Stochastic control and pricing under Swap measures

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Abstract

This paper relates to an approach described in [6] which, for the pricing of bonds and bond derivatives, is alternative to the classical approach that involves martingale measures and is based on the solution of a stochastic control problem, thereby avoiding a change of measure. It turns out that this approach can be extended to various situations where traditionally a change of measure is involved via a change of numeraire. In the present paper we study this extension for the case of Swap measures that are relevant in the classical approach to the pricing of Swaps and Swaptions.

1 Introduction

In a recent paper [6], a new approach has been proposed for the derivation of bonds and bond derivatives prices in a diffusion-type multivariate factor model for the term structure of interest rates which, while yielding the same arbitrage free prices, is alternative to the classical derivation. It is based on the solution of a stochastic control problem and its key feature can be described as follows. In the classical approach a fundamental tool are martingale measures that can be obtained by a Girsanov-type measure transformation. The latter implies a change of drift in the dynamics of the factors which however preserves the trajectories. Now, the drift of a diffusion-type factor process can also be changed by a feedback control as it is done in stochastic control. With the latter approach the trajectories are changed, but the measure remains unchanged.

An immediate implication of this key feature is the novel insight that it becomes equivalent to compute prices either on the basis of a traditional measure change or by solving an optimal stochastic control problem. In fact, since the values that one ultimately observes are the prices, it is irrelevant whether these values are generated by considering the same trajectories of the factors under a different measure or by considering different trajectories (which one does not even observe) under the same measure. What is relevant is that in both ways one generates the same prices. The major novelty of our approach can thus be seen in the linking of stochastic optimal control theory with the classical martingale approach thereby providing an alternative representation of the prices of bonds and interest rate derivatives under a multifactor term structure. The use of system theoretic tools also allows for much simpler formulae for computing bond derivatives prices.

In [6] the approach via a stochastic control interpretation is worked out in detail for prices and forward prices of bonds and then generalized also to forward measures in view of the pricing of more general derivative products. This generalization to forward measures hints at the possibility to extend the approach also to different situations. One such possible extensions concerns Swap measures that are relevant in interest rate derivative pricing, in particular for Swaps and Swaptions. The major purpose of the present paper is now to work out this latter extension.

In order to describe the approach for Swap measures, it is unavoidable to summarize the main steps of the approach in [6]. This is done in the first five sections of the paper, where we also specify the model and the assumptions and recall some basic facts from arbitrage pricing and stochastic control. Furthermore, in addition to recalling in these five sections the approach in [6], in subsection 5.1 we also present a control interpretation of expectations under forward measures, which will then be useful also in the case of Swap measures. Finally, sections 6 and 7 contain the novel part with respect to [6], namely the stochastic control interpretation of Swap rates and of expectations under Swap measures respectively.

2 Arbitrage-free term structure

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ be a given filtered probability space. Consider the l-dimensional Markovian factor process $x(\cdot)$, evolving under $\mathbb{Q}$, according to the dynamics

$$dx(t) = \mathbf{f}(t, x(t))dt + \mathbf{g}(t, x(t))dw_t, \qquad t \in [0, T], \qquad x(0) = 0, \tag{1}$$

where $T > 0$, $\mathbf{f}$ is an l-dimensional vector function and $\mathbf{g}$ is a matrix function of dimensions $l \times k$, w is a k-dimensional $(\mathbb{Q}, \mathcal{F}_t)$ Wiener process.

For the bond prices we consider a notation of the form $p(t, T, x(t))$, where t is the time variable, T is the date of maturity and $x(t)$ is the value of the factor process $x(\cdot)$ at time t. Analogously, the forward rate corresponding to $p(t, T, x)$ will be denoted by $f(t, T, x) := -\frac{\partial}{\partial T} \ln p(t, T, x)$ and the short rate by $r(t, x) := f(t, t, x)$.

We shall make the following

Assumption 2.1 *There exists some constant $M > 0$ such that, uniformly in $t \in [0, T]$:*

- $||\mathbf{f}(t, x)|| \leq M(1 + ||x||), \qquad ||\mathbf{g}(t, x)|| \leq M$
- $|r(t, x)| \leq M(1 + ||x||^2)$

We have the well-known Term Structure Equation (see e.g. [1], [2])

Theorem 2.2 *In an arbitrage-free bond market, with the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$, the function $p(t, T, x)$ is the unique solution (see Remark 2.3 below) of the PDE*

$$\begin{cases} \frac{\partial}{\partial t} p(t, T, x) + \mathbf{f}'(t, x) \nabla_x p(t, T, x) + \frac{1}{2} \mathrm{tr}(\mathbf{g}'(t, x) \nabla_{xx} p(t, T, x) \mathbf{g}(t, x)) \\ \qquad -r(t, x) p(t, T, x) = 0 \\ p(T, T, x) = 1 \end{cases} \tag{2}$$

Remark 2.3 *It is possible to prove (see [3], Ch.6, section 4) that, under Assumption 2.1, the stochastic differential equation* (1) *has a unique strong solution and that the solution to* (2)*, if it exists, is unique within the class of functions satisfying the growth condition $|p(t, T, x)| \leq Ce^{C||x||^2}$ for all $t \leq T$ and all $x \in \mathbb{R}^l$, where C is a positive constant possibly depending on T.*

3 Stochastic control in interest rate derivative pricing

As mentioned in the Introduction, the traditional pricing techniques are based on measure changes, where the trajectories of the stochastic processes involved are preserved, but a modification in their drift term is implicit in the measure transformation.

Our approach, instead, makes this drift modification for the factor process explicit, while maintaining the original martingale measure $\mathbb{Q}$. Moreover, we obtain drift changes by introducing a control process and choosing a suitable objective function. We shall show that the prices arising from a suitable stochastic control formulation are the same as those calculated by the usual methods.

These control problems are obtained in the following three steps, illustrated here for the case of bond prices:

- apply a *logarithmic transform* to the bond price;
- use the available pricing equations to obtain a PDE for the transformed price;
- identify an HJB equation, and the corresponding stochastic control problem, associated to such a PDE.

As a first instance of our argument, in this section we investigate the connection between bond prices and stochastic optimal control, following the approach in [4]. Here we assume that the factor process evolves, under the standard martingale measure $\mathbb{Q}$, according to the general dynamics (1).

Now we put

$$W(t,T,x) := -\ln p(t,T,x) \tag{3}$$

Remembering (2), we obtain that the function $W(t,T,x)$ in (3) satisfies

$$\begin{cases} \frac{\partial}{\partial t}W(t,T,x) + \mathbf{f}'(t,x)\nabla_x W(t,T,x) \\ \qquad -\frac{1}{2}\nabla_x W'(t,T,x)\mathbf{g}\mathbf{g}'(t,x)\nabla_x W(t,T,x) \\ \qquad +\frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx}W(t,T,x)\mathbf{g}(t,x)\right) + r(t,x) = 0 \\ W(T,T,x) = 0 \end{cases} \tag{4}$$

Remark 3.1 *As usual, it is easy to check that the bond price $p(t,T,x)$ is a solution to* (2) *and, in view of Remark 2.3, it is unique. Notice then that* (4) *is the equation satisfied by a one-to-one transformation of $p(t,T,x)$, and thus it also has a unique solution.*

Consider next the following stochastic control problem:

$$\begin{cases} dx(t) = \left[\mathbf{f}(t,x(t)) + \mathbf{g}(t,x(t))u(t)\right]dt + \mathbf{g}(t,x(t))dw_t \\ W(t,T,x) = \inf_{u(\cdot)\in\mathcal{U}} \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\int_t^T \left(\frac{1}{2}u'(s)u(s) + r(s,x(s))\right)ds\right\} \end{cases} \tag{5}$$

where $\mathcal{U}$ denotes the class of admissible control laws, namely the control processes for which the first equation in (5) has a unique solution in probability law and the expected cost, namely $J(t,T,x,u(\cdot)) := \mathbb{E}^{\mathbb{Q}}_{t,x}\big\{\int_t^T \big(\frac{1}{2}u'(s)u(s) + r(s,x(s))\big)ds\big\}$, has finite value.

It is possible to prove (see [4]) the following

Proposition 3.2 *The bond price $p(t,T,x)$ can be expressed as*

$$p(t,T,x) = \exp\left[-W(t,T,x)\right],$$

where $W(t,T,x)$ is the optimal value function of the stochastic control problem (5).

In the field of stochastic control, we have the following (see [5])

Sufficient Condition for Admissibility: *Given a process $u(\cdot)$, suppose that there exist some constants M and K such that:*

- $||u(t,x)|| \leq M(1+||x||)$ *for all* $(t,x) \in [0,T] \times \mathbb{R}^l$*;*
- *for any bounded* $B \subset \mathbb{R}^l$ *and any* T_0 *in* $(0,T)$*,*

$$||u(t,x) - u(t,y)|| \leq K||x-y||$$

  for all $x, y \in B$ *and* $0 \leq t \leq T_0$*.*

Then $u(\cdot)$ is an admissible control law. Notice that K may depend on B and T_0, while both M and K may depend on $u(\cdot)$.

Thus, a possible choice in order to have admissibility for the optimal control law in (5) is to make the following

Assumption 3.3 *The gradient of $W(t,T,x)$, solution of (4), satisfies a linear growth condition, i.e.*

$$||\nabla_x W(t,T,x)|| \leq M(1+||x||) \quad \textit{for all } x \in \mathbb{R}^l,$$

for some constant $M > 0$, uniformly in $t \in [0,T]$.

Remark 3.4 *Notice that such a hypothesis is not void: it is satisfied, for example, in the case of linear dynamics as discussed in [6].*

Remark 3.5 *Assumption 3.3 will turn out to be sufficient in order to assure that the optimal control law of problem (5) is an admissible control law, in the sense of the Sufficient Condition for Admissibility. However, such a hypothesis is not strictly needed: it can be substituted by another one implying just admissibility for the optimal control law.*

4 Forward prices

Since now we have a complete control interpretation for the bond prices maturing at a given T, we consider the more complex problem of pricing derivatives on these bonds that have a maturity τ, with $t \leq \tau \leq T$. For this purpose, in this section we first consider computing the expected value at time t of the T-bond price at time τ. We refer to such a quantity as the *forward price* of the T-bond. More precisely, using the forward measure Q^τ, the one with $p(t,\tau,x(t))$ as numeraire, we want to calculate

$$\mathbb{E}_{t,x}^{\mathbb{Q}^\tau} \{p(\tau, T, x(\tau))\} \tag{6}$$

Our purpose in this section is to obtain a control description for the forward price (6). For this purpose we define the process $x^\tau(t)$, with dynamics

$$\begin{aligned} &dx^\tau(t) = [\mathbf{f}(t,x^\tau(t)) - \mathbf{gg}'(t,x^\tau(t))\nabla_x W(t,\tau,x^\tau(t))]\,dt + \mathbf{g}(t,x^\tau(t))dw_t, \\ &x^\tau(0) = 0, \end{aligned} \tag{7}$$

where the function $W(t,\tau,x)$ is the unique solution of the PDE in (4), with $T=\tau$.

Moreover, let us put

$$p^{\tau}(t,T,x) := \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{p(\tau,T,x^{\tau}(\tau))\right\} = \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\exp\left[-W(\tau,T,x^{\tau}(\tau))\right]\right\}, \tag{8}$$

where the second equality comes from (3). The Kolmogorov backward equation associated to (8) is

$$\begin{cases} \frac{\partial}{\partial t}p^{\tau}(t,T,x) + \left[\mathbf{f}'(t,x) - (\nabla_x W)'(t,\tau,x)\mathbf{g}\mathbf{g}'(t,x)\right]\nabla_x p^{\tau}(t,T,x) \\ \qquad\qquad + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx}p^{\tau}(t,T,x)\mathbf{g}(t,x)\right) = 0 \\ \\ p^{\tau}(\tau,T,x) = \exp[-W(\tau,T,x)] \end{cases} \tag{9}$$

Remark 4.1 *Differently from what concerns equation* (2)*, Assumption 2.1 is not sufficient in order to guarantee uniqueness of the solution to* (9)*. However, since it is sufficient to require $\nabla_x W(t,\tau,x)$ to have at most linear growth in x, under Assumption 3.3, we indeed have uniqueness.*

Putting

$$W^{\tau}(t,T,x) := -\ln p^{\tau}(t,T,x), \tag{10}$$

analogously to what has been made in the previous section, the PDE (9) becomes

$$\begin{cases} \frac{\partial}{\partial t}W^{\tau}(t,T,x) + \left[\mathbf{f}'(t,x) - (\nabla_x W)'(t,\tau,x)\mathbf{g}\mathbf{g}'(t,x)\right]\nabla_x W^{\tau}(t,T,x) \\ \qquad\qquad - \frac{1}{2}\left(\nabla_x W^{\tau}\right)'(t,T,x)\mathbf{g}\mathbf{g}'(t,x)\nabla_x W^{\tau}(t,T,x) \\ \qquad\qquad + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx}W^{\tau}(t,T,x)\mathbf{g}(t,x)\right) = 0 \\ \\ W^{\tau}(\tau,T,x) = W(\tau,T,x) \end{cases} \tag{11}$$

For reasons of admissibility of the optimal control law (13) below, as explained in Remark 3.5, we make the following

Assumption 4.2 *The gradient of $W^{\tau}(t,T,x)$, solution of* (11)*, has at most linear growth, i.e.*

$$||\nabla_x W^{\tau}(t,T,x)|| \leq M(1+||x||) \quad \text{for all } x \in \mathbb{R}^l,$$

for some constant $M>0$, uniformly in $t \in [0,T]$.

Since it has a similar structure to (4), also the PDE (11) can be seen as resulting from a HJB equation, namely (dropping the arguments of the functions)

$$\begin{cases} \frac{\partial}{\partial t}W^{\tau} + \inf_{u\in\mathbb{R}^k}\left\{\left[\mathbf{f}' - (\nabla_x W)'\mathbf{g}\mathbf{g}' + u'\mathbf{g}'\right]\nabla_x W^{\tau} \right. \\ \qquad\qquad \left. + \frac{1}{2}\mathrm{tr}(\mathbf{g}'\nabla_{xx}W^{\tau}\mathbf{g}) + \frac{1}{2}u'u\right\} = 0, \\ \\ W^{\tau}(\tau,T,x) = W(\tau,T,x) \end{cases} \tag{12}$$

with the usual solution

$$u^*(t,x;W^\tau) = -\mathbf{g}'(t,x)\nabla_x W^\tau(t,T,x) \tag{13}$$

Thus, equation (12) is the HJB equation originating from the following stochastic control problem

$$\begin{cases} dx^\tau(t) = \big[\mathbf{f}(t,x^\tau(t)) - \mathbf{g}\mathbf{g}'(t,x^\tau(t))\nabla_x W(t,\tau,x^\tau(t))dt \\ \qquad\qquad +\mathbf{g}(t,x^\tau(t))u(t)\big] + \mathbf{g}(t,x^\tau(t))dw_t \\ \\ W^\tau(t,T,x) = \inf\limits_{u(\cdot)\in\mathcal{U}} \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\displaystyle\int_t^\tau \frac{1}{2}u'(s)u(s)ds + W(\tau,T,x^\tau(\tau))\right\} \end{cases} \tag{14}$$

The symbol $\mathcal{U}$ denotes the class of the admissible control laws, for which the first equation in (14) has a unique solution in probability law and the expected cost $J(t,\tau,x,u(\cdot)) := \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\int_t^\tau \frac{1}{2}u'(s)u(s)ds + W(\tau,T,x^\tau(\tau))\right\}$ has finite value.

Remark 4.3 *For the well-poseness of a stochastic control problem, also a condition on the terminal cost is needed. More precisely, we have to require it to have at most polynomial growth (again, see [5]). Notice that, thanks to Assumption 3.3, the stochastic control problem* (14) *satisfies this requirement. If we substitute Assumption 3.3 with another one implying just admissibility in* (5), *we need to require additionally that* $W(t,T,x)$ *has at most polynomial growth in* x.

Now we are ready to give the control interpretation promised above for the forward prices, based on problem (14). Indeed, it is possible to prove the following (see [6])

Proposition 4.4 *For* $t \le \tau$, *it holds*

$$\mathbb{E}^{\mathbb{Q}^\tau}_{t,x}\left\{p(\tau,T,x(\tau))\right\} = p^\tau(t,T,x) = \exp\left[-W^\tau(t,T,x)\right] \tag{15}$$

5 Forward measures and a general pricing formula

In this section we show the existence of a close connection between the factor process $x^\tau(\cdot)$ defined in (7) and the forward measure $\mathbb{Q}^\tau$, for each $\tau > 0$. More precisely, for a given expectation taken with respect to the forward measure $\mathbb{Q}^\tau$, we are interested in expressing such an expected value by using the standard martingale measure $\mathbb{Q}$, by means of a suitable modification to the original factor process $x(\cdot)$, evolving according to (1). This is specified in the following two Propositions.

Proposition 5.1 *Given* $\tau > 0$, *let* t *be a fixed time-instant, with* $0 \le t \le \tau$, *and let* x *be a fixed vector in* $\mathbb{R}^l$. *Let* $x(s)$, $s \in [t,\tau]$, *be the process satisfying* (1) *with*

$x(t) = x$, and let $x^\tau(s)$, $s \in [t, \tau]$, be the process satisfying the dynamics in (14) with $x^\tau(t) = x$. Then the random variable $x(\tau)$ has the same distribution under the forward measure $\mathbb{Q}^\tau$ (the one with numeraire $p(t, \tau, x(t))$) as the random variable $x^\tau(\tau)$ under the standard martingale measure $\mathbb{Q}$ (the one with numeraire $B(t)$).

This proposition can be proved analogously to Proposition 4.1. in [6].

The following proposition can now be obtained (see always [6]).

Proposition 5.2 *Given a date of maturity τ and a τ-claim $F(x(\tau))$, its arbitrage-free price at time t, with $t \leq \tau$, is*

$$\begin{aligned}\pi(t) &= \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\exp\left[-\int_t^\tau r(s, x(s))ds\right] \cdot F(x(\tau))\right\} \\ &= p(t, \tau, x) \cdot \mathbb{E}^{\mathbb{Q}^\tau}_{t,x}\left\{F(x(\tau))\right\}\end{aligned}$$

Then, if $W(t, \tau, x)$ is the unique solution of (4) with $T = \tau$, we have the following representation for $\pi(t)$:

$$\pi(t) = \exp\left[-W(t, \tau, x)\right] \cdot \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{F\left(x^\tau(\tau)\right)\right\}$$

5.1 Control interpretation for expectations under forward measures

Proposition 5.1 allows one to obtain a control description for expected values with respect to forward measures. It will be achieved by introducing a further control problem, obtained by adding a control term to the dynamics (7), i.e. by adding a second control term to the original factor process dynamics (1) (indeed, dynamics (7) originate from problem (5)). The result in this section somehow generalizes what has been made in Section 4 for forward prices. More precisely, we have the following: considering expectations under a forward measure, let $Y(x)$ be a real-valued positive function and define

$$d(t, \tau, x) := \mathbb{E}^{\mathbb{Q}^\tau}_{t,x}\left\{Y(x(\tau))\right\} \tag{16}$$

Thanks to Proposition 5.1, we get an alternative representation for $d(t, \tau, x)$, namely

$$d(t, \tau, x) = \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{Y(x^\tau(\tau))\right\} \tag{17}$$

The Kolmogorov backward equation associated to (17) is

$$\begin{cases} \frac{\partial}{\partial t} d(t, \tau, x) + \left[\mathbf{f}'(t, x) - (\nabla_x W)'(t, \tau, x)\mathbf{g}\mathbf{g}'(t, x)\right] \nabla_x d(t, \tau, x) \\ \qquad\qquad + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t, x)\nabla_{xx} d(t, \tau, x)\mathbf{g}(t, x)\right) = 0 \\ \\ d(\tau, \tau, x) = Y(x) \end{cases} \tag{18}$$

As usual, we apply the logarithmic transform to $d(t, \tau, x)$, and so we define

$$W^Y(t, \tau, x) := -\ln d(t, \tau, x) \tag{19}$$

We have the following PDE for $W^Y(t,\tau,x)$

$$\begin{cases} \frac{\partial}{\partial t}W^Y(t,\tau,x) + \left[\mathbf{f}'(t,x) - (\nabla_x W)'(t,\tau,x)\mathbf{g}\mathbf{g}'(t,x)\right]\nabla_x W^Y(t,\tau,x) \\ \qquad -\frac{1}{2}(\nabla_x W^Y)'(t,\tau,x)\mathbf{g}\mathbf{g}'(t,x)\nabla_x W^Y(t,\tau,x) \\ \qquad +\frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx}W^Y(t,\tau,x)\mathbf{g}(t,x)\right) = 0 \\ \\ W^Y(\tau,\tau,x) = -\ln Y(x) \end{cases} \tag{20}$$

As in the previous sections, also this PDE results from a HJB equation, in particular the one originating from the stochastic control problem

$$\begin{cases} dx^\tau(t) = \left[\mathbf{f}(t,x^\tau(t)) - \mathbf{g}\mathbf{g}'(t,x^\tau(t))\nabla_x W(t,\tau,x^\tau(t)) \right. \\ \qquad \left. +\mathbf{g}(t,x^\tau(t))u(t)\right]dt + \mathbf{g}(t,x^\tau(t))dw_t \\ \\ W^Y(t,\tau,x) = \inf_{u(\cdot)\in\mathcal{U}} \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\int_t^\tau \frac{1}{2}u'(s)u(s)ds - \ln Y\left(x^\tau(\tau)\right)\right\} \end{cases} \tag{21}$$

where $\mathcal{U}$ denotes the class of the control processes for which the first equation in (21) has a unique solution in probability law and the expected cost $J(t,\tau,x,u(\cdot)) := \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\int_t^\tau \frac{1}{2}u'(s)u(s)ds - \ln Y\left(x^\tau(\tau)\right)\right\}$ has finite value.

Thus

$$d(t,\tau,x) = \exp\left[-W^Y(t,\tau,x)\right],$$

with $W^Y(t,\tau,x)$ the optimal value function in (21).

Remark 5.3 *Notice that we need to assume that $Y(x)$ is regular enough in order for $-\ln Y(x)$ to have at most polynomial growth and for $\nabla_x W^Y(t,\tau,x)$ to have at most linear growth (for reasons of well-poseness of the stochastic control problem* (21) *and admissibility of the resulting optimal control law, see Remarks 3.5 and 4.3).*

6 Swap rates

The main purpose of this paper is to apply the previous control approach to swap measures. We use notations taken from [1] (see also [2]). Given a set of increasing dates $T_0, T_1, \ldots, T_N$ and choosing $n \in \{0,\ldots,N-1\}$, a fundamental quantity arising in this context is the so-called *swap rate* $R^{n,N}(t,x(t))$, given by

$$R^{n,N}(t,x(t)) := \frac{p(t,T_n,x(t)) - p(t,T_N,x(t))}{C^{n,N}(t,x(t))},$$

where

$$C^{n,N}(t,x(t)) := \sum_{i=n+1}^{N} \alpha_i \cdot p(t,T_i,x(t)),$$

with $\alpha_i := T_i - T_{i-1}$. Let $\mathbb{Q}^{n,N}$ be the swap measure, namely a probability measure, equivalent to $\mathbb{Q}$, under which $R^{n,N}(t, x(t))$ is a martingale. We thus have

$$\mathbb{E}_{t,x}^{\mathbb{Q}^{n,N}}\left\{R^{n,N}(T_n, x(T_n))\right\} = \frac{p(t, T_n, x) - p(t, T_N, x)}{C^{n,N}(t, x)} \tag{22}$$

Remark 6.1 *In what follows we shall assume that*

$$p(t, T_n, x) > p(t, T_N, x) \quad \textit{for all } (t, x) \in [0, T_n] \times \mathbb{R}^l$$

This hypothesis is not restrictive at all, since (remember that $p(t, T_n, x) \geq p(t, T_N, x)$ if $r(t, x) \geq 0$) we have $p(t, T_n, x) = p(t, T_N, x)$ if and only if $r(s, x(s)) = 0$ a.e. in $[T_n, T_N]$. In other words, the above assumption is always satisfied in the real world.

We are interested in obtaining a control description for (22). First of all, we prove the following

Lemma 6.2 *Let $T_0, T_1, \ldots, T_N$ be a set of increasing maturities and let $p(t, T_i, x)$ be the price of the zero-coupon T_i-bond at time t, for $i = 0, 1, \ldots, N$. Let $k \in \{0, 1, \ldots, N-1\}$ and let $P(t, x)$ be an arbitrary linear combination of bonds evaluated at time t for $x(t) = x$, i.e.*

$$P(t, x) = \sum_{i=k}^{N} \beta_i \cdot p(t, T_i, x),$$

for some $\beta_i \in \mathbb{R}$, $i = k, \ldots, N$. Then $P(t, x)$ is the unique solution of the PDE

$$\begin{cases} \frac{\partial}{\partial t}P(t,x) + \mathbf{f}'(t,x)\nabla_x P(t,x) + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx}P(t,x)\mathbf{g}(t,x)\right) \\ \qquad\qquad -r(t,x)P(t,x) = 0 \\ P(T_k, x) = \sum_{i=k}^{N} \beta_i \cdot p(T_k, T_i, x) \end{cases}$$

Proof: From Theorem 2.2, each function $p(t, T_i, x)$ satisfies

$$\frac{\partial}{\partial t}p(t,T_i,x) + \mathbf{f}'(t,x)\nabla_x p(t,T_i,x) + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx}p(t,T_i,x)\mathbf{g}(t,x)\right)$$
$$-r(t,x)p(t,T_i,x) = 0$$

Since $P(t, x)$ is a linear combination of them, it is solution of the same PDE. The choice of the boundary condition is obvious, while the uniqueness of the solution comes from Assumption 2.1.

Using Lemma 6.2, $C^{n,N}(t, x)$ satisfies (from now on, we shall omit the superscript n, N in all the PDEs)

$$\begin{cases} \frac{\partial}{\partial t}C(t,x) + \mathbf{f}'(t,x)\nabla_x C(t,x) + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx}C(t,x)\mathbf{g}(t,x)\right) \\ \qquad\qquad -r(t,x)C(t,x) = 0 \\ C(T_n, x) = \sum_{i=n+1}^{N} \alpha_i \cdot p(T_n, T_i, x) \end{cases} \tag{23}$$

Let us put (making explicit the dependence on the instant T_n)

$$Z^{n,N}(t,T_n,x) := -\ln C^{n,N}(t,x) \tag{24}$$

From (23), $Z^{n,N}(t,T_n,x)$ is the unique (for the same reasons as in Remark 2.3) solution of

$$\begin{cases} \frac{\partial}{\partial t} Z(t,T_n,x) + \mathbf{f}'(t,x)\nabla_x Z(t,T_n,x) \\ \qquad -\frac{1}{2}\left(\nabla_x Z\right)'(t,T_n,x)\mathbf{g}\mathbf{g}'(t,x)\nabla_x Z(t,T_n,x) \\ \qquad +\frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx} Z(t,T_n,x)\mathbf{g}(t,x)\right) + r(t,x) = 0 \\ \\ Z(T_n,T_n,x) = -\ln \sum_{i=n+1}^{N} \alpha_i \cdot p(T_n,T_i,x) \end{cases} \tag{25}$$

It is easy to recognize that this PDE has the same form as (4), except for the terminal condition. Thus one can carry out the same arguments as in the proof of Proposition 3.2 (given in [6]), observing that (25) can be seen as originating from the HJB equation (dropping the arguments of the functions)

$$\frac{\partial}{\partial t} Z + \inf_{u\in\mathbb{R}^k}\left\{\left[\mathbf{f}' + u'\mathbf{g}'\right]\nabla_x Z + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}' \ \nabla_{xx} Z \ \mathbf{g}\right) + \frac{1}{2}u'u + r\right\} = 0, \tag{26}$$

whose optimal control is

$$u^*(t,x;Z^{n,N}) = -\mathbf{g}'(t,x)\nabla_x Z^{n,N}(t,T_n,x) \tag{27}$$

In order to guarantee admissibility for such a control law, in the sense of the Sufficient Condition for Admissibility in Section 3, we make the following

Assumption 6.3 *The gradient of $Z^{n,N}(t,T_n,x)$, solution of* (25)*, satisfies a linear growth condition, i.e.*

$$||\nabla_x Z^{n,N}(t,T_n,x)|| \le M(1+||x||) \quad \textit{for all } x \in \mathbb{R}^l,$$

for some constant $M > 0$, uniformly in $t \in [0,T]$.

Such a hypothesis is not too restrictive. Indeed, we have the following

Lemma 6.4 *Under Assumption 3.3, and supposing that $r(t,x) \ge 0$, $Z^{n,N}(t,T_n,x)$ has at most quadratic growth.*

Proof: Given two dates of maturity T_1, T_2, with $T_1 < T_2$, since the spot rate is non-negative, we have $p(t,T_1,x) \ge p(t,T_2,x)$. Using this fact, we get

$$p(t,T_N,x)\sum_{i=n+1}^{N}\alpha_i \le C^{n,N}(t,x) = \sum_{i=n+1}^{N}\alpha_i \cdot p(t,T_i,x) \le p(t,T_{n+1},x)\sum_{i=n+1}^{N}\alpha_i$$

So, remembering (3), we obtain

$$-\ln\sum_{i=n+1}^{N}\alpha_i + W(t,T_{n+1},x) \le Z^{n,N}(t,T_n,x) \le -\ln\sum_{i=n+1}^{N}\alpha_i + W(t,T_N,x)$$

Since Assumption 3.3 implies that $W(t,T_{n+1},x)$ and $W(t,T_N,x)$ have at most quadratic growth, the proof is concluded.

The function $Z^{n,N}(t,T_n,x)$ in (24) is now the optimal value function of the stochastic control problem

$$\begin{cases} dx(t) = [\mathbf{f}(t,x(t)) + \mathbf{g}(t,x(t))u(t)]\,dt + \mathbf{g}(t,x(t))dw_t \\ \\ Z^{n,N}(t,T_n,x) \\ \qquad = \inf_{u(\cdot)\in\mathcal{U}} \mathbb{E}^{\mathbb{Q}}_{t,x}\{\int_t^{T_n}\left(\frac{1}{2}u'(s)u(s) + r(s,x(s))\right)ds \\ \qquad -\ln\sum_{i=n+1}^{N}\alpha_i\cdot p(T_n,T_i,x(T_n))\} \end{cases} \tag{28}$$

where $\mathcal{U}$ is the class of the admissible control laws, for which the first equation in (28) has a unique solution in probability law and the expected cost $J(t,T_n,x,u(\cdot)) := \mathbb{E}^{\mathbb{Q}}_{t,x}\{\int_t^{T_n}\left(\frac{1}{2}u'(s)u(s)+r(s,x(s))\right)ds - \ln\sum_{i=n+1}^{N}\alpha_i \cdot p(T_n,T_i,x(T_n))\}$ has finite value.

Substituting (27) into the dynamics in (28), we obtain

$$\begin{aligned} dx^{n,N}(t) = &\left[\mathbf{f}(t,x^{n,N}(t)) - \mathbf{gg}'(t,x^{n,N}(t))\nabla_x Z^{n,N}(t,T_n,x^{n,N}(t))\right]dt \\ &+\mathbf{g}(t,x^{n,N}(t))dw_t \end{aligned} \tag{29}$$

In order to give an idea of the significance of the dynamics (29), we compute the Girsanov kernel $L^{n,N}$ of the measure transformation from $\mathbb{Q}$ to the forward measure $\mathbb{Q}^{n,N}$. By using (25), we first have

$$\begin{aligned} dZ(t,T_n,x) &= \frac{\partial}{\partial t}Z(t,T_n,x)\ dt + \mathbf{f}'(t,x)\nabla_x Z(t,T_n,x)\ dt \\ &\quad +\frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx}Z(t,T_n,x)\mathbf{g}(t,x)\right)dt \\ &\quad +(\nabla_x Z)'(t,T_n,x)\mathbf{g}(t,x)\ dw_t \\ &= \left[\frac{1}{2}(\nabla_x Z)'(t,T_n,x)\mathbf{gg}'(t,x)\nabla_x Z(t,T_n,x) - r(t,x)\right]dt \\ &\qquad +(\nabla_x Z)'(t,T_n,x)\mathbf{g}(t,x)\ dw_t \end{aligned} \tag{30}$$

Thus

$$\begin{aligned} dC(t,x) &= de^{-Z(t,T_n,x)} = -C(t,x)\ dZ(t,T_n,x) + \\ &\qquad \frac{1}{2}\ C(t,x)(\nabla_x Z)'(t,T_n,x)\mathbf{gg}'(t,x)\nabla_x Z(t,T_n,x)\ dt \\ &= -C(t,x)\left[\frac{1}{2}(\nabla_x Z)'(t,T_n,x)\mathbf{gg}'(t,x)\nabla_x Z(t,T_n,x) - r(t,x)\right]dt \\ &\qquad -C(t,x)(\nabla_x Z)'(t,T_n,x)\mathbf{g}(t,x)\ dw_t \\ &\qquad +\frac{1}{2}\ C(t,x)(\nabla_x Z)'(t,T_n,x)\mathbf{gg}'(t,x)\nabla_x Z(t,T_n,x)\ dt \\ &= C(t,x)r(t,x)\ dt - C(t,x)(\nabla_x Z)'(t,T_n,x)\mathbf{g}(t,x)\ dw_t, \end{aligned} \tag{31}$$

and so, recalling (see [1]) that $L^{n,N}(t)$ is given by $L^{n,N}(t) = \frac{C(t,x)}{B(t)\,C(0,0)}$, namely by the normalized ratio of the numeraires,

$$\begin{aligned} dL^{n,N}(t) &= d\left(\frac{C(t,x)}{B(t)C(0,0)}\right) \\ &= \frac{dC(t,x)}{B(t)C(0,0)} + \frac{C(t,x)}{C(0,0)}\, d\left(\frac{1}{B(t)}\right) \qquad (32) \\ &= \frac{C(t,x)}{B(t)C(0,0)} r(t,x)\ dt - \frac{C(t,x)}{B(t)C(0,0)} (\nabla_x Z)'(t,T_n,x)\mathbf{g}(t,x)\ dw_t \\ &\qquad - \frac{C(t,x)}{B(t)C(0,0)} r(t,x)\ dt \\ &= -L^{n,N}(t)(\nabla_x Z)'(t,T_n,x)\mathbf{g}(t,x)\ dw_t \end{aligned}$$

Then the Girsanov kernel is exactly $-\mathbf{g}'(t,x(t))\nabla_x Z^{n,N}(t,T_n,x(t))$, i.e. the minimizer (27). It follows that (29) represents the factor process dynamics under the forward measure $\mathbb{Q}^{n,N}$. This leads us to claim that the expected value

$$\mathbb{E}_{t,x}^{\mathbb{Q}^{n,N}}\left\{R^{n,N}(T_n, x(T_n))\right\}$$

can be computed as an expectation with respect to the standard martingale measure $\mathbb{Q}$, assuming that the factor process evolves according to (29), instead of (1), $x(\cdot)$ and $x^{n,N}(\cdot)$ having the same initial condition x and $Z^{n,N}(t,T_n,x)$ being the solution of (25).

Indeed, defining the quantity

$$\mathcal{R}^{n,N}(t,x) := \mathbb{E}_{t,x}^{\mathbb{Q}}\left\{R^{n,N}(T_n, x^{n,N}(T_n))\right\}, \qquad (33)$$

we can prove the following

Proposition 6.5 *For $t \le T_n$, it holds*

$$\mathbb{E}_{t,x}^{\mathbb{Q}^{n,N}}\left\{R^{n,N}(T_n, x(T_n))\right\} = \mathcal{R}^{n,N}(t,x) \qquad (34)$$

Proof: From (22), we have to show that

$$\mathcal{R}^{n,N}(t,x) = \frac{p(t,T_n,x) - p(t,T_N,x)}{C^{n,N}(t,x)} \qquad (35)$$

Inspired by the proof of Proposition 4.4 (given in [6]), let $M(t,x) := p(t,T_n,x) - p(t,T_N,x)$. From Lemma 6.2, $M(t,x)$ satisfies

$$\begin{cases} \frac{\partial}{\partial t} M(t,x) + \mathbf{f}'(t,x)\nabla_x M(t,x) + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx} M(t,x)\mathbf{g}(t,x)\right) \\ \qquad\qquad -r(t,x)M(t,x) = 0 \\ \\ M(T_n,x) = p(T_n,T_n,x) - p(T_n,T_N,x) = 1 - p(T_n,T_N,x) \end{cases}$$

Defining (notice that, according to Remark 6.1, we may assume $M(t,x) > 0$)

$$D(t,x) := -\ln M(t,x), \tag{36}$$

this $D(t,x)$ is the solution of

$$\begin{cases} \frac{\partial}{\partial t} D(t,x) + \mathbf{f}'(t,x)\nabla_x D(t,x) - \frac{1}{2}\left(\nabla_x D\right)'(t,x)\mathbf{g}\mathbf{g}'(t,x)\nabla_x D(t,x) \\ \qquad\qquad +\frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx} D(t,x)\mathbf{g}(t,x)\right) + r(t,x) = 0 \\ D(T_n,x) = -\ln\left(1 - p(T_n,T_N,x)\right) \end{cases} \tag{37}$$

Moreover, the Kolmogorov backward equation associated to $\mathcal{R}^{n,N}(t,x)$ is (we write only $\mathcal{R}$ instead of $\mathcal{R}^{n,N}$)

$$\begin{cases} \frac{\partial}{\partial t}\mathcal{R}(t,x) + \left[\mathbf{f}'(t,x) - (\nabla_x Z)'(t,x)\mathbf{g}\mathbf{g}'(t,x)\right]\nabla_x \mathcal{R}(t,x) \\ \qquad\qquad +\frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx}\mathcal{R}(t,x)\mathbf{g}(t,x)\right) = 0 \\ \mathcal{R}(T_n,x) = \frac{1-p(T_n,T_N,x)}{\sum_{i=n+1}^{N}\alpha_i \cdot p(T_n,T_i,x)} \end{cases}$$

Again, applying a logarithmic transform to $\mathcal{R}^{n,N}(t,x)$ and making explicit the dependence on the time-instant T_n, we put

$$W^{\mathcal{R}}(t,T_n,x) := -\ln \mathcal{R}^{n,N}(t,x) \tag{38}$$

The function $W^{\mathcal{R}}(t,T_n,x)$ satisfies

$$\begin{cases} \frac{\partial}{\partial t} W^{\mathcal{R}}(t,T_n,x) + \left[\mathbf{f}'(t,x) - (\nabla_x Z)'(t,T_n,x)\mathbf{g}\mathbf{g}'(t,x)\right]\nabla_x W^{\mathcal{R}}(t,T_n,x) \\ \qquad -\frac{1}{2}(\nabla_x W^{\mathcal{R}})'(t,T_n,x)\mathbf{g}\mathbf{g}'(t,x)\nabla_x W^{\mathcal{R}}(t,T_n,x) \\ \qquad +\frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx} W^{\mathcal{R}}(t,T_n,x)\mathbf{g}(t,x)\right) = 0 \\ W^{\mathcal{R}}(T_n,T_n,x) = -\ln \frac{1-p(T_n,T_N,x)}{\sum_{i=n+1}^{N}\alpha_i \cdot p(T_n,T_i,x)} \end{cases} \tag{39}$$

In order to prove (35), it suffices to show that

$$\exp\left[-W^{\mathcal{R}}(t,T_n,x)\right] = \frac{\exp\left[-D(t,x)\right]}{\exp\left[-Z^{n,N}(t,T_n,x)\right]},$$

i.e. that

$$W^{\mathcal{R}}(t,T_n,x) + Z^{n,N}(t,T_n,x) = D(t,x) \tag{40}$$

Let

$$\tilde{W}(t,x) := W^{\mathcal{R}}(t,T_n,x) + Z^{n,N}(t,T_n,x)$$

From equations (39) for $W^{\mathcal{R}}(t,T_n,x)$ and (25) for $Z^{n,N}(t,T_n,x)$, we have

$$\begin{aligned}
-\frac{\partial}{\partial t}\tilde{W} = -\frac{\partial}{\partial t}W^{\mathcal{R}} - \frac{\partial}{\partial t}Z &= \mathbf{f}'\nabla_x W^{\mathcal{R}} - (\nabla_x Z)'\,\mathbf{g}\mathbf{g}'\nabla_x W^{\mathcal{R}} \\
&\quad -\frac{1}{2}\left(\nabla_x W^{\mathcal{R}}\right)'\mathbf{g}\mathbf{g}'\nabla_x W^{\mathcal{R}} + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'\nabla_{xx}W^{\mathcal{R}}\mathbf{g}\right) \\
&\quad +\mathbf{f}'\nabla_x Z - \frac{1}{2}(\nabla_x Z)'\,\mathbf{g}\mathbf{g}'\nabla_x Z + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'\nabla_{xx}Z\mathbf{g}\right) + r = \\
&= \mathbf{f}'\left[\nabla_x W^{\mathcal{R}} + \nabla_x Z\right] + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'\left[\nabla_{xx}W^{\mathcal{R}} + \nabla_{xx}Z\right]\mathbf{g}\right) \\
&\quad -\frac{1}{2}\left[\nabla_x W^{\mathcal{R}} + \nabla_x Z\right]'\mathbf{g}\mathbf{g}'\left[\nabla_x W^{\mathcal{R}} + \nabla_x Z\right] + r = \\
&= \mathbf{f}'\nabla_x\tilde{W} + \frac{1}{2}\mathrm{tr}(\mathbf{g}'\nabla_{xx}\tilde{W}\mathbf{g}) - \frac{1}{2}(\nabla_x\tilde{W})'\mathbf{g}\mathbf{g}'\nabla_x\tilde{W} + r
\end{aligned}$$

The boundary condition is

$$\begin{aligned}
\tilde{W}(T_n,x) &= W^{\mathcal{R}}(T_n,T_n,x) + Z^{n,N}(T_n,T_n,x) \\
&= -\ln\frac{1-p(T_n,T_N,x)}{\sum_{i=n+1}^{N}\alpha_i\cdot p(T_n,T_i,x)} - \ln\sum_{i=n+1}^{N}\alpha_i\cdot p(T_n,T_i,x) \\
&= -\ln\left(1-p(T_n,T_N,x)\right) \\
&= D(T_n,x)
\end{aligned}$$

Thus, $\tilde{W}(t,x)$ satisfies the same PDE as $D(t,x)$, namely the PDE in (37), and has the same terminal value. Since the equation in (37) has unique solution (for the same reasons as in Remark 2.3) and is satisfied by both $\tilde{W}(t,x)$ and $D(t,x)$, we get

$$D(t,x) = \tilde{W}(t,x) = W^{\mathcal{R}}(t,T_n,x) + Z^{n,N}(t,T_n,x) \qquad \text{for } t \leq T_n$$

Proposition 6.5 leads to a control interpretation for expectations on swap rates as in (22). As in Sections 3 and 4, in order to have also for (28) an admissible control problem we make the following

Assumption 6.6 *The gradient of $W^{\mathcal{R}}(t,T_n,x)$, solution of* (39)*, has at most linear growth, i.e.*

$$||\nabla_x W^{\mathcal{R}}(t,T_n,x)|| \leq M(1+||x||) \quad \textit{for all } x\in\mathbb{R}^l,$$

for some constant $M>0$, uniformly in $t\in[0,T]$.

The PDE in (39) can be seen as resulting from the HJB equation

$$\begin{aligned}
&\frac{\partial}{\partial t}W^{\mathcal{R}} + \inf_{u\in\mathbb{R}^k}\Big\{[\mathbf{f}' - (\nabla_x Z)'\mathbf{g}\mathbf{g}' + u'\mathbf{g}'] \\
&\qquad\qquad \nabla_x W^{\mathcal{R}} + \frac{1}{2}\mathrm{tr}\left(\mathbf{g}'\nabla_{xx}W^{\mathcal{R}}\mathbf{g}\right) + \frac{1}{2}u'u\Big\} = 0
\end{aligned}$$

Thus, we obtain

$$\mathbb{E}_{t,x}^{\mathbb{Q}^{n,N}}\left\{R^{n,N}(T_n, x(T_n))\right\} = \exp\left[-W^{\mathcal{R}}(t, T_n, x)\right],$$

where $W^{\mathcal{R}}(t, T_n, x)$ is the optimal value function of the stochastic control problem

$$\begin{cases} dx^{n,N}(t) = [\mathbf{f}(t, x^{n,N}(t)) - \mathbf{g}\mathbf{g}'(t, x^{n,N}(t))\nabla_x Z^{n,N}(t, T_n, x^{n,N}(t)) \\ \qquad\qquad +\mathbf{g}(t, x^{n,N}(t))u(t)]dt + \mathbf{g}(t, x^{n,N}(t))dw_t \\ \\ W^{\mathcal{R}}(t, T_n, x) \\ \quad = \inf_{u(\cdot)\in\mathcal{U}} \mathbb{E}_{t,x}^{\mathbb{Q}}\left\{\int_t^{T_n} \frac{1}{2}u'(s)u(s)ds - \ln \frac{1-p(T_n,T_N,x^{n,N}(T_n))}{\sum_{i=n+1}^{N}\alpha_i\cdot p(T_n,T_i,x^{n,N}(T_n))}\right\} \end{cases}$$

with $\mathcal{U}$ denoting again the class of the admissible control laws, for which the first equation in the control problem above has a unique solution in probability law and the expected cost $J(t, T_n, x, u(\cdot)) := \mathbb{E}_{t,x}^{\mathbb{Q}}\{\int_t^{T_n} \frac{1}{2}u'(s)u(s)ds - \ln \frac{1-p(T_n,T_N,x^{n,N}(T_n))}{\sum_{i=n+1}^{N}\alpha_i\cdot p(T_n,T_i,x^{n,N}(T_n))}\}$ has finite value.

7 Swap measures and a general pricing formula

Analogously to what has been made in Section 5, it is possible to establish a connection between the factor process $x^{n,N}(\cdot)$ and the swap measure $\mathbb{Q}^{n,N}$; more precisely, such a process is the key element in order to calculate expectations under $\mathbb{Q}^{n,N}$ by using the standard martingale measure $\mathbb{Q}$. Indeed, we have the following

Proposition 7.1 *Let $T_0, T_1, \ldots, T_N$ be a set of increasing maturities. Fix a vector x in $\mathbb{R}^l$, $n \in \{0, 1, \ldots, N-1\}$ and a time-instant t, with $0 \le t \le T_n$. Let $x(s)$, $s \in [t, T_n]$, be the process satisfying (1) with $x(t) = x$, and let $x^{n,N}(s)$, $s \in [t, T_n]$, be the process satisfying (29) with $x^{n,N}(t) = x$. Then the random variable $x(T_n)$ has the same distribution under the swap measure $\mathbb{Q}^{n,N}$ (the one with numeraire $C^{n,N}(t, x(t))$) as the random variable $x^{n,N}(T_n)$ under the standard martingale measure $\mathbb{Q}$ (the one with numeraire $B(t)$).*

Proof: The proof is analogous to the one of Proposition 4.1. in [6] and we outline here its main steps. We recall that (see (32))

$$dL^{n,N}(s) = -L^{n,N}(s)(\nabla_x Z)'(s, T_n, x)\mathbf{g}(s, x)\, dw_s,$$

i.e. the Girsanov kernel of the measure transformation from $\mathbb{Q}$ to $\mathbb{Q}^{n,N}$ is

$$\sigma(s) = -\mathbf{g}'(s, x(s))\nabla_x Z^{n,N}(s, T_n, x(s))$$

Thus, the process $w_s^{Q^{n,N}}$, defined by

$$dw_s^{Q^{n,N}} = dw_s^Q + \mathbf{g}'(s, x(s))\nabla_x Z^{n,N}(s, T_n, x(s))ds, \tag{41}$$

is a Wiener process under $\mathbb{Q}^{n,N}$. Consider the factor process $x(\cdot)$, satisfying (1) under $\mathbb{Q}$. Substituting (41) into (1), we get (29) under $\mathbb{Q}^{n,N}$. Since $x(t) = x^{n,N}(t) = x$, the distribution is the same.

We also deduce a pricing equation in the following analog of Proposition 5.2.

Proposition 7.2 *Given a date of maturity T_n, a T_n-claim $F(x(T_n))$, whose arbitrage-free price at time t, with $t \leq T_n$, is*

$$
\begin{aligned}
\pi(t) &= \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\exp\left[-\int_t^{T_n} r(s,x(s))ds\right]\cdot F(x(T_n))\right\} \\
&= C^{n,N}(t,x)\cdot \mathbb{E}^{\mathbb{Q}^{n,N}}_{t,x}\left\{\frac{F(x(T_n))}{C^{n,N}(T_n,x(T_n))}\right\},
\end{aligned}
$$

this price $\pi(t)$ admits a representation of the form

$$
\pi(t) = \exp\left[-Z^{n,N}(t,T_n,x)\right]\cdot \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\frac{F(x^{n,N}(T_n))}{C^{n,N}(T_n,x^{n,N}(T_n))}\right\},
$$

where $Z^{n,N}(t,T_n,x)$ is the unique solution of (25).

7.1 Control interpretation for expectations under swap measures

By proceeding exactly as in subsection 5.1, it is possible to obtain a control description for expectations with respect to swap measures. Given a positive function $V(x)$, we put

$$
\mathcal{E}(t,T_n,x) := \mathbb{E}^{\mathbb{Q}^{n,N}}_{t,x}\left\{V(x(T_n))\right\}
$$

and making use of Proposition 7.1, we get

$$
\mathcal{E}(t,T_n,x) = \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{V(x^{n,N}(T_n))\right\} \tag{42}
$$

Following the standard procedure, we write the Kolmogorov backward equation associated to (42), and then put

$$
W^V(t,T_n,x) := -\ln \mathcal{E}(t,T_n,x) \tag{43}
$$

The function W^V satisfies

$$
\begin{cases}
\frac{\partial}{\partial t}W^V(t,T_n,x) + \\
\qquad \left[\mathbf{f}'(t,x) - (\nabla_x Z^{n,N})'(t,T_n,x)\mathbf{g}\mathbf{g}'(t,x)\right]\nabla_x W^V(t,T_n,x) \\
\qquad -\frac{1}{2}(\nabla_x W^V)'(t,T_n,x)\mathbf{g}\mathbf{g}'(t,x)\nabla_x W^V(t,T_n,x) \\
\qquad +\frac{1}{2}\mathrm{tr}\left(\mathbf{g}'(t,x)\nabla_{xx}W^V(t,T_n,x)\mathbf{g}(t,x)\right) = 0 \\
W^V(T_n,T_n,x) = -\ln V(x)
\end{cases} \tag{44}
$$

Our control interpretation is given by the fact that $W^V(t,T_n,x)$ can be seen also as the optimal value function of the stochastic control problem

$$\begin{cases} dx^{n,N}(t) = [\mathbf{f}(t,x^{n,N}(t)) - \mathbf{gg}'(t,x^{n,N}(t))\nabla_x Z^{n,N}(t,T_n,x^{n,N}(t)) \\ \qquad\qquad + \mathbf{g}(t,x^{n,N}(t))u(t)]dt + \mathbf{g}(t,x^{n,N}(t))dw_t \\ W^V(t,T_n,x) = \inf_{u(\cdot)\in\mathcal{U}} \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\int_t^{T_n} \frac{1}{2}u'(s)u(s)ds - \ln V\left(x^{n,N}(T_n)\right)\right\} \end{cases} \tag{45}$$

where $\mathcal{U}$ denotes the class of the control processes for which the first equation in (45) has a unique solution in probability law and the expected cost $J(t,T_n,x,u(\cdot)) := \mathbb{E}^{\mathbb{Q}}_{t,x}\left\{\int_t^{T_n} \frac{1}{2}u'(s)u(s)ds - \ln V\left(x^{n,N}(T_n)\right)\right\}$ has finite value. As in subsection 5.1, we need to assume that $V(x)$ is regular enough in order to have a well-posed stochastic control problem and an admissible optimal control law, i.e. in order for $-\ln V(x)$ to have at most polynomial growth and for the gradient of $W^V(t,T_n,x)$ to have at most linear growth.

Remark 7.3 *It is easy to see that the entire argument in Sections 6 and 7 works exactly as for the forward measures. Indeed, in both cases:*

- *the logarithm of the numeraire is the optimal value function of a stochastic control problem (see* (5) *for forward measures and* (28) *for swap measures);*
- *the optimal control law $u^*(\cdot)$ coincides with the Girsanov kernel of the measure transformation from $\mathbb{Q}$ to the new martingale measure, when discussing forward prices and swap rates respectively;*
- *the optimally controlled factor process is distributed under $\mathbb{Q}$ as the original factor process under the equivalent martingale measure (compare Propositions 5.1 and 7.1);*
- *a second control problem (see* (21) *and* (45)*), obtained by further controlling the factor process, permits to give a control interpretation to all expectations.*

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