac{1}{2}) + \psi_2(\frac{1}{2}) = \frac{10}{9}v \\ \\ \frac{1}{2}\phi_2(\frac{1}{2}) + \psi_2(\frac{1}{2}) = \frac{25}{36}v \end{cases}$$

Notice now that the system in (ϕ, ψ) has the same left hand side as before in i), while the right hand side is multiplied by $\frac{5}{8}$. It follows that

$$(\phi_2(\frac{1}{2}), \psi_2(\frac{1}{2})) = \frac{5}{8}(\phi_2(2), \psi_2(2)) = \left(\frac{5}{18}v, \frac{5}{9}v\right)$$

implying that the optimal portfolio value in period $t = 1$ and with a price of the underlying $S_1 = \frac{1}{2}$ is

$$V_1^*(\frac{1}{2}) = \phi_2(\frac{1}{2}) + \psi_2(\frac{1}{2}) = \frac{15}{18}v$$

Again, if we compute the fraction invested in the risky asset we obtain as before for its optimal value

$$\pi_1^*(\frac{1}{2}) = \frac{5}{18}v \cdot \frac{18}{15}\frac{1}{v} = \frac{1}{3}$$

Finally, we come to

iii) $t = 0,\ S_0 = 1$

Taking into account the requirement imposed on the strategy in the generic period t and expressed in (28), as well as the optimal portfolio values computed for $t = 1$ in the previous two cases i) and ii), we obtain the system of equations

$$\begin{cases} 2\alpha_1^1(1) + \alpha_1^0(1) = \frac{12}{9}v \\ \\ \frac{1}{2}\alpha_1^1(1) + \alpha_1^0(1) = \frac{15}{18}v \end{cases} \Leftrightarrow \begin{cases} 2\phi_1(1) + \psi_1(1) = \frac{12}{9}v \\ \\ \frac{1}{2}\phi_1(1) + \psi_1(1) = \frac{15}{18}v \end{cases}$$

and notice, again, that the system in (ϕ, ψ) has the left hand side the same as before in ii), while the right hand side is multiplied by $\frac{6}{5}$. It follows that

$$(\phi_1(1), \psi_1(1)) = \frac{6}{5}(\phi_2(\frac{1}{2}), \psi_2(\frac{1}{2})) = \left(\frac{1}{3}v, \frac{2}{3}v\right)$$

and, as expected, the optimal portfolio value in period $t = 0$ results in

$$V_0^* = v$$

while the optimal fraction invested in the risky asset is

$$\pi_0^*(1) = \frac{1}{3}$$

To conclude this direct approach notice that only one system of linear equations had actually to be solved.

6.2 Trinomial market model

We consider the following model with data analogous to the binomial model : let the horizon be $T = 2$ and $S_0 = 1$. Furthermore,

$$r = 0,\ u = 2,\ m = 1,\ d = \tfrac{1}{2},$$
$$p_1 = p_2 = p_3 = \tfrac{1}{3}$$

and we have $\Omega = \{\omega^1, \cdots, \omega^9\}$ with

$$\begin{cases} \omega^1 = (u,u), & \omega^2 = (u,m), & \omega^3 = (u,d) \\ \omega^4 = (m,u), & \cdots, & \omega^6 = (m,d) \\ \omega^7 = (d,u), & \cdots, & \omega^9 = (d,d) \end{cases} \tag{36}$$

and $P\{\omega^i\} = \frac{1}{9}$, $i = 1, \cdots, 9$.

$$\underline{\text{Objective}} \quad \begin{cases} \max E\{\log V_2\} \\ \\ E^{Q^j}\{V_2\} = v, \quad Q^j \ : \text{extremal MM} \end{cases}$$

namely it consists, again, in maximizing the expected log-utility of the terminal value of a self financing portfolio strategy starting from an initial capital of $V_0 = v$.

6.2.1 Dynamic programming

The procedure is completely analogous to that for the Binomial model, except that here we have three possible states of nature. More precisely, we have

$$\begin{aligned}
i) \quad U_2(v) &= \log v \\
ii) \quad U_1(v) &= \log v + \max_{\pi_1}\left[\frac{1}{3}\log(1+\pi_1) + \frac{1}{3}\log 1 + \frac{1}{3}\log\left(1 - \frac{\pi_1}{2}\right)\right] \\
&= \log v + \tfrac{1}{3}\max_{\pi_1}\left[\log(1+\pi_1) + \log\left(1 - \frac{\pi_1}{2}\right)\right] \\
&= \log v + \tfrac{1}{3}\max_{\pi_1}\Gamma(\pi_1) = \log v + \frac{1}{3}\Gamma(\pi_1^*) \\
iii) \quad U_0(v) &= \log v + \tfrac{1}{3}\Gamma(\pi_1^*) \\
&+ \max_{\pi_0}\left[\frac{1}{3}\log(1+\pi_0) + \frac{1}{3}\log 1 + \frac{1}{3}\log\left(1 - \frac{\pi_0}{2}\right)\right] \\
&= \log v + \tfrac{1}{3}\Gamma(\pi_1^*) + \tfrac{1}{3}\Gamma(\pi_0^*)
\end{aligned}$$

thereby obtaining the maximizing values $\pi_1^* = \pi_0^* = \frac{1}{2}$ so that, again, the optimal strategy requires investing in the risky asset in each period and in every state the same fraction of the wealth, this time equal to $\frac{1}{2}$. Furthermore,

$$\Gamma(\pi_1^*) = \log\frac{3}{2} + \log\frac{3}{4}$$

and the optimal value of expected terminal utility is

$$\log v + \frac{1}{3}\left(\Gamma(\pi_0^*) + \Gamma(\pi_1^*)\right) = \log v - \frac{1}{3}[6\log 2 - 4\log 3]$$

6.2.2 Martingale method

According to section 3.2, we have four extremal martingale measures $\bar{Q}^j$ $(j = 1, \cdots, 4)$ and, having assumed $m \leq 1 + r = 1$, according to that same section we have (with ω^i as in (36))

$$\begin{array}{llll} \bar{Q}^1(\omega^i) = 0 & \text{for } i \neq 5 & \text{and } \bar{Q}^1(\omega^5) = 1 & \\ \bar{Q}^2(\omega^i) = 0 & \text{for } i \neq 2, 8 & \text{and } \bar{Q}^2(\omega^2) = \frac{1}{3}, & \bar{Q}^2(\omega^8) = \frac{2}{3} \\ \bar{Q}^3(\omega^i) = 0 & \text{for } i \neq 4, 6 & \text{and } \bar{Q}^3(\omega^4) = \frac{1}{3}, & \bar{Q}^3(\omega^6) = \frac{2}{3} \\ \bar{Q}^4(\omega^i) = 0 & \text{for } i \neq 1, 3, 7, 9 & \text{and} & \\ \bar{Q}^4(\omega^1) = \frac{1}{9}, & \bar{Q}^4(\omega^3) = \frac{2}{9}, & \bar{Q}^4(\omega^7) = \frac{2}{9}, & \bar{Q}^4(\omega^9) = \frac{4}{9} \end{array} \tag{37}$$

Computation of the optimal value (Steps 1 and 2)

From (29) we have that the optimal terminal wealth is

$$V^* = \frac{(1+r)^2}{\sum_{i=1}^4 \lambda_i L^i} = \frac{1}{\sum_{i=1}^4 \lambda_i L^i}$$

where (see (30)) the values of λ_i satisfy the system $(j = 1, \cdots, 4)$

$$v = E\left\{\frac{\bar{Q}^j}{\sum_{i=1}^4 \lambda_i \bar{Q}^i}\right\} = \sum_{k=1}^9 P\{\omega^k\} \frac{\bar{Q}^j(\omega^k)}{\sum_{i=1}^4 \lambda_i \bar{Q}^i(\omega^k)}$$

According to the values of $\bar{Q}^j(\omega^k)$ this becomes

$$\begin{cases} v = \frac{1}{9}\left[\frac{1}{\lambda_1}\right] \\ v = \frac{1}{9}\left[\frac{1}{\lambda_2} + \frac{1}{\lambda_2}\right] \\ v = \frac{1}{9}\left[\frac{1}{\lambda_3} + \frac{1}{\lambda_3}\right] \\ v = \frac{1}{9}\left[\frac{1}{\lambda_4} + \frac{1}{\lambda_4} + \frac{1}{\lambda_4} + \frac{1}{\lambda_4}\right] \end{cases}$$

implying that

$$\lambda_1 = \frac{1}{9v},\ \lambda_2 = \frac{2}{9v},\ \lambda_3 = \frac{2}{9v},\ \lambda_4 = \frac{4}{9v}.$$

It follows (ω^k is as in (36))

$$
\begin{aligned}
V^*(\omega^k) &= \frac{1}{\frac{1}{9v}\frac{\bar{Q}^1(\omega^k)}{P(\omega^k)} + \frac{2}{9v}\frac{\bar{Q}^2(\omega^k)}{P(\omega^k)} + \frac{2}{9v}\frac{\bar{Q}^3(\omega^k)}{P(\omega^k)} + \frac{4}{9v}\frac{\bar{Q}^4(\omega^k)}{P(\omega^k)}} \\
&= \frac{v}{\bar{Q}^1(\omega^k) + 2\bar{Q}^2(\omega^k) + 2\bar{Q}^3(\omega^k) + 4\bar{Q}^4(\omega^k)}
\end{aligned}
\tag{38}
$$

and the optimal value of expected utility of terminal wealth is therefore

$$
\begin{aligned}
E\{\log V^*\} &= \log v \\
&- \tfrac{1}{9}\textstyle\sum_{k=1}^{9} \log\left[\bar{Q}^1(\omega^k) + 2\bar{Q}^2(\omega^k) + 2\bar{Q}^3(\omega^k) + 4\bar{Q}^4(\omega^k)\right] \\
&= \log v - \tfrac{1}{9}\left[\log\tfrac{4}{9} + \log\tfrac{2}{3} + \log\tfrac{8}{9} + \log\tfrac{2}{3} + \log 1\right. \\
&+ \left.\log\tfrac{4}{3} + \log\tfrac{8}{9} + \log\tfrac{4}{3} + \log\tfrac{16}{9}\right] \\
&= \log v - \tfrac{1}{9}\left[18\log 2 - 12\log 3\right]
\end{aligned}
$$

Computation of the optimal strategy (Step 3)

Recalling that for our example we have taken the horizon to be $T = 2$, as in (27) we have to start imposing that for all "states of nature"

$$\alpha_T^1 S_T + \alpha_T^0 = V_T = V^*$$

with V^* the optimal terminal wealth. Since for the trinomial model we have three states of nature corresponding to ξ being equal to u, m or d, we would end up with three equations in the two unknowns (α_2^1, α_2^0). However, according to what was mentioned at the end of section 5.2, it suffices that

$$\alpha_2^1 S_2 + \alpha_2^0 = V^* \tag{39}$$

holds in two of the states of nature.

We choose as states of nature (for the driving random process ξ_t, $t = 0, 1$) the following $\omega^k = (\omega_0^k, \omega_1^k)$ with $\omega_1^k \in \{u, m\} = \{2, 1\}$ (i.e. choosing for period $t = 1$ only the two alternatives u, m) and we verify afterwards that the condition (39), in the form of (40) below, is then automatically satisfied also for $\omega_1^k = d = \frac{1}{2}$.

For the actual computations we follow a backwards recursion procedure as for the independent direct approach of the previous section 6.1. Instead of (α_t^1, α_t^0) we thus determine equivalently (ϕ_t, ψ_t), i.e. the amounts invested in the risky and non-risky assets respectively and for which condition (39) becomes

$$\phi_2(S_1)\omega_1^k + \psi_2(S_1) = V^*(\omega_1^k) \tag{40}$$

We start from

i) $t = 1,\ S_1 = 2$ (corresponding to $\omega_0 = u$)

Taking into account the optimal portfolio values in the last period as expressed in (38) with $\bar{Q}^j(\omega^k)$ as in (37), we obtain the system of equations

$$\begin{cases} 2\phi_2(2) + \psi_2(2) = V(\omega^1) = \frac{v}{4/9} = \frac{9}{4}v \\ \\ \phi_2(2) + \psi_2(2) = V(\omega^2) = \frac{v}{2/3} = \frac{3}{2}v \end{cases}$$

having as solution

$$\phi_2(2) = \frac{3}{4}v\,, \quad \psi_2(2) = \frac{3}{4}v$$

and implying that the optimal portfolio value in period $t = 1$ and with $S_1 = 2$ is $V_1^*(2) = \frac{3}{2}v$. For the optimal fraction of wealth invested in the risky asset we obtain

$$\pi_1^*(2) = \frac{1}{2}$$

which is independent of the period t and the value S of the underlying and coincides with the value obtained by the method of Dynamic programming.

It remains to verify that the condition (40) is satisfied also for $\omega_1^k = d$ i.e. (see (36)) for ω^3. We have in fact

$$\frac{1}{2}\phi_2(2) + \psi_2(2) = \frac{9}{8}v = V(\omega^3) \qquad \text{o.k.}$$

Next we perform the computations for

ii) $t = 1,\ S_1 = 1$ (corresponding to $\omega_0 = m$)

Proceeding analogously to point i), this time we obtain the system

$$\begin{cases} 2\phi_2(1) + \psi_2(1) = V(\omega^4) = \frac{v}{2/3} = \frac{3}{2}v \\ \\ \phi_2(1) + \psi_2(1) = V(\omega^5) = v \end{cases} \tag{41}$$

Notice now that, as was the case with the independent direct approach in the Binomial model, the left hand side of the system in (ϕ, ψ) is as before in i), while the right hand side is multiplied by $\frac{2}{3}$. It then follows that

$$(\phi_2(1), \psi_2(1)) = \frac{2}{3}(\phi_2(2), \psi_2(2)) = (\frac{1}{2}v, \frac{1}{2}v)$$

implying that the optimal portfolio value in period $t = 1$, and with a price of the underlying $S_1 = 1$, is

$$V_1^*(1) = v$$

and that the optimal fraction of wealth invested in the risky asset in period $t = 1$ is

$$\pi_1^*(1) = \frac{1}{2}$$

again as was obtained by the method of Dynamic programming.

We verify here too that the condition (40) is satisfied also for $\omega_1^k = d$ i.e. (see (36)) for ω^6. We have in fact

$$\frac{1}{2}\phi_2(1) + \psi_2(1) = \frac{3}{4}v = V(\omega^6) \qquad \text{o.k.}$$

We go on to

iii) $t = 1$, $S_1 = \frac{1}{2}$ (corresponding to $\omega_0 = d$)

Noticing that

$$\big(V(\omega^7), V(\omega^8)\big) = \frac{3}{4}\big(V(\omega^4), V(\omega^5)\big)$$

we can refer to (41) to obtain

$$(\phi_2(\frac{1}{2}), \psi_2(\frac{1}{2})) = \frac{3}{4}(\phi_2(1), \psi_2(1)) = (\frac{3}{8}v, \frac{3}{8}v)$$

implying that the optimal portfolio value in period $t = 1$ and with a price of the underlying $S_1 = \frac{1}{2}$ is

$$V_1^*(\frac{1}{2}) = \frac{3}{4}v$$

and that the optimal fraction of wealth invested in the risky asset in the given situation is again

$$\pi_1^*(1) = \frac{1}{2}.$$

Verifying the condition (40) also for $\omega_1^k = d$ i.e. (see (36)) for ω^9 we obtain

$$\frac{1}{2}\phi_2(\frac{1}{2}) + \psi_2(\frac{1}{2}) = \frac{9}{16}v = V(\omega^9) \qquad \text{o.k.}$$

The last case is

iv) $t = 0$, $S_0 = 1$

Taking into account the requirement for the strategy (α_t^1, α_t^0) in the generic period t expressed in (28) that for (ϕ_t, ψ_t) and $t = 0$ becomes

$$\phi_1(S_0)\omega_0^k + \psi_1(S_0) = V^*(\omega_0^k) \tag{42}$$

as well as the optimal portfolio values for period $t = 1$ with $S_1 = 2, 1$ (cases i) and ii)) we obtain the system

$$\begin{cases} 2\phi_1(1) + \psi_1(1) = V_1(2) = \frac{3}{2}v \\ \\ \phi_1(1) + \psi_1(1) = V_1(1) = v \end{cases}$$

and notice that it is the same as for the case $t = 1, S_1 = 1$ in ii) so that the solution is, as there,

$$(\phi_1(1), \psi_1(1)) = (\frac{1}{2}v, \frac{1}{2}v)$$

and, again as expected, the optimal portfolio value in period $t = 0$ results in

$$V_0^* = v$$

while the optimal fraction invested in the risky asset is the constant value

$$\pi_0^*(1) = \frac{1}{2}$$

We conclude by verifying also here that the condition corresponding to (28) for period $t = 0$ is satisfied, namely

$$\frac{1}{2}\phi_1(1) + \psi_1(1) = \frac{3}{4}v = V_1(\frac{1}{2}) \qquad \text{o.k.}$$

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