Interest rate derivatives pricing when the short rate is a continuous time finite state Markov process

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Abstract

The purpose of this study is to price interest rate derivatives by assuming the spot rate a continuous-time Markov chain with a finite state space. Our model is inspired by the paper Filipovic' and Zabczyk [5]; we extend their deterministic discrete time structure by one with random times and consider also the multifactor case. We are able to price with the same approach various interest rate derivatives, in particular bonds, caps and swaptions. We also present some numerical results.

Mathematics Subject Classification : Primary 91G30; Secondary 60J28, 91G20

Keywords : Interest rate models, multifactor models, continuous time Markov chains, interest rate derivatives, pricing.

1 Introduction

In their article [5] Filipovic' and Zabczyk present an approach to obtain in discrete time the analog of the affine term structure models in continuous time. They consider the spot rate $r(t)$ a Markov chain (MC) with a finite state space. Since the short rate is a MC in discrete time, the number of jumps in a fixed time interval is deterministic.

In real markets the spot rate does not generally change at deterministic times but it rather "jumps" at random times. This suggests to model the spot rate as a continuous time Markov chain (CTMC) with a finite state space $E = \{r_1, r_2, \ldots, r_N\}$, $N \in \mathbb{N}$, $r_i \in \mathbb{R}$, $i = 1, \ldots, N$. Under a martingale measure $\tilde{\mathbb{P}}$, equivalent to the physical measure $\mathbb{P}$, the transition intensity matrix of the chain is supposed to be given by $Q = \{q_{i,j}\}_{i,j=1,\cdots,N}$. Such a modeling approach appears also to be more realistic with respect to the traditional diffusion-type models for the short rate. The innovation introduced by this model with respect to [5] is represented by the fact that the number of jumps of the spot rate between an evaluation time t and a maturity

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T (denoted by $\nu_{t,T}$), namely the number of transition of the MC, is random and can take arbitrarily large values.

Continuous-time term structure models that allow also for jumps have already been considered in the literature. We limit ourselves to mention here just a couple of them. For the case of jump diffusions the article by Björk-Kabanov-Runggaldier [1] illustrates how, by assuming an appropriate affine structure, the bond price can be expressed in terms of solutions of a system of ODE's. This approach is theoretically interesting but does not consider derivative prices and it turns out to be difficult to implement in practice. For a more general Levy driven model, the article by Eberlein and Kluge [4] considers also Caps and Swaptions. Here the authors obtain explicit analytic solution formulae in the scalar case that require however rather sophisticated mathematical tools; moreover their numerical results do not concern the prices as such, which is our main goal.

On the other hand, in our setup where the short rate evolves as a continuous time Markov process, we are able to obtain explicit formulae for bond prices and derivatives such as caps and swaptions that can actually be implemented to obtain numerical results. In fact, the pricing of bonds and interest derivatives will be shown to be particular cases of the pricing of a fictitious financial product, namely the "Prototype product" that is an analog here of Arrow-Debreu prices and which represents a unified approach to the pricing of interest rate related products. We obtain a computable expression of the price of the Prototype product by using a technique based on a contracting operator and on the distribution of $\nu_{t,T}$. Even though by our approach we face a difficulty represented by the randomness of the jump times of the spot rate, we are able to give an explicit computable formula for the distribution of $\nu_{t,T}$.

We furthermore generalize the one-factor short rate model discussed above by considering a multi-factor short rate model in which the spot rate depends on several correlated CTMCs. Under a particular multi-factor short rate model, bonds, caps and swaptions can be viewed as particular cases of the Prototype product whose price admits a computable explicit formula also when the short rate is driven by more factors. The multi-factor short rate model can also be applied to the pricing of defaultable bonds, where the pricing formula depends on the short rate and default intensity processes. Finally we derive numerical results to illustrate the performance of our approach: we compare our method with the tree method suggested in [3] when both are considered for the pricing of a bond in a continuous time affine term structure model. We point out that, while tree methods work well for scalar models without jumps, our approach is applicable without substantially additional difficulties also to the multivariate case and it can be used to approximate prices in continuous time models involving jumps. We also would like to stress the fact that our approach is specifically designed for CTMC models for the short rate which appears to be more realistic than diffusion-type models or discrete time models with fixed time instants.

In Section 2 we discuss the pricing approach based on the Prototype product when the short rate is a CTMC; in Section 3 we derive the prices of bonds, caps and swaptions by using the results obtained for the Prototype product pricing. In

Section 4, by assuming that the short rate depends on several CTMCs, we discuss the pricing of both bonds and other interest rate derivatives, including defaultable bonds, by starting from the pricing of the Prototype product under a particular multi-factor short rate model. To conclude, in Section 5, numerical results are presented in support of the theory developed in the previous chapters.

2 Pricing of interest rate derivatives with a Markov short rate: the Prototype product

In this section we consider a market model under which the spot rate is a single CTMC: we introduce a fictitious financial product, the "Prototype product", an analog of Arrow-Debreu prices, and we show that, by using a technique based on a contracting operator and on the distribution of $\nu_{t,T}$, one obtains an explicit pricing formula. Furthermore we briefly discuss the Prototype product pricing under a more general setup when the spot rate is considered as a renewal process.

2.1 Market model

Let a filtered probability space be given by $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in\mathbb{R}}, \mathbb{P})$ where $\mathbb{P}$ is the physical measure.

To introduce the model, consider first the price $p(t,T)$ at time t of a zero coupon bond that matures in $T > t$. In a general setting the bond price has the following representation

$$p(t,T) = \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\int_t^T r(u)du\Big)|\mathcal{F}_t\Big]$$

where, in order to avoid arbitrage, $\widetilde{\mathbb{P}}$ is a martingale measure equivalent to $\mathbb{P}$. If we assume a Markov short rate, $p(t,T)$ can be expressed by means of specific quantities that we are going to introduce next. In particular, we consider the spot rate $r(t)$ a continuous time Markov chain (CTMC) with a state space $E = \{r_1, r_2, \ldots, r_N\}$, $N \in \mathbb{N}$ and $r_i \in \mathbb{R}^+,\ i = 1, \ldots, N$. Denote by

- $Q = (q_{i,j})_{1\le i,j\le N}$ the transition intensity kernel homogeneous with respect to the time;
- $q_i = \sum_{\substack{j=1\\ j\neq i}}^{N} q_{i,j},\ i = 1, \ldots, N$ the intensities associated with the state r_i; $i = 1, \cdots, N$
- the transition probabilities from the state r_i to r_j

$$\begin{cases} p_{i,j} = \frac{q_{i,j}}{q_i} & \text{if } i \neq j \\ p_{i,j} = 0 & \text{if } i = j. \end{cases} \tag{1}$$

Hence $r(t)$ is a stochastic process with right-continuous piecewise constant trajectories where the jump times T_i $(i = 1, 2, \ldots)$ are random variables and, conditionally on a generic value r_h $(h = 1, \ldots, N)$ of the process at time T_i, the interarrival times $\mathrm{T}_{i+1} - \mathrm{T}_i$ are exponentially distributed:

$$(\mathrm{T}_{i+1} - \mathrm{T}_i | r(\mathrm{T}_i) = r_h) \sim \mathcal{E}xp(q_h) \tag{2}$$

Therefore we may write (using temporarily $\bar{p}(t,T)$ to denote the bond price at time t with maturity T)

$$\bar{p}(t,T) = \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp(r_{\nu_t}(t - \mathrm{T}_{\nu_t}))exp\Big(- \sum_{i=\nu_t}^{\nu_T - 1} r_i(\mathrm{T}_{i+1} - \mathrm{T}_i) - r_{\nu_T}(T - \mathrm{T}_{\nu_T})\Big)|\mathcal{F}_t^r\Big] \tag{3}$$

where, for a generic time s, the random variable ν_s denotes the number of jumps of the Markov process until s and $r(s) = r_i$ for $s \in [T_i, T_{i+1})$. In what follows, given two generic times $\underline{s}$ and $\overline{s}$ with $\underline{s} < \overline{s}$, we denote by $\nu_{\underline{s},\overline{s}}$ the number of jumps in the interval $[\underline{s}, \overline{s})$.

2.1.1 Reduction to a simpler expression for bond prices

Notice first that, since $\{\mathrm{T}_i \leq s\} \in \mathcal{F}_s^r$ and T_i is an $\mathcal{F}^r$-stopping time, we may consider the family of $\sigma-$algebras, indexed by T_i, by putting

$$\mathcal{F}_{\mathrm{T}_i}^r = \{A \in \mathcal{F}_T^r | A \cap \{\mathrm{T}_i \leq s\} \in \mathcal{F}_s^r, \forall s \leq T\}. \tag{4}$$

For simplicity we denote by $\mathcal{F}_i^r$ the $\sigma-$algebra $\mathcal{F}_{\mathrm{T}_i}^r$.

Observe next that the factor $exp(r_{\nu_t}(t - \mathrm{T}_{\nu_t})$ in (3) can be determined on the basis of the information at time t. This allows us to assume, without loss of generality, that $t = \mathrm{T}_{\nu_t}$ and, consequently, the bond price in (3) can be written in the simpler form

$$\bar{p}(t,T) = \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(- \sum_{i=\nu_t}^{\nu_T - 1} r_i(\mathrm{T}_{i+1} - \mathrm{T}_i) - r_{\nu_T}(T - \mathrm{T}_{\nu_T})\Big)|\mathcal{F}_{\nu_t}^r\Big] \tag{5}$$

where $\mathcal{F}_{\nu_t}^r = \mathcal{F}_t^r$. In order to further simplify the expression in (5) we now show that in many cases of interest, in particular when the values of the jump intensities are large with respect to the values r_i of the short rate, we may drop the last term in the exponential in (5) and still obtain a value for bond price that is close to its value in (5). To this effect let $\bar{p}(t,T)$ be as in (5) and put

$$p(t,T) = \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(- \sum_{i=\nu_t}^{\nu_T - 1} r_i(\mathrm{T}_{i+1} - \mathrm{T}_i)\Big)|\mathcal{F}_{\nu_t}^r\Big] \tag{6}$$

so that $\bar{p}(t,T) \leq p(t,T)$. Furthermore, let

$$\begin{aligned} U &:= -\textstyle\sum_{i=\nu_t}^{\nu_T - 1} r_i(\mathrm{T}_{i+1} - \mathrm{T}_i) - r_{\nu_T}(T - \mathrm{T}_{\nu_T}) \\ W &:= -\textstyle\sum_{i=\nu_t}^{\nu_T - 1} r_i(\mathrm{T}_{i+1} - \mathrm{T}_i) \end{aligned}$$

implying that $U \leq W$. We have now

Proposition 2.1. *The difference between $p(t,T)$ and $\bar{p}(t,T)$ can be bounded from above as follows*

$$\Delta := p(t,T) - \bar{p}(t,T) \leq \frac{\bar{r}}{\bar{q}}$$

where $\frac{\bar{r}}{\bar{q}} := \max_{i \leq N} \frac{r_i}{q_{r_i}}$.

Proof Denoting by ξ a random variable taking a.s. values appropriately between 0 and $W-U$ and, using a Taylor expansion up to first order, we have

$$\begin{aligned}
\Delta &= \mathbb{E}^{\widetilde{\mathbb{P}}}\left\{e^{W} - e^{U} \mid \mathcal{F}^r_{\nu_t}\right\} = \mathbb{E}^{\widetilde{\mathbb{P}}}\left\{e^{U}\left(e^{W-U} - 1\right) \mid \mathcal{F}^r_{\nu_t}\right\} \\
&= \mathbb{E}^{\widetilde{\mathbb{P}}}\left\{e^{U}\left(1 + e^{\xi}(W-U) - 1\right) \mid \mathcal{F}^r_{\nu_t}\right\} = \mathbb{E}^{\widetilde{\mathbb{P}}}\left\{e^{U+\xi}(W-U) \mid \mathcal{F}^r_{\nu_t}\right\} \\
&\leq \mathbb{E}^{\widetilde{\mathbb{P}}}\left\{e^{W}(W-U) \mid \mathcal{F}^r_{\nu_t}\right\} \leq \mathbb{E}^{\widetilde{\mathbb{P}}}\left\{(W-U) \mid \mathcal{F}^r_{\nu_t}\right\} \\
&= \mathbb{E}^{\widetilde{\mathbb{P}}}\left\{r_{\nu_T}(T - T_{\nu_T}) \mid \mathcal{F}^r_{\nu_t}\right\} \leq \mathbb{E}^{\widetilde{\mathbb{P}}}\left\{r_{\nu_T}(T_{\nu_T+1} - T_{\nu_T}) \mid \mathcal{F}^r_{\nu_t}\right\} \\
&= \mathbb{E}^{\widetilde{\mathbb{P}}}\left\{\mathbb{E}^{\widetilde{\mathbb{P}}}\left\{r_{\nu_T}(T_{\nu_T+1} - T_{\nu_T}) \mid \mathcal{F}^r_{\nu_T}\right\} \mid \mathcal{F}^r_{\nu_t}\right\} = \mathbb{E}^{\widetilde{\mathbb{P}}}\left\{\frac{r_{\nu_T}}{q_{r_{\nu_T}}} \mid \mathcal{F}^r_{\nu_t}\right\} \leq \frac{\bar{r}}{\bar{q}}
\end{aligned}$$

□

Basically this result states that, whenever the jumps are sufficiently frequent, then one can work equally well with the simpler expression (6) for the bond prices and this is what we are going to do below defining accordingly also the price of a Prototype product.

2.2 The Prototype product

The expression (6) for the bond prices leads us to introduce a more general financial product that we shall call *Prototype product* and that is related to Arrow-Debreu prices. We shall show that bond prices as well as interest rate derivatives can be obtained either as special cases or as linear combinations of Prototype products.

Definition 2.2. *A* ***Prototype product*** *is a financial product which guarantees to deliver a certain payoff $\vartheta_0(r_{\nu_T})$ at maturity T. This payoff depends on the value taken by the spot rate at the date of maturity T. Its price at time $t < T$ is defined as*

$$V_{\vartheta_0,t,T}(r_{\nu_t}) = \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\nu_t}^{\nu_T-1} r_i(\mathrm{T}_{i+1} - \mathrm{T}_i)\Big)\vartheta_0(r_{\nu_T})|\mathcal{F}^r_{\nu_t}\Big] \tag{7}$$

and the ***Prototype payoff*** *$\vartheta_0(\cdot)$ is supposed to have the following form*

$$\vartheta_0(\cdot) = \sum_{i=1}^{N} w_i \mathbf{I}_{\{\cdot = r_i\}}, r_i \in E,\ \ w_i \in \{0\} \cup \mathbb{R}_+ \tag{8}$$

Remark 2.3. *Corresponding to the full bond pricing formula for* $\bar{p}(t,T)$ *in (5), for the Prototype product we would have the expression*

$$\bar{V}_{\vartheta_0,t,T}(r_{\nu_t}) = \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\nu_t}^{\nu_T-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i) - r_{\nu_T}(T-\mathrm{T}_{\nu_T})\Big)\vartheta_0(r_{\nu_T})|\mathcal{F}^r_{\nu_t}\Big] \quad (9)$$

Since $\vartheta_0(\cdot) \in [0, \max_{i\le N} w_i]$ *and in all our applications below* $w_i \le 1$, *the same approximation as in Proposition 2.1 for the bond prices holds here too. For this reason, also for the Prototype product we shall work with the simpler expression in (7).*

In the following we give a general representation of the price of Prototype product as defined in (7).

Proposition 2.4. *The price at time* t *of the Prototype product with maturity* T *admits the following representation*

$$V_{\vartheta_0,t,T}(r_{\nu_t}) = \sum_{k=0}^{+\infty} \vartheta_k(r_{\nu_t})\widetilde{\mathbb{P}}(\nu_{t,T}=k|r_{\nu_t}) \quad (10)$$

where $\nu_{t,T} = \nu_T - \nu_t$ *represents the number of jumps occurring in the interval* $[t,T]$ *and*

$$\vartheta_k(r_{\nu_t}) \triangleq \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\nu_t}^{\nu_t+k-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_0(r_{\nu_t+k})|\mathcal{F}^r_{\nu_t}\Big],\ k\in\mathbb{N}. \quad (11)$$

Proof The price of the Prototype product in (7) can be represented as follows

$$\begin{aligned}
V_{\vartheta_0,t,T}(r_{\nu_t}) &\overset{(7)}{=} \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\nu_t}^{\nu_T-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_0(r_{\nu_T})|\mathcal{F}^r_{\nu_t}\Big] \\
&= \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\nu_t}^{\nu_t+\nu_{t,T}-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_0(r_{\nu_T})|\mathcal{F}^r_{\nu_t}\vee\sigma\{\nu_{t,T}\}\Big]|\mathcal{F}^r_{\nu_t}\Big] \\
&= \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\nu_t}^{\nu_t+\nu_{t,T}-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_0(r_{\nu_T})|\mathcal{F}^r_{\nu_t}\vee\sigma\{\nu_{t,T}\}\Big]|r_{\nu_t}\Big] \\
&\overset{(11)}{=} \mathbb{E}^{\widetilde{\mathbb{P}}}[\vartheta_{\nu_{t,T}}(r_{\nu_t})|r_{\nu_t}]; \quad (12)
\end{aligned}$$

since $\nu_{t,T}$ is a discrete random variable, we can write the expectation as the following sum

$$V_{\vartheta_0,t,T}(r_{\nu_t}) = \sum_{k=0}^{+\infty} \vartheta_k(r_{\nu_t})\widetilde{\mathbb{P}}(\nu_{t,T}=k|r_{\nu_t}).$$

□

The representation (10) stresses the fact that we need to know both the distribution of $\nu_{t,T}$ and an explicit expression for the functions ϑ_k, which we shall study later on, and that to obtain a computable expression for $V_{\vartheta_0,t,T}$ one has to truncate the infinite sum. For this purpose we shall show that the functions ϑ_k defined in (11) admit, for each $k \in \mathbb{N}$, a recursive representation that will also allow in the next subsection to introduce a contracting operator which gives the possibility to approximate up to any level of precision the price of the Prototype product by a truncated series.

Lemma 2.5. *Let $r(t)$ be a CTMC with state space E and $\vartheta_0(\cdot)$ the* Prototype payoff *as in (8). For fixed $k, \eta \in \mathbb{N}$, the quantity*

$$\vartheta_k(r_\eta) = \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\eta}^{\eta+k-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_0(r_{\eta+k})|\mathcal{F}^r_\eta\Big] \tag{13}$$

can be computed recursively by

$$\begin{cases} \vartheta_h(r_{\eta+k-h}) = \mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-r_{\eta+k-h}(\mathrm{T}_{\eta+k-h+1}-\mathrm{T}_{\eta+k-h})}\vartheta_{h-1}(r_{\eta+k-h+1})\big|\mathcal{F}^r_{\eta+k-h}\big] \\ \forall h = 1,\dots,k \end{cases} \tag{14}$$

The proof of Lemma 2.5 is given in the Appendix.

2.3 A contraction mapping application

Inspired by formula (14) for the functions ϑ_k, we consider a function space $\mathfrak{M}$ defined by

$$\mathfrak{M} \triangleq \{\vartheta : E \to \{0\} \cup \mathbb{R}_+\} \tag{15}$$

so that $\vartheta(v) = \sum_{i=1}^N w_i \mathbf{I}_{\{v=r_i\}}$ where $w_i \in \{0\} \cup \mathbb{R}_+$, $\forall i = 1,\dots,N$. We introduce the operator $\mathcal{T}$ on $\mathfrak{M}$ by

$$\mathcal{T}\vartheta(v) \triangleq \mathbb{E}^{\widetilde{\mathbb{P}}}_v[e^{-v\mathfrak{T}}\vartheta(u)] \quad \text{with} \quad \mathfrak{T} \sim \mathcal{E}xp(q(v)), \quad \vartheta \in \mathfrak{M} \tag{16}$$

where, by considering a generic $i \in \mathbb{N}$, the quantities introduced in the above definition can be interpreted as follows

- v and u are the spot rate r_i at jump time T_i and r_{i+1} at jump time T_{i+1} respectively,
- $q(v)$ is the intensity associated with each v of state space E.

Remark 2.6. *By Lemma 2.5, the functions ϑ_h defined by (11) are elements of $\mathfrak{M}$ and they can be obtained by iterating the operator $\mathcal{T}$ in (16): $\vartheta_h = \mathcal{T}\vartheta_{h-1} \in \mathfrak{M}$.*

In the following Propositions we shall show that the contraction property of $\mathcal{T}$ allows to obtain a computable approximation of the actual Prototype product price $V_{\vartheta_0,t,T}$ by a finite sum and with arbitrary precision.

Proposition 2.7.

a) The function space $\mathfrak{M}$ is closed w.r.t. $\mathcal{T}$, *namely* $\mathcal{T} : \mathfrak{M} \to \mathfrak{M}$

b) $\mathcal{T}$ is a contracting operator: *For* $\vartheta,\ \vartheta' \in \mathfrak{M}$ *we have*

$$||\mathcal{T}\vartheta - \mathcal{T}\vartheta'|| \leq \gamma||\vartheta - \vartheta'|| \tag{17}$$

with the norm $||\cdot||$ *defined by* $||f|| \triangleq \sup_{v\in E}|f(v)|$, *E the finite state space and* $\gamma \triangleq \sup_{v\in E} \frac{q(v)}{v+q(v)} < 1$

c) The fixed point of $\mathcal{T}$ is identically equal to zero: $\mathcal{T}\vartheta^* = \vartheta^*$ *where* $\vartheta^* \equiv 0$.

Proof

a) By considering $\vartheta(u) = \sum_{m=1}^{N} b_m \mathbf{I}_{\{u=r_m\}} \in \mathfrak{M}$, we have that

$$\begin{aligned}\mathcal{T}\vartheta(v) &\overset{(16)}{=} \mathbb{E}_v^{\widetilde{\mathbb{P}}}[e^{-v\mathfrak{T}} \cdot \vartheta(u)] = \mathbb{E}_v^{\widetilde{\mathbb{P}}}\Big[e^{-v\mathfrak{T}} \cdot \sum_{m=1}^{N} b_m \mathbf{I}_{\{u=r_m\}}\Big] \\ &= \sum_{n=1}^{N} \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[e^{-v\mathfrak{T}} \sum_{m=1}^{N} b_m \mathbf{I}_{\{u=r_m\}} | v = r_n\Big]\mathbf{I}_{\{v=r_n\}} = \sum_{n=1}^{N} \widetilde{b}_n \mathbf{I}_{\{v=r_n\}}\end{aligned} \tag{18}$$

with $\widetilde{b}_n \triangleq \sum_{m=1}^{N} b_m \mathbb{E}^{\widetilde{\mathbb{P}}}[e^{-v\mathfrak{T}} \cdot \mathbf{I}_{\{u=r_m\}} | v = r_n] \in \mathbb{R},\ \forall n = 1, \ldots, N.$

b) By Jensen's inequality we have

$$||\mathcal{T}\vartheta(v) - \mathcal{T}\vartheta'(v)|| \leq \Big(\sup_{v\in E} \mathbb{E}_v[e^{-v\mathfrak{T}}]\Big) \cdot ||\vartheta(u) - \vartheta'(u)||$$

where

$$\sup_{v\in E} \mathbb{E}_v[e^{-v\mathfrak{T}}] = \sup_{v\in E} \int_0^{+\infty} e^{-vs} q(v) e^{-q(v)s} \mathrm{ds} = \sup_{v\in E} \frac{q(v)}{v + q(v)} \triangleq \gamma$$

Moreover $\gamma < 1$ because $q(v)$ and v are always positive quantities.

c) By considering $\vartheta(\cdot) = 0$ in definition (16) we obtain $\mathcal{T}\vartheta(v) = \mathbb{E}_v[e^{-v\mathfrak{T}} \cdot 0] = 0$. By its unicity the fixed point of the operator $\mathcal{T}$ is thus equal to zero. □

Proposition 2.8. *Let the functions* ϑ_k *be defined as in Lemma 2.5 for a given* ϑ_0. *For an arbitrarily small* ϵ, *for* γ *as in b) of Proposition 2.7 and for* $n_\epsilon \in \mathbb{N}$ *such that*

$$n_\epsilon \geq \left\lceil \frac{\log(\epsilon(1-\gamma))}{\log(\gamma)} - \frac{\sup_{v\in E}|\vartheta_1(v) - \vartheta_0(v)|}{\log(\gamma)} \right\rceil, \tag{19}$$

we have that

$$V^{\epsilon}_{\vartheta_0,t,T}(r_{\nu_t}) \triangleq \sum_{k=0}^{n_\epsilon} \vartheta_k(r_{\nu_t}) \widetilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t}) \tag{20}$$

approximates the actual price of the Prototype product defined as in (10) in the sense that

$$|V^{\epsilon}_{\vartheta_0,t,T}(r_{\nu_t}) - V_{\vartheta_0,t,T}(r_{\nu_t})| < \epsilon \textit{ uniformily in } (t, T, r_{\nu_t}) \tag{21}$$

Proof Follows directly from Proposition 2.7. □

2.4 The functions ϑ_k, explicit formula

In accordance with Remark 2.6, the functions ϑ_k defined by (11) are elements of the state space $\mathfrak{M}$ and consequently they admit the representation

$$\vartheta_k(\cdot) = \sum_{m=1}^{N} w_m^k \mathbf{I}_{\{\cdot = r_m\}}, \;\; k \geq 1 \tag{22}$$

where, for a fixed natural number k, the coefficients w_m^k are real values $\forall\ m = 1, \ldots, N$.

In the next Lemma we present a simpler representation for $\vartheta_k(\cdot)$ by introducing vector notation.

Definition 2.9. *Let $\underline{r} = [r_1, \ldots, r_N]'$ be the N-dimensional vector with components the values of state space E and define*

- $\theta_0(\underline{r}) \triangleq [w_1, \ldots, w_N]'$ *where the components correspond to the Prototype payoff $\vartheta_0(\cdot)$ in (8),*
- $\theta_k(\underline{r}) = [w_1^k, \ldots, w_N^k]'$, *with w_m^k corresponding to formula (22).*

In other terms, for $k \in \mathbb{N}$, $\{\vartheta_k(r_i)\}_{\{r_i \in E\}}$ is the collection of all possible values assumed by the function ϑ_k and, for a fixed $r_i \in E$, $\vartheta_k(r_i)$ is the i-th component of vector $\theta_k(\underline{r})$.

Lemma 2.10. *Let $r(t)$ be a CTMC with state space E and transition kernel $Q = (q_{i,j})_{1 \leq i,j \leq N}$. The vectors $\theta_k(\underline{r})$ in Definition 2.9 admit, for $k \in \mathbb{N}$, the representation*

$$\theta_k(\underline{r}) = \widetilde{Q}^k \cdot \theta_0(\underline{r}), k \in \mathbb{N} \tag{23}$$

with $\widetilde{Q}^0 \triangleq I_N$ the identity matrix and

$$\widetilde{Q} = \begin{bmatrix} 0 & \frac{q_{1,2}}{r_1+q_1} & \frac{q_{1,3}}{r_1+q_1} & \cdots & \frac{q_{1,N}}{r_1+q_1} \\ \frac{q_{2,1}}{r_2+q_2} & 0 & \frac{q_{2,3}}{r_2+q_2} & \cdots & \frac{q_{2,N}}{r_2+q_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{q_{N-1,1}}{r_{N-1}+q_{N-1}} & \frac{q_{N-1,2}}{r_{N-1}+q_{N-1}} & \cdots & 0 & \frac{q_{N-1,N}}{r_{N-1}+q_{N-1}} \\ \frac{q_{N,1}}{r_N+q_N} & \frac{q_{N,2}}{r_N+q_N} & \cdots & \frac{q_{N,N-1}}{r_N+q_N} & 0 \end{bmatrix}. \tag{24}$$

The proof of Lemma 2.10 is given in the Appendix.

2.5 Distribution of $\nu_{t,T}$

The discrete random variable $\nu_{t,T}$ represents the number of jumps of the process $r(\cdot)$ between t and T. We now compute $\widetilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t} = r_m)$, namely the probability of k jumps occurring in the interval $[t, T]$ when the process $r(\cdot)$ at time t is equal to r_m, for all $k \in \mathbb{N}$ and $r_m \in E,\ m \in \{1, \dots, N\}$.

Lemma 2.11. *Let $r(t)$ be a CTMC with state space E and transition kernel $Q = (q_{i,j})_{1 \le i,j \le N}$ then, for every positive $k \in \mathbb{N}$ and $r_m \in E$ with $m \in \{1, \dots, N\}$, we have*

$$\widetilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t} = r_m) = \sum_{\substack{i=1 \\ i \neq m}}^{N} q_{m,i} \int_t^T e^{-q_m (s-t)} \widetilde{\mathbb{P}}(\nu_{s,T} = k-1 | r_{\nu_s} = r_i) ds, \ \text{with } t < T. \tag{25}$$

Proof Let us denote the first jump time after t as the random variable $\widehat{\tau}$, that is $\widehat{\tau} = \inf\{u > 0 : \nu_{t+u} > \nu_{(t+u)^-}\}$. By properties of CTMC's we observe that $(\widehat{\tau} | r_{\nu_t} = r_m) \sim \mathcal{E}xp(q_m)$. Consider now the random variable τ defined as follows:

$$\tau \triangleq \widehat{\tau} + t;$$

since the density function of $\widehat{\tau}$ is $\widetilde{\mathbb{P}}(\widehat{\tau} \in ds | r_{\nu_t} = r_m) = q_m e^{-q_m s} ds$, the density function of τ is given by

$$\widetilde{\mathbb{P}}(\tau \in ds | r_{\nu_t} = r_m) = \widetilde{\mathbb{P}}(\widehat{\tau} + t \in ds | r_{\nu_t} = r_m) = q_m e^{-q_m (s-t)} ds, \ s > t. \tag{26}$$

We can now proceed to prove the statement. In fact, by the law of total probability, we have that

$$\widetilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t} = r_m) = \int_t^T \sum_{i=1}^{N} \widetilde{\mathbb{P}}(\tau \in ds, r_{\nu_\tau} = r_i | r_{\nu_t} = r_m) \widetilde{\mathbb{P}}(\nu_{s,T} = k-1 | r_{\nu_s} = r_i) \tag{27}$$

where r_i $(i = 1, \dots, N)$ are all possible states reachable at the jump time τ. By properties of CTMC's, the random variables τ and r_{ν_τ} are, conditionally on r_{ν_t}, independent and so

$$\begin{aligned} \widetilde{\mathbb{P}}(\tau \in ds, r_{\nu_\tau} = r_i | r_{\nu_t} = r_m) &= \widetilde{\mathbb{P}}(\tau \in ds | r_{\nu_t} = r_m) \widetilde{\mathbb{P}}(r_{\nu_\tau} = r_i | r_{\nu_t} = r_m) \\ &\overset{(26)}{=} q_m e^{-q_m (s-t)} p_{m,i} ds. \end{aligned}$$

Hence, by (27), we obtain

$$\begin{aligned} \widetilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t} = r_m) &= \int_t^T \sum_{i=1}^{N} p_{m,i} q_m e^{-q_m (s-t)} \widetilde{\mathbb{P}}(\nu_{s,T} = k-1 | r_{\nu_s} = r_i) ds \\ &\overset{(1)}{=} \sum_{\substack{i=1 \\ i \neq m}}^{N} q_{m,i} \int_t^T e^{-q_m (s-t)} \widetilde{\mathbb{P}}(\nu_{s,T} = k-1 | r_{\nu_s} = r_i) ds. \end{aligned}$$

□

We derive now an explicit expression for the probabilities defined in (25):

Proposition 2.12. *Let $r(t)$ be a CTMC with state space E and transition kernel $Q = (q_{i,j})_{1\le i,j\le N}$, then, for every positive $k \in \mathbb{N}$ and $r_{i_0} \in E$, $i_0 \in \{1,\dots,N\}$ one has that*

$$\begin{cases} \widetilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t} = r_{i_0}) = \displaystyle\sum_{\substack{i_1,\dots,i_k=1 \\ i_1\neq i_0, i_2\neq i_1,\dots,i_k\neq i_{k-1}}}^{N} e^{-q_{i_k}(T-t)}\varphi_k(Q)\cdot\Psi_k(0,T-t,Q) \\ \widetilde{\mathbb{P}}(\nu_{t,T} = 0 | r_{\nu_t} = r_{i_0}) = e^{-q_{i_0}(T-t)} \end{cases} \tag{28}$$

with Ψ_k the following multiple integral

$$\Psi_k(\underline{s},\overline{s},Q) \triangleq \int_{\underline{s}}^{\overline{s}} e^{(q_{i_1}-q_{i_0})t_1} \int_{t_1}^{\overline{s}} e^{(q_{i_2}-q_{i_1})t_2} \cdots \int_{t_{k-1}}^{T} e^{(q_{i_k}-q_{i_{k-1}})t_k} dt_k \dots dt_2 dt_1, \ \forall 0 \le \underline{s} < \overline{s} \tag{29}$$

and

$$\varphi_k(Q) \triangleq q_{i_0,i_1}\cdot \dots \cdot q_{i_{k-1},i_k}. \tag{30}$$

The proof of Proposition 2.12 is given in the Appendix.

2.6 The explicit pricing formula

We are now able to give an explicit representation of the approximation of the price of the Prototype product introduced in Proposition 2.8:

Theorem 2.13. *Under the same hypotheses and notations of Proposition 2.8 and Lemma 2.10 and assuming that at the evaluation time t the spot rate is equal to a fixed $r_i \in E$, the approximated price $V^{\epsilon}_{\vartheta_0,t,T}(r_{\nu_t})$ of the Prototype product admits the following explicit representation*

$$V^{\epsilon}_{\vartheta_0,t,T}(r_{\nu_t})|_{r_{\nu_t}=r_i} = \sum_{k=0}^{n_\epsilon} [\widetilde{Q}^k\cdot\theta_0(\underline{r})]_i \widetilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t} = r_i), \tag{31}$$

where $\widetilde{Q}$ is defined in (24), $\theta_0(\underline{r})$ is as in Definition 2.9 and $[v]_i$ denotes, for a generic vector v, its i-th element.

Proof Follows directly from the definition of $V^{\epsilon}_{\vartheta_0,t,T}(r_{\nu_t})$ in Proposition 2.8 and formula (23). □

Remark 2.14. *By (10) and (23) it follows that the actual bond price can be expressed as $V_{\vartheta_0,t,T}(r_{\nu_t})|_{r_{\nu_t}=r_i} = \sum_{k=0}^{+\infty}[\widetilde{Q}^k\cdot\theta_0(\underline{r})]_i\widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t} = r_i)$ which is just the expression in (31) for the sum over k extended up to ∞. This expression can equivalently be written as*

$$V_{\vartheta_0,t,T}(r_{\nu_t})|_{r_{\nu_t}=r_i} = \mathbb{E}^{\widetilde{\mathbb{P}}}[[\widetilde{Q}^{\nu_{t,T}}\cdot\theta_0(\underline{r})]_i | r_{\nu_t} = r_i]. \tag{32}$$

Remark 2.15. *Under the assumption that $\widetilde{Q}$ defined by (24) is diagonalizable, $V^{\epsilon}_{\vartheta_0,t,T}(r_{\nu_t})$ admits also a matrix representation*

$$V^{\epsilon}_{\vartheta_0,t,T}(r_{\nu_t})|_{r_{\nu_t}=r_i} = e_i' \cdot S \cdot \begin{bmatrix} \mathbb{E}^{\widetilde{\mathbb{P}}}_{\epsilon}[d_1^{\nu_{t,T}}|r_{\nu_t}=r_i] & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathbb{E}^{\widetilde{\mathbb{P}}}_{\epsilon}[d_N^{\nu_{t,T}}|r_{\nu_t}=r_i] \end{bmatrix} \cdot S^{-1} \cdot \theta_0(\underline{r})$$

where S is an $N \times N$ matrix the columns of which are the eigenvectors of $\widetilde{Q}$, $(d_j)_{j=1,\dots,N}$ are the eigenvalues of $\widetilde{Q}$, e_i is the i^{th} unit vector, $\theta_0(\underline{r})$ is as in Definition 2.9 and $\mathbb{E}^{\widetilde{\mathbb{P}}}_{\epsilon}[d^{\nu_{t,T}}|r_{\nu_t}=r_i] \triangleq \sum_{k=0}^{n_\epsilon} d^k \widetilde{\mathbb{P}}(\nu_{t,T}=k|r_{\nu_t}=r_i)$ for a real number d and n_ϵ given by (19).

In the following Remark we hint at the possibility to generalize this framework by considering the interest rate as a renewal process.

Remark 2.16. *The pricing formula for the Prototype Product $V_{\vartheta_0,t,T}(r_{\nu_t}) = \sum_{k=0}^{+\infty} \vartheta_k(r_{\nu_t}) \cdot \widetilde{\mathbb{P}}(\nu_{t,T}=k|r_{\nu_t})$ given by (10) holds also when the spot rate $r(t)$ is a general renewal process which takes values in a finite set $E=\{r_1,\dots,r_N\}$. We have in fact that*

- *the functions $\vartheta_k(r_{\nu_t})$ can be represented recursively, as in Remark 2.6, by using an operator $\mathcal{T}$ defined similarly to the one introduced in (16), namely*

  $$\mathcal{T}\vartheta(v) \triangleq \mathbb{E}^{\widetilde{\mathbb{P}}}_{v}[e^{-v\mathfrak{T}}\vartheta(u)] \quad \textit{with} \quad \mathfrak{T} \sim F(q(v)), \quad \vartheta \in \mathfrak{M}$$

  where $\mathfrak{T}$ represents the interarrival time with a general distribution F which depends on the parameter $q(v)$ and $\mathfrak{M}$ is defined by (15);

- *the probabilities $\widetilde{\mathbb{P}}(\nu_{t,T}=k|r_{\nu_t})$ can be obtained by a formula similar to (25). More precisely, while in the proof of Lemma 2.11 the time τ is exponentially distributed (see formula (26)), in this case τ is distributed according to F.*

3 Bond, Cap and Swaption pricing with a Markov short rate

Once we have presented the Prototype product pricing under the assumption that the spot rate is a CTMC, we are able to give, for specific contracts (bonds, caps, swaptions, bond options,etc.), a representation of the price as a linear combination of Prototype products. We consider as price of the Prototype product the one given by the simpler formula (7) which (see Remark 2.3) can be considered as a valid alternative to the full pricing formula (9) especially when jumps are sufficiently frequent in the sense that the values of the intensities q_i are large with respect to the values r_i of the short rate. Below we shall limit ourselves to bonds, caplets and swaptions letting the matrix $\widetilde{Q}$ be as in (24) and the distribution of the random variable $\nu_{t,T}$ conditionally on $r(t)$ as given by (28).

3.1 Bond pricing

The bond price $p(t,T)$ is simply a Prototype product with a particular payoff $\vartheta_0 \equiv 1$, that is $\vartheta_0(\cdot) = \sum_{i=1}^N w_i \mathbf{I}_{\{\cdot = r_i\}}$ where $w_i = 1$, $i = 1,\ldots,N$. We shall here denote $p(t,T)$ by $p(t,T;r(t))$ because a T-bond price evaluated at time t depends on the value of the spot rate at time t. Hence a computable approximation of the bond price can be obtained from the pricing formula of a Prototype product, namely we have

Proposition 3.1. *Let $r(t)$ be a CTMC with state space E and transition kernel $Q = (q_{i,j})_{1\le i,j\le N}$; let its value, at the initial time t when already ν_t jumps have occurred, be $r(t) = r_i$ for a fixed $i = 1,\ldots,N$. For an arbitrarily small ϵ and $n_\epsilon \in \mathbb{N}$ such that*

$$n_\epsilon \ge \left\lceil \frac{\log(\epsilon(1-\gamma))}{\log(\gamma)} - \frac{\sup_{v\in E} |\sum_{i=1}^N \left(\frac{q_i}{r_i+q_i} - 1\right)\mathbf{I}_{\{v=r_i\}}|}{\log(\gamma)} \right\rceil, \tag{33}$$

it follows that, denoting by $[v]_i$ the i-th element of a generic vector v and given $\theta_0(\underline{r}) = [1,\ldots,1]' \in \mathbb{R}^N$, the quantity

$$p_\epsilon(t,T;r(t)) \triangleq \sum_{k=0}^{n_\epsilon} [\widetilde{Q}^k \cdot \theta_0(\underline{r})]_i \widetilde{\mathbb{P}}(\nu_{t,T} = k | r(t) = r_i) \tag{34}$$

approximates the bond price $p(t,T;r(t))$ in the sense that

$$|p_\epsilon(t,T;r(t)) - p(t,T;r(t))| < \epsilon \textit{ uniformly in } (t,T,r(t)). \tag{35}$$

Proof Follows directly from Theorem 2.13 for the vector $\theta_0(\underline{r}) = [1,\ldots,1]' \in \mathbb{R}^N$. □

It is interesting to observe that, by considering the spot rate $r(\cdot)$ as an homogeneous time CTMC, by Proposition 2.12 the distribution of $\nu_{t,T}$ does not depend separately on t and T but only on the length $T-t$ and consequently the quantity $p_\epsilon(t,T;r(t))$ defined in Proposition 3.1 has the same property. For this reason we shall use the following notation:

Notation 3.2. *The approximated price $p_\epsilon(t,T;r(t))$ of a bond evaluated at the time t with the initial value of the spot rate equal to $r_m \in E$ admits the following representation*

$$p_\epsilon(t,T;r(t))|_{r(t)=r_m} \quad = \quad p_\epsilon(r_m, T-t) \tag{36}$$

where $p_\epsilon(r_m,T-t) \triangleq p_\epsilon(0,T-t;r(0)|_{r(0)=r_m}$.

3.2 Cap pricing

Following the notations in Brigo-Mercurio [2], let us consider a set of payment dates $S_{\alpha,\beta} = \{S_{\alpha+1}, \ldots, S_\beta\}$, $\alpha < \beta \in \mathbb{N}$ such that, for a fixed date $t > 0$, $t < S_\alpha < S_{\alpha+1} < \ldots < S_\beta$ and this implies a set of tenors $\{s_i \triangleq S_i - S_{i-1}; i = \alpha+1, \ldots, \beta\}$. For the cap pricing we limit ourselves to the caplets because the cap is viewed as a sum of caplets. For a fixed $i \in \{\alpha+1, \ldots, \beta\}$, the i-th caplet is a call option on the Libor rate $L_i(t) \triangleq L(t, S_{i-1}, S_i) = \frac{1}{s_i}\big(\frac{p(t,S_{i-1})}{p(t,S_i)} - 1\big)$. Assuming a unitary nominal capital, we have on the given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in\mathbb{R}}, \widetilde{\mathbb{P}})$ with $\widetilde{\mathbb{P}}$ a martingale measure

$$Cpl(t, S_i) = s_i \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\int_t^{S_i} r(u)du\Big)(L_i(S_{i-1}) - K)^+\Big|\mathcal{F}_t\Big] \tag{37}$$

where K is the strike price. By double conditioning and using the fact that $p(S_{i-1}, S_i) = \mathbb{E}^{\widetilde{\mathbb{P}}}[e^{-\int_{S_{i-1}}^{S_i} r(u)du}|\mathcal{F}_{S_{i-1}}]$, we obtain the alternative representation

$$Cpl(t, S_i) = (1 + Ks_i)\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[e^{-\int_t^{S_{i-1}} r(u)du}\Big(\frac{1}{1+Ks_i} - p(S_{i-1}, S_i)\Big)^+\Big|\mathcal{F}_t\Big] \tag{38}$$

We shall here denote $Cpl(t, S_i)$ by $Cpl(t, S_i; r(t))$ because the price of the i-th caplet, evaluated at time t, depends on the value of the spot rate at time t. Considering the spot rate as a CTMC, we can particularize formula (38) as we have done for the bond pricing in (6), namely

$$\begin{aligned} &Cpl(t, S_i; r(t)) = (1 + Ks_i) \\ &\cdot\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\exp\Big(-\textstyle\sum_{j=\nu_t}^{\nu_{S_{i-1}}-1} r_j(\mathrm{T}_{j+1} - \mathrm{T}_j)\Big)\Big(\frac{1}{1+Ks_i} - p(S_{i-1}, S_i; r(S_{i-1}))\Big)^+\Big|\mathcal{F}^r_{\nu_t}\Big] \end{aligned} \tag{39}$$

We show now that the above pricing formula for the caplet can be viewed as a linear combination of N different Prototype products, whereby the infinite sum is approximated by a finite sum (see Theorem 2.13). We have in fact

Proposition 3.3. *Let $r(t)$ be a CTMC with state space E and transition kernel $Q = (q_{i,j})_{1\le i,j\le N}$; let its value, at the initial time t when ν_t jumps have already occurred, be $r(t) = r_i$ for a fixed $i = 1, \ldots, N$. Let us consider an arbitrarily small ϵ and $(n_\epsilon^m)_{m=1,\ldots,N} \in \mathbb{N}$ such that*

$$\begin{cases} n_\epsilon^m \ge \left\lceil \dfrac{\log(\epsilon(1-\gamma))}{\log(\gamma)} - \dfrac{\sup_{v\in E}\left|\psi_1^m(v) - \psi_0^m(v)\right|}{\log(\gamma)} \right\rceil, \; with \\ \psi_1^m(v) \triangleq \displaystyle\sum_{\substack{i_0,i_1=1 \\ i_0\ne i_1}}^N w_{i_0}(m)\frac{q_{i_1,i_0}}{r_{i_1} + q_{i_1}}\mathbf{I}_{\{v=r_{i_1}\}},\; \psi_0^m(v) \triangleq \displaystyle\sum_{i_0=1}^N w_{i_0}(m)\mathbf{I}_{\{v=r_{i_0}\}} \quad m = 1, \ldots, N, \\ w_{i_0}(m) = \begin{cases} 0, & i_0 \ne m \\ 1, & i_0 = m, \end{cases} \end{cases} \tag{40}$$

then, letting $V^{\epsilon}_{\psi_0^m,t,S_{i-1}}$ be as in (31) for $\vartheta_0 = \psi_0^m$, $T = S_{i-1}$ and $p_\varepsilon(r_m, s_i)$ as in (36), we have that

$$Cpl_\epsilon(t, S_i; r(t))|_{r(t)=r_l} \triangleq (1 + Ks_i) \sum_{m=1}^{N} \left(\frac{1}{1 + Ks_i} - p_\epsilon(r_m, s_i) \right)^+ V^{\epsilon}_{\psi_0^m,t,S_{i-1}}(r_l) \tag{41}$$

is a good approximation of the caplet price $Cpl(t, S_i; r(t))$ in the sense that

$$Cpl_\epsilon(t, S_i; r(t)) \xrightarrow{\epsilon \to 0} Cpl(t, S_i; r(t)) \text{ uniformly in } (t, S_{i-1}, S_i, r(t)). \tag{42}$$

Proof The price of a caplet can be written as follows

$$\begin{aligned}
&\tfrac{Cpl(t,S_i;r(t))}{1+Ks_i} \\
&= \mathbb{E}^{\widetilde{\mathbb{P}}}\left[\exp\Big(-\textstyle\sum_{j=\nu_t}^{\nu_{S_{i-1}}-1} r_j(\mathrm{T}_{j+1} - \mathrm{T}_j)\Big) \Big(\tfrac{1}{1+Ks_i} - p(S_{i-1}, S_i; r(S_{i-1})) \Big)^+ \Big| \mathcal{F}^r_{\nu_t} \right] \\
&= \mathbb{E}^{\widetilde{\mathbb{P}}}\left[\textstyle\sum_{m=1}^{N} \exp\Big(-\sum_{j=\nu_t}^{\nu_{S_{i-1}}-1} r_j(\mathrm{T}_{j+1} - \mathrm{T}_j)\Big) \Big(\tfrac{1}{1+Ks_i} - p(S_{i-1}, S_i; r_m) \Big)^+ \mathbf{I}_{\{r_{\nu_{S_{i-1}}} = r_m\}} \Big| \mathcal{F}^r_{\nu_t} \right] \\
&\overset{(36)}{=} \textstyle\sum_{m=1}^{N} \left\{ \Big(\tfrac{1}{1+Ks_i} - p(r_m, s_i) \Big)^+ \mathbb{E}^{\widetilde{\mathbb{P}}}\left[\exp\Big(-\sum_{j=\nu_t}^{\nu_{S_{i-1}}-1} r_j(\mathrm{T}_{j+1} - \mathrm{T}_j)\Big) \mathbf{I}_{\{r_{\nu_{S_{i-1}}} = r_m\}} \Big| \mathcal{F}^r_{\nu_t} \right] \right\} \\
&= \textstyle\sum_{m=1}^{N} \left\{ \Big(\tfrac{1}{1+Ks_i} - p(r_m, s_i) \Big)^+ \mathbb{E}^{\widetilde{\mathbb{P}}}\left[\exp\Big(-\sum_{j=\nu_t}^{\nu_{S_{i-1}}-1} r_j(\mathrm{T}_{j+1} - \mathrm{T}_j)\Big) \psi_0^m(r_{\nu_{S_{i-1}}}) \Big| \mathcal{F}^r_{\nu_t} \right] \right\} \\
&= \textstyle\sum_{m=1}^{N} \Big(\tfrac{1}{1+Ks_i} - p(r_m, s_i) \Big)^+ V_{\psi_0^m,t,S_{i-1}}(r_{\nu_t})
\end{aligned}$$

Hence, by using the result of Propositions 2.8 and 3.1 by which

1. $p_\epsilon(r_m, S_i - S_{i-1}) \xrightarrow{\epsilon \to 0} p(r_m, S_i - S_{i-1})$ uniformly in (S_{i-1}, S_i, r_m)
2. $V^{\epsilon}_{\psi_0^m,t,S_{i-1}}(r_l) \xrightarrow{\epsilon \to 0} V_{\psi_0^m,t,S_{i-1}}(r_l)$ uniformly in (t, S_{i-1}, r_l), $\forall m = 1, \ldots, N$

we obtain the statement. □

3.3 Swaption pricing

Using the same notations as in Subsection 3.2, a swaption is the option of entering, at a specific date S_α, a *Payer Forward Swap* (PFS), a contract in which the owner pays the "fixed leg" $\sum_{i=\alpha+1}^{\beta} Ks_i p(t, S_i)$ (that is the value at time t of the total amount to be paid with a *fixed* interest rate K along the set of payments dates $S_{\alpha,\beta}$) and receives the "floating leg" $\sum_{i=\alpha+1}^{\beta} L_i(t) s_i p(t, S_i)$ (that is the value at time t of the total amount to be received at the Libor rate $L_i(t)$ rate along $S_{\alpha,\beta}$). The swaption price is given by

$$Swopt_t(S_\alpha, S_{\alpha,\beta}) = \mathbb{E}^{\widetilde{\mathbb{P}}}\left[exp\Big(-\int_t^{S_\alpha} r(u)du \Big) (PFS^{\beta}_{\alpha}(S_\alpha, K))^+ \Big| \mathcal{F}^r_{\nu_t} \right] \tag{43}$$

where

$$PFS_\alpha^\beta(t,K) \triangleq p(t,S_\alpha) - p(t,S_\beta) - K\sum_{h=\alpha+1}^{\beta} s_h p(t,S_h) \tag{44}$$

is the value at time t of the PFS. When the short rate is given by a CTMC, the swaption price can be written as follows in accordance with what we had done for the bond pricing in (6)

$$Swopt_t(S_\alpha,S_{\alpha,\beta};r(t)) = \mathbb{E}^{\widetilde{\mathbb{P}}}\left[\exp\Big(-\sum_{j=\nu_t}^{\nu_{S_\alpha}-1} r_j(\mathrm{T}_{j+1}-\mathrm{T}_j)\Big)(PFS_\alpha^\beta(S_\alpha,K))^+\Big|\mathcal{F}_{\nu_t}^r\right] \tag{45}$$

and, similarly to the caplet price, it can be written as a linear combination of N Prototype products, whereby the infinite sum is approximated by a finite sum (see again Theorem 2.13).

Proposition 3.4. *Let $r(t)$ be a CTMC with state space E and transition kernel $Q=(q_{i,j})_{1\le i,j\le N}$; let its value, at the initial time t when ν_t jumps have already occurred, be $r(t)=r_i$ for a fixed $i=1,\ldots,N$. Let us consider an arbitrarily small ϵ and $(n_\epsilon^m)_{m=1,\ldots,N}\in\mathbb{N}$ such that*

$$\begin{cases} n_\epsilon^m \ge \left\lceil \dfrac{\log(\epsilon(1-\gamma))}{\log(\gamma)} - \dfrac{\sup_{v\in E}\left|\psi_1^m(v)-\psi_0^m(v)\right|}{\log(\gamma)}\right\rceil, \text{ with} \\ \psi_1^m(v) \triangleq \displaystyle\sum_{\substack{i_0,i_1=1\\ i_0\neq i_1}}^{N} w_{i_0}(m)\frac{q_{i_1,i_0}}{r_{i_1}+q_{i_1}}\mathbf{I}_{\{v=r_{i_1}\}},\ \psi_0^m(v) \triangleq \sum_{i_0=1}^{N} w_{i_0}(m)\mathbf{I}_{\{v=r_{i_0}\}} \quad m=1,\ldots,N, \\ w_{i_0}(m) = \begin{cases} 0, & i_0\neq m \\ 1, & i_0=m, \end{cases} \end{cases}$$

then, letting $V^\epsilon_{\psi_0^m,t,S_\alpha}$ be as in (31) for $\vartheta_0=\psi_0^m$, $T=S_\alpha$ and $p_\varepsilon(r_m,S_h-S_\alpha)$ as in (36), we have that

$$\begin{aligned} &Swopt_t^\epsilon(S_\alpha,S_{\alpha,\beta};r(t))|_{r(t)=r_l} \\ &\quad = \textstyle\sum_{m=1}^{N}\Big(1-p_\epsilon(r_m,S_\beta-S_\alpha)-K\sum_{h=1+\alpha}^{\beta} s_h p_\epsilon(r_m,S_h-S_\alpha)\Big)^+ \cdot V^\epsilon_{\psi_0^m,t,S_\alpha}(r_l) \end{aligned} \tag{46}$$

is a good approximation of the swaption price $Swopt_t^\epsilon(S_\alpha,S_{\alpha,\beta};r(t))$ in the sense that

$$Swopt_t^\epsilon(S_\alpha,S_{\alpha,\beta};r(t)) \xrightarrow{\epsilon\to 0} Swopt_t(S_\alpha,S_{\alpha,\beta};r(t)) \text{ uniformly in } (t,S_\alpha,S_{\alpha,\beta},r(t)). \tag{47}$$

Proof Analogous to the proof of Proposition 3.3 using (44). □

4 Pricing of interest rate derivatives when the short rate depends on several correlated CTMCs: a multi-factor approach

Until now we have considered the pricing of interest rate derivatives when the short rate is given by a single CTMC. However one can suppose that the value of the short rate depends on several factors (such as e.g. the credit spread, inflation rate, etc.) to obtain a more flexible model for the evolution of the spot rate. In the following we shall present a two-factor model in which the short rate is represented by a linear combination of two correlated CTMC's and we shall see how the approach developed in Section 2 can be generalized when more factors are considered. As a specific application of a two-factor model one can consider defaultable bonds.

4.1 Market model

As a means to introduce correlation we consider two CTMCs X and Y with the respective transition kernels dependent on a discrete random variable Z taking values in $\mathcal{Z} = \{z_1, \ldots, z_M\}$ with distribution $\pi = \{\pi_1, \ldots, \pi_M\} = \{\widetilde{\mathbb{P}}(Z = z_1), \ldots, \widetilde{\mathbb{P}}(Z = z_M)\}$, where $\widetilde{\mathbb{P}}$ is a martingale measure (*martingale modeling*). We make the following assumptions

Assumption 4.1. *$X(t; Z)$ denotes a CTMC with state space $E^X = \{x_1, \ldots, x_N\}$ and transition intensity matrix $Q^X(Z) = (q(Z)^X_{i,j})_{1\le i,j\le N}$ in the following sense: given a fixed value $\bar{z} \in \mathcal{Z}$, the process $X(t) \triangleq (X(t; Z)|Z = \bar{z})$ is a CTMC with state space $E^X = \{x_1, \ldots, x_N\}$ ($N \in \mathbb{N}$ and $x_i \in \mathbb{R}^+$ for each $i = 1, \ldots, N$) where*

- *$Q^X(\bar{z}) = (q(\bar{z})^X_{i,j})_{1\le i,j\le N}$ is the transition kernel homogeneous with respect the time (with $q(\bar{z})^X_{i,j} \in \mathbb{R}$),*
- *$q(\bar{z})^X_i = \sum_{\substack{j=1\\ j\neq i}}^N q(\bar{z})^X_{i,j}$, $i = 1, \ldots, N$ is the intensity associated with the state x_i.*

Moreover, being T^X_i the random time at which the i-th jump of $X(t; Z)$ occurs, we have that

- *given a generic value x_h ($h = 1, \ldots, N$) of the process $X(t; \bar{z})$ at time T^X_i, the interarrival time $\mathrm{T}^X_{i+1} - \mathrm{T}^X_i$ is exponentially distributed with parameter $q(\bar{z})^X_h$ under the measure $\widetilde{\mathbb{P}}$ (namely $(\mathrm{T}^X_{i+1} - \mathrm{T}^X_i | X(\mathrm{T}_i) = x_h) \sim \mathcal{E}xp(q(\bar{z})^X_h)$);*
- *for a generic time $s \ge 0$, $\nu(\bar{z})^X_s$ denotes the number of jumps of $X(t; Z)$ until s;*
- *The filtration generated by the process X up to the stopping time T^X_i is defined as (see (4))*

$$\mathcal{H}_{T^X_i} = \mathcal{H}^X_i = \{A \in \mathcal{F}^X_T | A \cap \{\mathrm{T}^X_i \le s\} \in \mathcal{F}^X_s, \forall s \le T\}$$

  and, for $s \in [T^X_i, T^X_{i+1})$, we put $X_i \triangleq X(s, Z)$ (similarly $X_i(\bar{z}) \triangleq X(s, \bar{z})$ if we consider a realization $\bar{z}$ of the r.v. Z);

- *for two generic times* $\underline{s}$ *and* $\overline{s}$ *such that* $\underline{s} < \overline{s}$*, the random variable* $\nu(\bar{z})^X_{\underline{s},\overline{s}}$ *denotes the number of jumps of* X *in the interval* $[\underline{s}, \overline{s})$*.*

Assumption 4.2. $Y(t;Z)$ *(analogously to the definition of* $X(t;Z)$*) denotes a CTMC with state space* $E^Y = \{y_1, \dots, y_{\widetilde{N}}\}$ *(*$\widetilde{N} \in \mathbb{N}$*) and transition intensity matrix* $Q^Y(Z) = (q(Z)^Y_{i,j})_{1 \le i,j \le \widetilde{N}}$*. The notations introduced for the stochastic process* X *are also valid for* $Y(t;Z)$*, but now the filtration generated by the process* Y *up to the stopping time* T^Y_i *is*

$$\mathcal{G}_{T^Y_i} = \mathcal{G}^Y_i = \{A \in \mathcal{F}^Y_T | A \cap \{\mathrm{T}^Y_i \le s\} \in \mathcal{F}^Y_s, \forall s \le T\}$$

and, for $s \in [T^Y_i, T^Y_{i+1})$*, we put* $Y_i \triangleq Y(s,Z)$ *(similarly* $Y_i(\bar{z}) \triangleq Y(s,\bar{z})$ *if we consider a realization* $\bar{z}$ *of the r.v.* Z*).*

Notice that, by the above definitions, the CTMCs X and Y are, conditionally on Z, mutually independent.

Let us now consider the short rate as given by

$$r(t) = a\,X(t;Z) + b\,Y(t;Z)\,, \quad a,b \in \mathbb{R}, \quad t \ge 0. \tag{48}$$

For the representation of bond prices in this market model we follow the same considerations as in Subsection 2.1.1. In particular, we shall assume without loss of generality, but also with some abuse of notation, that $t = T^X_{\nu^X_t} = T^Y_{\nu^Y_t}$ (notice that, although the event $T^X_{\nu^X_t} = T^Y_{\nu^Y_t}$ has probability zero, we make this formal assumption in the sense that, analogously to what discussed in subsection 2.1.1, the contribution to the price at time t coming from the intervals $[T^X_{\nu^X_t}, t]$ or $[T^Y_{\nu^Y_t}, t]$ can be separately precalculated on the basis of the information available at time t). Furthermore, defining on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \widetilde{\mathbb{P}})$ with $\mathcal{F}_t \triangleq \mathcal{H}_t \vee \mathcal{G}_t = \mathcal{H}_{T^X_{\nu^X_t}} \vee \mathcal{G}_{T^Y_{\nu^Y_t}}$ and $\widetilde{\mathbb{P}}$ a martingale measure

$$\bar{p}(t,T;X_{\nu^X_t},Y_{\nu^Y_t}) = \mathbb{E}^{\widetilde{\mathbb{P}}}\Bigg[exp\Big(-a\sum_{i=\nu^X_t}^{\nu(Z)^X_T-1} X_i(\mathrm{T}^X_{i+1} - \mathrm{T}^X_i) - b\sum_{j=\nu^Y_t}^{\nu(Z)^Y_T-1} Y_j(\mathrm{T}^Y_{j+1} - \mathrm{T}^Y_j)\Big)$$
$$\cdot \exp\Big(-aX_{\nu^X_T}(T - T_{\nu^X_T}) - bY_{\nu^Y_T}(T - T_{\nu^Y_T})\Big)|\mathcal{F}_t\Bigg] \tag{49}$$

and

$$p(t,T;X_{\nu^X_t},Y_{\nu^Y_t}) =$$
$$\mathbb{E}^{\widetilde{\mathbb{P}}}\Bigg[exp\Bigg(-a\sum_{i=\nu^X_t}^{\nu(Z)^X_T-1} X_i(\mathrm{T}^X_{i+1} - \mathrm{T}^X_i) - b\sum_{j=\nu^Y_t}^{\nu(Z)^Y_T-1} Y_j(\mathrm{T}^Y_{j+1} - \mathrm{T}^Y_j)\Bigg)|\mathcal{F}_t\Bigg]\,, \tag{50}$$

we obtain the following analog of Proposition 2.1, namely

Proposition 4.3. *We have*

$$\Delta := p(t,T;X_{\nu_t^X},Y_{\nu_t^Y}) - \bar{p}(t,T;X_{\nu_t^X},Y_{\nu_t^Y}) \le a\,\frac{\bar{x}}{\bar{q}^X} + b\,\frac{\bar{y}}{\bar{q}^Y}$$

where $\frac{\bar{x}}{\bar{q}^X} := \max_{i\le N}\,\frac{x_i}{q_i^X}$; $\frac{\bar{y}}{\bar{q}^Y} := \max_{j\le\tilde{N}}\,\frac{y_j}{q_j^Y}$.

Proof Putting here

$$\begin{aligned} U &:= -a\sum_{i=\nu_t^X}^{\nu(Z)_T^X-1} X_i(\mathrm{T}_{i+1}^X - \mathrm{T}_i^X) - b\sum_{j=\nu_t^Y}^{\nu(Z)_T^Y-1} Y_j(\mathrm{T}_{j+1}^Y - \mathrm{T}_j^Y) \\ &- aX_{\nu_T^X}(T - T_{\nu_T^X}) - bY_{\nu_T^Y}(T - T_{\nu_T^Y}) \\ W &:= -a\sum_{i=\nu_t^X}^{\nu(Z)_T^X-1} X_i(\mathrm{T}_{i+1}^X - \mathrm{T}_i^X) - b\sum_{j=\nu_t^Y}^{\nu(Z)_T^Y-1} Y_j(\mathrm{T}_{j+1}^Y - \mathrm{T}_j^Y) \end{aligned}$$

and following the same considerations as in the proof of Proposition 2.1, we arrive at

$$\begin{aligned} \Delta &:= p(t,T;X_{\nu_t^X},Y_{\nu_t^Y}) - \bar{p}(t,T;X_{\nu_t^X},Y_{\nu_t^Y}) \le \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[(W-U) \mid \mathcal{F}_{\nu_t}^r\Big] \\ &= \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[aX_{\nu_T^X}(T - T_{\nu_T^X}) + bY_{\nu_T^Y}(T - T_{\nu_T^Y}) \mid \mathcal{F}_{\nu_t}^r\Big] \\ &\le \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[aX_{\nu_T^X}(T_{\nu_T^X+1} - T_{\nu_T^X}) \mid \mathcal{F}_{\nu_T}^r\Big] \mid \mathcal{F}_{\nu_t}^r\Big] \\ &\quad +\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[bY_{\nu_T^Y}(T_{\nu_T^Y+1} - T_{\nu_T^Y}) \mid \mathcal{F}_{\nu_T}^r\Big] \mid \mathcal{F}_{\nu_t}^r\Big] \\ &= \mathbb{E}^{\widetilde{\mathbb{P}}}\left[a\,\frac{X_{\nu_T^X}}{q_{\nu_T^X}^X} + b\,\frac{Y_{\nu_T^Y}}{q_{\nu_T^Y}^Y} \mid \mathcal{F}_{\nu_t}^r\right] \le a\,\frac{\bar{x}}{\bar{q}^X} + b\,\frac{\bar{y}}{\bar{q}^Y} \end{aligned}$$

□

Analogously to the scalar case, also this result here states basically that, whenever the jumps of X and Y are sufficiently frequent, then one can work equally well with the simpler expression (50) for the bond prices rather than with the full expression in (49) and this is what we are going to do below, defining accordingly also the price of the Prototype product.

4.2 The Prototype product and an explicit representation for the pricing formula

In order to generalize the Definition 2.2 of the Prototype product to the present situation, we present a result which will be useful to reduce the problem of the Prototype Product pricing to one that is simpler to treat:

Lemma 4.4. *Let $X(Z) \doteq (X(s,Z), \mathcal{H}_s)_{s\in[t,T]}$ and $Y(Z) \doteq (Y(s,Z), \mathcal{G}_s)_{s\in[t,T]}$ be two stochastic processes of which the dynamics depend on a random variable Z taking values in $\mathcal{Z} = \{z_1, \ldots, z_M\}$ with distribution $\pi = \{\pi_1, \ldots, \pi_M\}$. By assuming that, conditionally on Z, the processes X and Y are independent, it follows that*

$$\mathbb{E}^{\widetilde{\mathbb{P}}}[f(X(Z))g(Y(Z))|\mathcal{H}_t \vee \mathcal{G}_t] = \sum_{h=1}^{M} \mathbb{E}^{\widetilde{\mathbb{P}}}[f(X(z_h))|\mathcal{H}_t]\mathbb{E}^{\widetilde{\mathbb{P}}}[g(Y(z_h))|\mathcal{G}_t]\pi_h, \ \forall T > t \geq 0 \tag{51}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are two generic functions.

Proof By the *tower property* of conditional expectations it follows that

$$\begin{aligned} &\mathbb{E}^{\widetilde{\mathbb{P}}}[f(X(Z))g(Y(Z))|\mathcal{H}_t \vee \mathcal{G}_t] \\ &= \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbb{E}^{\widetilde{\mathbb{P}}}[f(X(Z))g(Y(Z))|\sigma\{Z\} \vee \mathcal{H}_t \vee \mathcal{G}_t]\Big|\mathcal{H}_t \vee \mathcal{G}_t\Big] \end{aligned} \tag{52}$$

and, by using arguments of independence,

$$\begin{aligned} (52) &= \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbb{E}^{\widetilde{\mathbb{P}}}[f(X(Z))|\sigma\{Z\} \vee \mathcal{H}_t \vee \mathcal{G}_t]\mathbb{E}^{\widetilde{\mathbb{P}}}[g(Y(Z))|\sigma\{Z\} \vee \mathcal{H}_t \vee \mathcal{G}_t]\Big|\mathcal{H}_t \vee \mathcal{G}_t\Big] \\ &= \sum_{h=1}^{M} \mathbb{E}^{\widetilde{\mathbb{P}}}[f(X(z_h))|\mathcal{H}_t]\mathbb{E}^{\widetilde{\mathbb{P}}}[g(Y(z_h))|\mathcal{G}_t]\pi_h. \end{aligned} \tag{53}$$

□

We introduce now the Prototype Product when the short rate is given by (48).

Definition 4.5. *A **Prototype product** is a financial product which guarantees to deliver a certain payoff Θ_0 at maturity T. This payoff depends on the value taken by the spot rate at the date of maturity T. Under the two-factor short-rate model (48) with the factors X and Y defined as in Assumptions 4.1 and 4.2, the price of the Prototype product at time $t < T$ is, analogously to Definition 2.2, represented by*

$$V_{\Theta_0,t,T}(X_{\nu_t^X}, Y_{\nu_t^Y}) = \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[DF(t,T;r) \cdot \Theta_0(X_{\nu(Z)_T^X}, Y_{\nu(Z)_T^Y})|\mathcal{H}_t \vee \mathcal{G}_t\Big] \tag{54}$$

where

- $DF(t,T;r) \triangleq exp\left(- a \sum_{i=\nu_t^X}^{\nu(Z)_T^X - 1} X_i(\mathrm{T}_{i+1}^X - \mathrm{T}_i^X) - b \sum_{j=\nu_t^Y}^{\nu(Z)_T^Y - 1} Y_j(\mathrm{T}_{j+1}^Y - \mathrm{T}_j^Y)\right)$ *is the discount factor;*

- Θ_0 *is the **Prototype payoff** supposed to have the following form*

$$\Theta_0(x,y) = \sum_{i=1}^{N}\sum_{j=1}^{\widetilde{N}} w_i \widetilde{w}_j \mathbf{I}_{\{x=x^i\}}\mathbf{I}_{\{y=y^j\}}, \ x^i \in E^X, \ y^j \in E^Y, \ w_i, \widetilde{w}_j \in \{0\} \cup \mathbb{R}_+ \tag{55}$$

Remark 4.6. *Analogously to the full expression of the bond price in (49) one can write a full expression $\bar{V}_{\Theta_0,t,T}(X_{\nu_t^X}, Y_{\nu_t^Y})$ also for the Prototype product and it corresponds to a discount factor $\overline{DF}(t,T;r)$ given by*

$$\overline{DF}(t,T;r) = DF(t,T;r)\exp\left[-aX_{\nu_T^X}(T-T_{\nu_T^X}) - bY_{\nu_T^Y}(T-T_{\nu_T^Y})\right]$$

Since here too in the applications we have $\Theta_0(\cdot)\in[0,1]$, the same approximation as in Proposition 4.3 for the bond prices holds also for the Prototype products. Analogously to the scalar case (see Remark 2.3) we shall thus work also here with the simpler expression (54) for the Prototype product.

The price of the Prototype Product defined above in (54) can be represented, by using the result of Lemma 4.4, by an expression similar to the pricing formula of the Prototype Product in the one-factor short-rate model (see Theorem 2.13). In the following we give some notations useful for the next Theorem

- $\widetilde{Q}^X(z_h) = (\widetilde{q}(z_h)^X_{i,j})_{1\le i,j,\le N} \triangleq \begin{cases} \frac{q(z_h)^X_{i,j}}{ax_i+q(z_h)^X_i}, & i\neq j \\ 0, & i=j \end{cases} \quad \forall h=1,\dots,M;$
- $\widetilde{Q}^Y(z_h) = (\widetilde{q}(z_h)^Y_{i,j})_{1\le i,j,\le \tilde{N}} \triangleq \begin{cases} \frac{q(z_h)^Y_{i,j}}{by_i+q(z_h)^Y_i}, & i\neq j \\ 0, & i=j \end{cases} \quad \forall h=1,\dots,M;$
- $\theta_0(\underline{X}) \triangleq [w_1,\dots,w_N]'$ and $\tilde{\theta}_0(\underline{Y}) \triangleq [\tilde{w}_1,\dots,\tilde{w}_{\tilde{N}}]'$ whose components w_i and $\tilde{w}_i$ are the coefficients of the function Θ_0 defined as in (55).

Theorem 4.7. *Let us suppose the dynamics of the short rate to be given by (48) and the factors X and Y to be defined as in Assumptions 4.1 and 4.2 respectively. Assuming that $X_{\nu_t^X} = x_n$ and $Y_{\nu_t^Y} = y_m$, for an arbitrarily small ϵ we define the approximation of the actual price of the Prototype product as*

$$V^\epsilon_{\Theta_0,t,T}(X_{\nu_t^X}, Y_{\nu_t^Y})|_{X_{\nu_t^X}=x_n, Y_{\nu_t^Y}=y_m} \triangleq \sum_{h=1}^{M} U^\epsilon_{\vartheta_0,t,T}(x_n,z_h)U^\epsilon_{\tilde{\vartheta}_0,t,T}(y_m,z_h)\pi_h \tag{56}$$

where, $\forall z_h\in\mathcal{Z}$,

$$U^\epsilon_{\vartheta_0,t,T}(x_n,z_h) = \sum_{k=0}^{n^X_\epsilon(z_h)} [\widetilde{Q}^X(z_h)^k\cdot\theta_0(\underline{X})]_n \widetilde{\mathbb{P}}(\nu(z_h)^X_{t,T}=k|X_{\nu_t^X}=x_n) \tag{57}$$

and

$$U^\epsilon_{\tilde{\vartheta}_0,t,T}(y_m,z_h) = \sum_{k=0}^{n^Y_\epsilon(z_h)} [\widetilde{Q}^Y(z_h)^k\cdot\tilde{\theta}_0(\underline{Y})]_m \widetilde{\mathbb{P}}(\nu(z_h)^Y_{t,T}=k|Y_{\nu_t^Y}=y_m) \tag{58}$$

with

$$\begin{cases} n^X_\epsilon(z_h) \ge \left\lceil \frac{\log(\epsilon(1-\gamma(z_h)))}{\log(\gamma(z_h))} - \frac{\sup_{i\in\{1,\dots,N\}}|\vartheta_1(x_i,z_h)-\vartheta_0(x_i)|}{\log(\gamma(z_h))} \right\rceil, \\ \gamma(z_h) \triangleq \sup_{i\in\{1,\dots,N\}} \frac{q(z_h)^X_i}{ax_i+q(z_h)^X_i}, \end{cases}$$

and

$$\begin{cases} n^Y_\epsilon(z_h) \geq \left\lceil \frac{\log(\epsilon(1-\gamma(z_h)))}{\log(\gamma(z_h))} - \frac{\sup_{j\in\{1,\dots,\widetilde{N}\}} |\widetilde{\vartheta}_1(y_j,z_h)-\widetilde{\vartheta}_0(y_j)|}{\log(\gamma(z_h))} \right\rceil, \\ \gamma(z_h) \triangleq \sup\limits_{j\in\{1,\dots,\widetilde{N}\}} \dfrac{q(z_h)^Y_j}{by_j + q(z_h)^Y_j}. \end{cases}$$

where, for a given value of z_h*, the quantities* $\vartheta_k(x_i, z_h)$ *and* $\tilde{\vartheta}_k(y_j, z_h)$ *are defined analogously to (11) and satisfy recursions analogous to (14). Moreover, the conditional probabilities in (57) and (58) admit a representation as in Proposition 2.12 where*

- q_i *and* $q_{i,j}$ *become* $q(z_h)^X_i$ *and* $q(z_h)^X_{i,j}$ *respectively for the random variable* $\nu(z_h)^X_{t,T}$ $(\forall i, j = 1, \dots, N)$*,*
- q_i *and* $q_{i,j}$ *become* $q(z_h)^Y_i$ *and* $q(z_h)^Y_{i,j}$ *respectively for the random variable* $\nu(z_h)^Y_{t,T}$ $(\forall i, j = 1, \dots, \widetilde{N})$*.*

Then $V^\epsilon_{\Theta_0,t,T}(X_{\nu^X_t}, Y_{\nu^Y_t})$ *is a good approximation of* $V_{\Theta_0,t,T}(X_{\nu^X_t}, Y_{\nu^Y_t})$ *in the sense that*

$$V^\epsilon_{\Theta_0,t,T}(X_{\nu^X_t}, Y_{\nu^Y_t}) \xrightarrow{\epsilon\to 0} V_{\Theta_0,t,T}(X_{\nu^X_t}, Y_{\nu^Y_t}) \text{ uniformily in } (t, T, X_{\nu^X_t}, Y_{\nu^Y_t}). \tag{59}$$

Proof See Lemma 4.4 and Theorem 2.13. □

Remark 4.8. *Theorem 4.7 also implies that all the results worked out for the Prototype product pricing under a one-factor short rate model (Section 2), including the representation in Remark 2.15, carry over to the individual terms* $U^\epsilon_{\vartheta_0,t,T}$ *and* $U^\epsilon_{\widetilde{\vartheta}_0,t,T}$*.*

4.3 Bond, Cap and Swaption pricing under a two-factor short-rate model

As we have done in Section 3 when we have considered the short rate as a CTMC, we are now going to show that bonds, caps and swaptions can be represented as linear combinations of Prototype products also in the present two-factor setting; additionally, we give a representation for defaultable bonds.

We first obtain the following computable pricing formula for a bond:

Proposition 4.9. *Let us consider the same hypotheses and notations of Theorem 4.7. For an arbitrarily small* ϵ*, we define the approximation of the price of a* T*-bond at the date of evaluation* t *as*

$$p_\epsilon(t, T; X_{\nu^X_t}, Y_{\nu^Y_t})|_{X_{\nu^X_t}=x_n, Y_{\nu^Y_t}=y_m} \triangleq \sum_{h=1}^{M} U^\epsilon_{\vartheta_0,t,T}(x_n, z_h) U^\epsilon_{\widetilde{\vartheta}_0,t,T}(y_m, z_h)\pi_h \tag{60}$$

where, $\forall z_h \in \mathcal{Z}$, $U^{\epsilon}_{\vartheta_0,t,T}(x_n, z_h)$ and $U^{\epsilon}_{\tilde{\vartheta}_0,t,T}(y_m, z_h)$ are given by (57) and (58) with $\theta_0(\underline{X}) = [1,\ldots,1]' \in \mathbb{R}^N$ and $\tilde{\theta}_0(\underline{Y}) = [1,\ldots,1]' \in \mathbb{R}^{\tilde{N}}$ respectively. *The value* $p_\epsilon(t,T;X_{\nu_t^X},Y_{\nu_t^Y})$ *is a good approximation of the actual bond price* $p(t,T;X_{\nu_t^X},Y_{\nu_t^Y})$ *in the sense that*

$$p_\epsilon(t,T;X_{\nu_t^X},Y_{\nu_t^Y}) \xrightarrow{\epsilon\to 0} p(t,T;X_{\nu_t^X},Y_{\nu_t^Y}) \text{ uniformily in } (t,T,X_{\nu_t^X},Y_{\nu_t^Y}). \tag{61}$$

Remark 4.10. *Due to the time homogeneity of the CTMCs* $X(t,Z)$ *and* $Y(t,Z)$, *by analogy with Notation 3.2, we shall put* $p_\epsilon(s_i;x_n,y_m) \triangleq p_\epsilon(0,s_i;x_n,y_m)$.

As regards the caplet and the swaption pricing formulae when the short rate is given by (48), we obtain the following results

Proposition 4.11. *Let us consider the spot rate given by relation (48) and that, at the date of evaluation* t, $X_{\nu_t^X} = x_n$ *and* $Y_{\nu_t^Y} = y_m$ *for fixed* $n \in \{1,\ldots,N\}$ *and* $m \in \{1,\ldots,\tilde{N}\}$. *For an arbitrarily small* ϵ, *we define the approximation of the price of the* i*-th caplet* $Cpl(t,S_i;X_{\nu_t^X},Y_{\nu_t^Y})$ *as*

$$\frac{Cpl_\epsilon(t,S_i;X_{\nu_t^X},Y_{\nu_t^Y})|_{X_{\nu_t^X}=x_n,Y_{\nu_t^Y}=y_m}}{1+Ks_i}$$

$$\triangleq \sum_{l=1}^{N}\sum_{j=1}^{\tilde{N}}\left(\frac{1}{1+Ks_i} - p_\epsilon(s_i;x_l,y_j)\right)^+ V^{\epsilon}_{\Psi_0^{l,j},t,S_{i-1}}(x_n,y_m)$$

where, for each pair of indexes (l,j), $V^{\epsilon}_{\Psi_0^{l,j},t,S_{i-1}}$ *is defined by (56) with* $T = S_{i-1}$ *and*

$$\begin{cases} \Theta_0 = \Psi_0^{l,j}(\cdot,*) \triangleq \sum_{i_0=1}^{N}\sum_{i_1=1}^{\tilde{N}} w_{i_0}(l)\tilde{w}_{i_1}(j)\mathbf{I}_{\{\cdot=x^{i_0}\}}\mathbf{I}_{\{*=y^{i_1}\}}, \\ w_{i_0}(l) = \begin{cases} 0, & i_0 \neq l \\ 1, & i_0 = l \end{cases}, \tilde{w}_{i_1}(j) = \begin{cases} 0, & i_1 \neq j \\ 1, & i_1 = j. \end{cases} \end{cases} \tag{62}$$

The value $Cpl_\epsilon(t,S_i;X_{\nu_t^X},Y_{\nu_t^Y})$ *is a good approximation of the actual price* $Cpl(t,S_i;X_{\nu_t^X},Y_{\nu_t^Y})$ *in the sense that*

$$Cpl_\epsilon(t,S_i;X_{\nu_t^X},Y_{\nu_t^Y}) \xrightarrow{\epsilon\to 0} Cpl(t,S_i;X_{\nu_t^X},Y_{\nu_t^Y}) \text{ uniformly in } (t,S_{i-1},S_i,X_{\nu_t^X},Y_{\nu_t^Y}).$$

Proposition 4.12. *Under the assumptions of the previous Proposition we define, for an arbitrarily small* ϵ, *the approximation of the price of the swaption* $Swopt_t(S_\alpha,S_{\alpha,\beta};X_{\nu_t^X},Y_{\nu_t^Y})$ *as*

$$Swopt_t^{\epsilon}(S_\alpha,S_{\alpha,\beta};X_{\nu_t^X},Y_{\nu_t^Y})|_{X_{\nu_t^X}=x_n,Y_{\nu_t^Y}=y_m}$$

$$= \sum_{l=1}^{N}\sum_{j=1}^{\tilde{N}}(g^\epsilon(S_\alpha,S_{\alpha,\beta};x_l,y_j))^+ V^{\epsilon}_{\Psi_0^{l,j},t,S_{i-1}}(x_n,y_m)$$

where $g^\epsilon(S_\alpha,S_{\alpha,\beta};x_l,y_j) = 1 - p_\epsilon(S_\beta - S_\alpha;x^l,y^j) - K\sum_{h=\alpha+1}^{\beta} s_h p_\epsilon(S_h - S_\alpha;x_l,y_j)$ *(see Proposition 3.4) and, for each pair of indexes* (l,j), $V^{\epsilon}_{\Psi_0^{l,j},t,S_\alpha}$ *is defined by (56)* *with* $\Theta_0 = \Psi_0^{l,j}$ *as given in (62) and* $T = S_\alpha$*. The value* $Swopt_t^\epsilon(S_\alpha, S_{\alpha,\beta}; X_{\nu_t^X}, Y_{\nu_t^Y})$ *is a good approximation of the actual price* $Swopt_t(S_\alpha, S_{\alpha,\beta}; X_{\nu_t^X}, Y_{\nu_t^Y})$ *in the sense that*

$$Swopt_t^\epsilon(S_\alpha, S_{\alpha,\beta}; X_{\nu_t^X}, Y_{\nu_t^Y}) \xrightarrow{\epsilon \to 0} Swopt_t^\epsilon(S_\alpha, S_{\alpha,\beta}; X_{\nu_t^X}, Y_{\nu_t^Y}) \tag{63}$$

uniformly in $(t, S_\alpha, S_{\alpha,\beta}, X_{\nu_t^X}, Y_{\nu_t^Y})$*.*

Furthermore, the Prototype Product pricing approach of Subsection 4.2 can be used to price defaultable bonds, as mentioned in the following

Remark 4.13. *In a general setting of the reduced form approach to credit risk, the price of a defaultable bond at time of today* t *and maturity* T *can be written as*

$$\Pi(t,T) = \mathbf{I}_{\{\tau > t\}} \mathbb{E}^{\widetilde{\mathbb{P}}}[e^{-\int_t^T r(s)+\lambda(s)ds} | \mathcal{F}_t]$$

where the processes r *and* λ *represent the spot rate and the default intensity respectively,* τ *is the time of default and* $\mathcal{F}_t$ *is the filtration generated by the two-dimensional process* (r, λ)*. If we suppose that* $r(t) \equiv X(t,Z)$ *as defined in Assumption 4.1 and the default intensity* $\lambda(t) \equiv Y(t,Z)$ *as defined in Assumption 4.2, then the price of a defaultable bond* $\Pi(t,T)$ *admits a representation as in (50) with* $a = b = 1$ *which is the pricing formula of a default free bond when the spot rate depends on the two correlated CTMCs* $X(t,Z)$ *and* $Y(t,Z)$*.*

5 Numerical results when the short rate is a single-factor CMTC

The aim of this Section is to test numerically the pricing approach that we developed in the paper and that we call ***"Prototype Product Approach"***. We limit ourselves to a single factor CTMC where we compare our results with the exact ones in a case where explicit formulae are available. Further numerical results concerning a two-factor short rate model can be found in [9].

For the numerical tests we shall treat only the pricing of zero-coupon bonds because, as seen in the previous sections, in our approach the prices of caps and swaptions can be written as functions of prices of bonds and other Prototype Products (this holds also for bond options). We shall test numerically the validity of our approach by proceeding as follows: consider a continuous time short-rate model for which the bond price admits an explicit closed formula and compare this exact price with the one obtained by the Prototype Product Approach after approximating the short rate by a CTMC.

5.1 A continuous time model and its CTMC approximation (Kushner approximation)

Let us choose the following continuous-time affine model for the short rate, known as the Cox-Ingersoll-Ross (**CIR**) or square-root model

$$\begin{cases} dr(t) = k(\theta - r(t))dt + \sigma\sqrt{r(t)}dW_t \\ r(0) = \widetilde{r} \end{cases} \tag{64}$$

where W_t is a Wiener process under an equivalent martingale measure $\tilde{\mathbb{P}}$ as introduced in Section 2 and the long-run mean θ, the rate of mean reversion k, the volatility σ and the initial spot rate $\tilde{r}$ are positive constants. Moreover, to ensure that the process remains positive, the following condition has to be satisfied

$$2k\theta > \sigma^2. \tag{65}$$

We approximate this square-root process by a CTMC using a suitable approximation that we call *"Kushner approximation"* and that we summarize next (for details see [8], in particular [7]). Denote by $r^{h,n}(t)$ the CTMC obtained by first discretizing with respect to the space variable (with spatial step length h) the infinitesimal generator of the diffusion $r(t)$, thus obtaining a denumerable CTMC $\{r^h(t)\}$, and then stopping $r^h(t)$ at the boundary of the interval $I = (0, N)$ (with $N \triangleq hn$ suitably chosen where n represents the number of subintervals into which I is divided). We have that the state space of $r^{h,n}(t)$ is given by

$$E^{h,n} = \{r_0, \ldots, r_N\} = \{0, h, \ldots, h(n-1), hn\} \in \mathbb{R}^{N+1} \tag{66}$$

and the transition intensity kernel is represented by the matrix $Q^{h,n} = (q^{h,n}_{i,j})_{\{1 \le i,j \le N+1\}}$ with the first and last rows identically equal to zero (absorption at the boundary) and with the i-th row given by

$$[0, \ldots, 0, q^{h,n}_-(r_i), q^{h,n}(r_i), q^{h,n}_+(r_i), 0, \ldots, 0], \tag{67}$$

where $q^{h,n}(r_i)$ is in the diagonal and

$$\begin{cases} q^{h,n}_-(r_i) = \frac{(k(\theta - r_i))^-}{h} + \frac{\sigma^2 r_i}{2h^2} \\ q^{h,n}(r_i) = -\frac{|k(\theta - r_i)|}{h} - \frac{\sigma^2 r_i}{h^2} \\ q^{h,n}_+(r_i) = \frac{(k(\theta - r_i))^+}{h} - \frac{\sigma^2 r_i}{2h^2} \end{cases} \tag{68}$$

with $(\cdot)^+$ and $(\cdot)^-$ denoting the positive and negative parts respectively. Moreover, the intensity associated with a generic state $r_i \in E^{h,n}$ can be represented by

$$q^{h,n}_i = \sum_{\substack{j=1 \\ j \ne i}}^{N+1} q^{h,n}_{i,j} = -q^{h,n}(r_i). \tag{69}$$

The CTMC $r^{h,n}(t)$ converges to $r(t)$ as $n \to +\infty$ and $h \to 0$ in the sense of weak convergence of the induced probability measures.

Once discretized, the short rate becomes a CTMC and so we can compute bond prices with the Prototype Product Approach.

Remark 5.1. *If the spot rate is a CTMC $r^{h,n}(t)$ given by the Kushner approximation, then*

$$\frac{r_i}{q^{h,n}(r_i)} = \frac{h(i-1)}{\frac{|k(\theta - h(i-1))|}{h} + \frac{\sigma^2}{h^2}} \qquad i = 1, \ldots, N+1. \tag{70}$$

By considering h at most of the order of 10^{-2} and $h(i-1)$ at most equal to 0.03 as well as suitable CIR parameters as in the numerical results below, the bound on the difference between the full and the simplified bond prices in Proposition 2.1 as well as on the difference between the Prototype products (see Remark 2.3), namely $\frac{\bar{r}}{\bar{q}} := \max_{i \leq N} \frac{r_i}{q_{r_i}}$ is close to zero. In fact, as can be seen from the numerical results below, the prices computed by the Prototype Product Approach and based on the simplified bond pricing formula (6) are very close to their exact values computed with the explicit closed formula for the original continuous time model.

5.2 Computing and comparing bond prices

We shall price zero-coupon bonds with the Prototype Product Approach both by computing the explicit formulae, derived in the previous part of the paper, as well as by a full simulation approach based on Monte Carlo techniques. Moreover, in order to have a further possibility of comparison, we shall also consider a widely used approach, namely the *lattice method*, to compute approximations of the bond price starting from the continuous-time affine short rate model. We shall thus compute prices in the following four ways:

a) ***Explicit Closed formula***

b) ***Lattice Method***

c) ***Prototype Product Approach*** (after approximating the diffusion by a CTMC with the ***Kushner approximation***):

- **c.1)** ***Explicit Formulae***
- **c.2)** ***Monte Carlo simulations***

We are now going to describe in more detail each of the just mentioned alternatives.

5.2.1 Explicit Closed formula

Under the CIR affine term structure model, the price at time t of a zero-coupon bond with maturity T is given by

$$p(t,T) = A(t,T)e^{-B(t,T)r(t)} \tag{71}$$

with

$$\begin{cases} A(t,T) = \left(\frac{2he^{\frac{(k+h)(T-t)}{2}}}{2h+(k+h)(e^{h(T-t)}-1)}\right)^{\frac{2k\theta}{\sigma^2}} \\ B(t,T) = \frac{2e^{(e^{h(T-t)}-1)}}{2h+(k+h)(e^{h(T-t)}-1)} \\ h = \sqrt{k^2+2\sigma^2}. \end{cases} \tag{72}$$

For more details see Brigo-Mercurio [2].

5.2.2 Lattice Method

The lattice method is widely used in finance and it consists in building a recombining tree which approximates the evolution of a diffusion process (in this case the short rate as given by the CIR model).

Here we consider the lattice algorithm suggested in Costabile-Leccadito-Massabó [3] who propose an approach based on a direct discretization of the process $r(t)$ by means of a recombining binomial tree with a number of nodes that grows linearly with the number of steps; then, by an argument based on absence of arbitrage, they compute the bond price by working backwards along the tree. To solve a frequent problem in lattice methods, namely that the transition probabilities have to belong in $[0, 1]$, the authors introduce *multiple upward* and *downward jumps* that satisfy an appropriate matching condition.

5.2.3 Prototype Product Approach

Starting from a CIR affine term structure model as in (64) for the short rate, in order to apply the Prototype Product Approach to the bond pricing, we have first to approximate the short rate $r(t)$, which is a diffusion process, by a CTMC and for this we use the Kushner approximation that we had summarized in section 5.1. We shall denote h and n by ***h(K-A)*** and ***n(K-A)*** respectively. Below we refer to the first alternative of the Prototype Product Approach (see (c.1)) as ***"Explicit Formulae"*** and the second (see (c.2)) as ***"Monte Carlo simulations"***.

Prototype Product Approach (Explicit Formulae)

According to Proposition 3.1 an ϵ-approximation of the bond price is given by the explicitly computable formula

$$p_\epsilon(t, T; r^{h,n}(t))|_{r^{h,n}(t)=r_i} = \sum_{k=0}^{n_\epsilon} [\widetilde{Q^{h,n}}^k \cdot \theta_0(\underline{r}^{h,n})]_i \widetilde{\mathbb{P}}(\nu_{t,T} = k | r^{h,n}_{\nu_t} = r_i) \tag{73}$$

where the spot rate is $r^{h,n}(t)$ and we have denoted here by $\widetilde{Q^{h,n}}$ the matrix $\widetilde{Q}$ that was introduced in (24). With respect to its general definition in (24) the matrix $\widetilde{Q^{h,n}}$ takes here a simpler form because of the tridiagonal structure of the transition kernel $Q^{h,n}$: the first and the last rows are identically equal to zero and the j-th row is given by

$$\left[0, \ldots, 0, \frac{q^{h,n}_-(r_j)}{r_j - q^{h,n}(r_j)}, 0, \frac{q^{h,n}_+(r_j)}{r_j - q^{h,n}(r_j)}, 0, \ldots, 0\right]$$

where the $(j + 1)$-th and the $(j - 1)$-th terms are different from zero.

The most demanding part in the computations according to (73) are the probabilities $\widetilde{\mathbb{P}}(\nu_{t,T} = k | r^{h,n}_{\nu_t} = r_i)$. This can be avoided by the next alternative based on Monte Carlo simulations. Details can be found in [9].

Prototype Product Approach (Monte Carlo simulations)

By (32) we have that

$$p(t,T;r^{h,n}(t))|_{r^{h,n}(t)=r_i} = \mathbb{E}^{\widetilde{\mathbb{P}}}[[\widetilde{Q^{h,n}}^{\nu_{t,T}} \cdot \theta_0(\underline{r}^{h,n})]_i | r^{h,n}_{\nu_t} = r_i]. \tag{74}$$

An approach to compute the price in the above expression can then also be obtained by using the *MonteCarlo* technique, that is based on

$$\frac{1}{M}\sum_{l=1}^{M}[\widetilde{Q^{h,n}}^{\nu^l_{t,T}} \cdot \theta_0(\underline{r}^{h,n})]_i \overset{M\to\infty}{\longrightarrow} p(t,T;r^{h,n}(t))|_{r^{h,n}(t)=r_i} \quad \widetilde{\mathbb{P}} - \text{a.s.} \tag{75}$$

where $\nu^l_{t,T}$ is the l-th simulation outcome of the random variable $\nu_{t,T}$.

5.3 The actual numerical results

We present now some tables where the bond prices, for several maturities and several values of the CIR parameters in (64), are obtained as follows

CF: the exact closed formula;

RBT: the lattice method, namely the recombining binomial tree according to [3], where we have chosen a number of steps ***"stepsRBT"*** always equal to 500;

PPA(EF)+K-A: the Prototype Product Approach after discretizing the short rate with the Kushner approximation (**K-A**) and by using the explicit formulae (**EF**) discussed in our study;

PPA(MC)+K-A: the Prototype Product Approach after discretizing the short rate with the Kushner approximation (**K-A**) and by using a full simulation approach based on the Monte Carlo technique (**MC**). We have chosen the number of steps for the Monte Carlo simulations ***"stepsRBT"***, namely M in formula (75), always equal to 500.

As regards the other parameters, we have considered the date of today as $t = 0$ years, three different times of maturity T (namely 0.5, 2 and 5 years) and the parameters of the CIR model such that the condition (65) is verified. In Table 1 and Table 2 the numerical results relative to **CF**, **RBT** and **PPA(MC)+K-A** are presented when the values of the initial spot rate $\widetilde{r}$ and the mean-reversion constant θ in formula (64) are of the order of one hundredth; in Table 3 we present also some results relative to **PPA(EF)+K-A** when $\widetilde{r}$ and θ are of the order of one tenth.

We have performed the numerical simulations on a single core Intel x86 Linux machine equipped with 2GB of RAM and we have implemented a C/C++ framework by using the well-known GNU Scientific Library to handle the data structure.

Table 1: bond prices with **CF**, **RBT** and **PPA(MC)+K-A** (***stepsMC***=***stepsRBT***=500)

T(years)	0.5	2	5	0.5	2	5
$\widetilde{r}$	0.01	0.01	0.01	0.02	0.02	0.02
θ	0.01	0.01	0.01	0.02	0.02	0.02
k	0.8	0.8	0.8	0.5	0.5	0.5
σ	0.1	0.1	0.1	0.05	0.05	0.05
n(K-A)	600	600	700	600	600	700
h(K-A)	0.00005	0.00005	0.00005	0.0001	0.0001	0.0001
CF	0.995014	0.980245	0.951463	0.990051	0.960822	0.905047
RBT	0.995042	0.980302	0.951556	0.99007	0.960898	0.905226
PPA(MC)+K-A	**0.995024**	**0.980276**	0.951621	0.990143	0.960734	0.905318

Table 2: bond prices with **CF**, **RBT** and **PPA(MC)+K-A** (***stepsMC=stepsRBT***=500)

T(years)	0.5	2	5	0.5	2	5
$\widetilde{r}$	0.03	0.03	0.03	0.02	0.02	0.02
θ	0.03	0.03	0.03	0.02	0.02	0.02
k	1.1	1.1	1.1	1.2	1.2	1.2
σ	0.1	0.1	0.1	0.1	0.1	0.1
n(K-A)	600	600	700	600	600	700
h(K-A)	0.00015	0.00015	0.00015	0.0001	0.0001	0.0001
CF	0.985116	0.941861	0.861095	0.990053	0.960849	0.905072
RBT	0.985146	0.941974	0.86135	0.990072	0.960926	0.905251
PPA(MC)+K-A	**0.985128**	**0.941968**	**0.861319**	**0.990059**	0.95647	0.90193

Table 3: bond prices with **CF**, **RBT**, **PPA(MC)+K-A** and **PPA(EF)+K-A** (***stepsMC=stepsRBT***=500)

T(years)	0.5	0.5	0.5	0.5
$\widetilde{r}$	0.1	0.1	0.2	0.3
θ	0.1	0.1	0.2	0.3
k	0.1	0.4	0.2	0.3
σ	0.1	0.05	0.2	0.3
n(K-A)	300	300	300	300
h(K-A)	0.01	0.01	0.02	0.03
CF	0.951249	0.951234	0.904977	0.86114
RBT	0.951343	0.951329	0.905157	0.861394
PPA(MC)+K-A	0.951022	0.950859	0.905229	0.861104
PPA(EF)+K-A	**0.951324**	0.951723	**0.905012**	0.861756

Remark 5.2. *Both results with* ***PPA(EF)+K-A*** *and* ***PPA(MC)+K-A*** *are competitive with the* ***RBT*** *method and they generally differ only at the fourth decimal digit. In fact, in spite of the Kushner approximation which is necessary for the comparison with the prices of the continuous affine term structure model given in (71)-(72), our methods based on the Prototype Product pricing work roughly as the lattice method which does not require previously any approximation to be applied and consequently does not feel the effect of the error due to* ***K-A****. Furthermore,* ***PPA(EF)+K-A*** *and* ***PPA(MC)+K-A*** *work sometimes better than the lattice methods (see results in bold). In any case our approach is designed for r given directly by a CTMC and the Kushner approximation was introduced only for comparison purposes.*

Moreover it is known that lattice methods work well under a one-factor short rate model, but it becomes more difficult to implement them if the short rate depends on several correlated processes (for an up-to-date description see [6]). On the contrary, the Prototype Product Approach applies well also to a particular multi-factor short rate model (see Section 4) and in this case it can be numerically implemented (see [9]).

5.4 Conclusions

We briefly sum up the results obtained in this chapter: if we consider a one-factor short rate model, the Prototype Product Approach, by using either the explicit formulae or the Monte Carlo simulations, is competitive with the lattice method which is widely used to compute the price of zero-coupon bonds. To allow for such a comparison we had to start from a continuous time diffusion model which required a preliminary discretization to obtain a CTMC for which our methods are designed. Moreover we are able to obtain numerical results for prices of caps, swaptions and bond options with the same complexity as required for the computation of bond prices (considered as a particular case of Prototype Product) because all the prices of these interest rate derivatives can be viewed as linear combinations of Prototype Product prices (see Subsections 3.2-3.3).

Appendix

Proof of Lemma 2.5.
Recall the definition of $\mathcal{F}^r_{\mathrm{T}_i}$ in (4) and that, for simplicity of notation, we have put $\mathcal{F}^r_i = \mathcal{F}^r_{\mathrm{T}_i}$. Also put $\mathcal{F}^{\mathrm{T}}_k = \sigma\{\mathrm{T}_i; i \leq k\}$. Inspired by Filipović-Zabczyk [5], we can divide the proof into k steps:

1st STEP

$$
\begin{aligned}
& \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\eta}^{\eta+k-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_0(r_{\eta+k})|\mathcal{F}^r_\eta\Big] \\
= \; & \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbb{E}^{\widetilde{\mathbb{P}}}\big[exp\Big(-\sum_{i=\eta}^{\eta+k-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_0(r_{\eta+k})|\mathcal{F}^r_{\eta+k-1}\vee\mathcal{F}^{\mathrm{T}}_{\eta+k-1}\big]\Big|\mathcal{F}^r_\eta\Big] \\
= \; & \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-r_{\eta+k-1}(\mathrm{T}_{\eta+k}-\mathrm{T}_{\eta+k-1})}exp\Big(-\sum_{i=\eta}^{\eta+k-2} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_0(r_{\eta+k})|\mathcal{F}^r_{\eta+k-1}\vee\mathcal{F}^{\mathrm{T}}_{\eta+k-1}\big]\Big|\mathcal{F}^r_\eta\Big] \\
= \; & \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\eta}^{\eta+k-2} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-r_{\eta+k-1}(\mathrm{T}_{\eta+k}-\mathrm{T}_{\eta+k-1})}\vartheta_0(r_{\eta+k})|\mathcal{F}^r_{\eta+k-1}\vee\mathcal{F}^{\mathrm{T}}_{\eta+k-1}\big]\Big|\mathcal{F}^r_\eta\Big] \\
= \; & \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\eta}^{\eta+k-2} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-r_{\eta+k-1}(\mathrm{T}_{\eta+k}-\mathrm{T}_{\eta+k-1})}\vartheta_0(r_{\eta+k})|r_{\eta+k-1}\big]\Big|\mathcal{F}^r_\eta\Big] \qquad (76)
\end{aligned}
$$

where the last passage is due to the fact that, for a generic $i \in \mathbb{N}$, conditionally on $\mathcal{F}^r_i \vee \mathcal{F}^{\mathrm{T}}_i$, both the distributions of the interarrival time $\mathrm{T}_{i+1} - \mathrm{T}_i$ and of the visited state r_{i+1} depend only on the initial state r_i by the properties of the CTMC's; hence

$$
(76) \overset{(14)}{=} \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\eta}^{\eta+k-2} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_1(r_{\eta+k-1})\Big|\mathcal{F}^r_\eta\Big]. \qquad (77)
$$

2^{nd} STEP

$$
\begin{aligned}
(76)\quad &= \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\eta}^{\eta+k-2} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_1(r_{\eta+k-1})\Big|\mathcal{F}^r_\eta\Big] \\
= \quad & \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbb{E}^{\widetilde{\mathbb{P}}}\big[exp\Big(-\sum_{i=\eta}^{\eta+k-2} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_1(r_{\eta+k-1})\big|\mathcal{F}^r_{\eta+k-2}\vee\mathcal{F}^{\mathrm{T}}_{\eta+k-2}\big]\Big|\mathcal{F}^r_\eta\Big] \\
= \quad & \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-r_{\eta+k-2}(\mathrm{T}_{\eta+k-1}-\mathrm{T}_{\eta+k-2})}\vartheta_1(r_{\eta+k-1})exp\Big(-\sum_{i=\eta}^{\eta+k-3} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\big|\mathcal{F}^r_{\eta+k-2}\vee\mathcal{F}^{\mathrm{T}}_{\eta+k-2}\big]\Big|\mathcal{F}^r_\eta\Big] \\
= \quad & \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\eta}^{\eta+k-3} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-r_{\eta+k-2}(\mathrm{T}_{\eta+k-1}-\mathrm{T}_{\eta+k-2})}\vartheta_1(r_{\eta+k-1})\big|\mathcal{F}^r_{\eta+k-2}\vee\mathcal{F}^{\mathrm{T}}_{\eta+k-2}\big]\Big|\mathcal{F}^r_\eta\Big] \\
= \quad & \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\eta}^{\eta+k-3} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-r_{\eta+k-2}(\mathrm{T}_{\eta+k-1}-\mathrm{T}_{\eta+k-2})}\vartheta_1(r_{\eta+k-1})\big|r_{\eta+k-2}\big]\Big|\mathcal{F}^r_\eta\Big] \\
= \quad & \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\eta}^{\eta+k-3} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_2(r_{\eta+k-2})\Big|\mathcal{F}^r_\eta\Big] \qquad (78)
\end{aligned}
$$

where the last passage is again justified by the properties of the CTMC's recalled in the first step; hence, recursively until the last step, we obtain

$\mathbf{k}^{\text{th}}$ STEP

$$
\begin{aligned}
&\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\eta}^{\eta+k-1} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_0(r_{\eta+k})|\mathcal{F}^r_\eta\Big] = \cdots = \\
&\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[exp\Big(-\sum_{i=\eta}^{\eta+k-k} r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)\Big)\vartheta_{k-1}(r_{\eta+k-(k-1)})\Big|\mathcal{F}^r_\eta\Big] = \\
&\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[e^{r_\eta(\mathrm{T}_{\eta+1}-\mathrm{T}_\eta)}\vartheta_{k-1}(r_{\eta+1})\Big|\mathcal{F}^r_\eta\Big] \overset{(14)}{=} \vartheta_k(r_\eta). \qquad \square
\end{aligned}
$$

Proof of Lemma 2.10.
At first we prove by induction that the coefficients w^k_m of the function ϑ_k have, for $m = 1, \ldots, N$, the following explicit representation

$$
\begin{cases}
w^1_m = \displaystyle\sum_{\substack{i_0=1\\ i_0\neq m}}^{N} w_{i_0}\frac{q_{m,i_0}}{r_m+q_m}, & k=1\\[2ex]
w^k_m = \displaystyle\sum_{\substack{i_0,\ldots,i_{k-1}=1\\ i_0\neq i_1,\ldots,i_{k-1}\neq m}}^{N} w_{i_0}\frac{q_{m,i_{k-1}}}{r_m+q_m}\left[\prod_{h=1}^{k-1}\frac{q_{i_h,i_{h-1}}}{r_{i_h}+q_{i_h}}\right], & k>1
\end{cases}
\tag{79}
$$

We consider w.l.o.g the functions ϑ_k evaluated in r_i, the state of the rate process at a generic transition time T_i.

Base Case $(k=1)$

$$\begin{aligned}
\vartheta_1(r_i) &\overset{(14)}{=} \mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)}\vartheta_0(r_{i+1})\big|r_i\big] \\
&\overset{(8)}{=} \mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)}\sum_{i_0=1}^{N} w_{i_0}\mathbf{I}_{\{r_{i+1}=r_{i_0}\}}\big|r_i\big] \\
&= \sum_{i_0,i_1=1}^{N} w_{i_0}\mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)}\mathbf{I}_{\{r_{i+1}=r_{i_0}\}}\big|r_i=r_{i_1}\big]\mathbf{I}_{\{r_i=r_{i_1}\}}
\end{aligned} \tag{80}$$

Now

$$\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)}\mathbf{I}_{\{r_{i+1}=r_{i_0}\}}\Big|r_i=r_{i_1}\Big] = \mathbb{E}^{\widetilde{\mathbb{P}}}\Big[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)}\Big|r_i=r_{i_1}\Big]\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbf{I}_{\{r_{i+1}=r_{i_0}\}}\Big|r_i=r_{i_1}\Big] \tag{81}$$

because, conditionally on r_i, the interarrival time $\mathrm{T}_{i+1}-\mathrm{T}_i$ and r_{i+1} (the value of the process at the transition time T_{i+1}) are independent by the properties of the CTMCs. Moreover we have that

- $\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)}\Big|r_i=r_{i_1}\Big] = \int_0^\infty e^{-r_{i_1}u}q_{i_1}e^{-q_{i_1}u}du = \frac{q_{i_1}}{r_{i_1}+q_{i_1}}$ because $\mathrm{T}_{i+1}-\mathrm{T}_i$ is exponentially distributed in accordance with (2),
- $\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[\mathbf{I}_{\{r_{i+1}=r_{i_0}\}}\Big|r_i=r_{i_1}\Big] = \widetilde{\mathbb{P}}(r_{i+1}=r_{i_0}|r_i=r_{i_1}) = p_{i_1,i_0}$ the transition probability from state r_{i_1} to r_{i_0};

hence

$$\begin{aligned}
\mathbb{E}^{\widetilde{\mathbb{P}}}\Big[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)}\mathbf{I}_{\{r_{i+1}=r_{i_0}\}}\Big|r_i=r_{i_1}\Big] &\overset{(81)}{=} p_{i_1,i_0}\frac{q_{i_1}}{r_{i_1}+q_{i_1}} \\
&\overset{(1)}{=} \frac{q_{i_1,i_0}}{r_{i_1}+q_{i_1}}, \quad \forall i_0\neq i_1
\end{aligned} \tag{82}$$

We obtain thus that

$$\vartheta_1(r_i) \overset{(80)}{=} \sum_{\substack{i_0,i_1=1\\ i_0\neq i_1}}^{N} w_{i_0}\frac{q_{i_1,i_0}}{r_{i_1}+q_{i_1}}\mathbf{I}_{\{r_i=r_{i_1}\}} = \sum_{i_1=1}^{N} w_{i_1}^1\mathbf{I}_{\{r_i=r_{i_1}\}} \tag{83}$$

where $w_{i_1}^1 = \sum_{\substack{i_0=1\\ i_0\neq i_1}}^{N} w_{i_0}\frac{q_{i_1,i_0}}{r_{i_1}+q_{i_1}}$ defined as in (79).

Inductive step
By Lemma 2.5

$$\vartheta_k(r_i) = \mathbb{E}^{\widetilde{\mathbb{P}}}\big[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)}\vartheta_{k-1}(r_{i+1})\big|r_i\big] = \mathbb{E}^{\widetilde{\mathbb{P}}}\Bigg[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)}\sum_{i_{k-1}=1}^{N} w_{i_{k-1}}^{k-1}\mathbf{I}_{\{r_{i+1}=r_{i_{k-1}}\}}\Bigg]. \tag{84}$$

By the induction hypothesis

$$(84) \quad = \mathbb{E}^{\widetilde{\mathbb{P}}}\left[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)} \sum_{\substack{i_0,\dots,i_{k-1}=1 \\ i_0\neq i_1,\dots,i_{k-2}\neq i_{k-1}}}^{N} w_{i_0} \left[\prod_{h=1}^{k-1} \frac{q_{i_h,i_{h-1}}}{r_{i_h}+q_{i_h}} \right] \mathbf{I}_{\{r_{i+1}=r_{i_{k-1}}\}} \big| r_i \right]$$

$$= \sum_{\substack{i_0,\dots,i_{k-1}=1 \\ i_0\neq i_1,\dots,i_{k-2}\neq i_{k-1}}}^{N} w_{i_0} \left[\prod_{h=1}^{k-1} \frac{q_{i_h,i_{h-1}}}{r_{i_h}+q_{i_h}} \right] \mathbb{E}^{\widetilde{\mathbb{P}}}\left[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)} \mathbf{I}_{\{r_{i+1}=r_{i_{k-1}}\}} \big| r_i \right]$$

$$= \sum_{i_k=1}^{N} \Bigg\{ \sum_{\substack{i_0,\dots,i_{k-1}=1 \\ i_0\neq i_1,\dots,i_{k-2}\neq i_{k-1}}}^{N} w_{i_0} \left[\prod_{h=1}^{k-1} \frac{q_{i_h,i_{h-1}}}{r_{i_h}+q_{i_h}} \right]$$

$$\mathbb{E}^{\widetilde{\mathbb{P}}}\left[e^{-r_i(\mathrm{T}_{i+1}-\mathrm{T}_i)} \mathbf{I}_{\{r_{i+1}=r_{i_{k-1}}\}} \big| r_i = r_{i_k} \right] \mathbf{I}_{\{r_i=r_{i_k}\}} \Bigg\}$$

$$\overset{(82)}{=} \sum_{i_k=1}^{N} \Bigg\{ \sum_{\substack{i_0,\dots,i_{k-1}=1 \\ i_0\neq i_1,\dots,i_{k-1}\neq i_k}}^{N} w_{i_0} \frac{q_{i_k,i_{k-1}}}{r_{i_k}+q_{i_k}} \left[\prod_{h=1}^{k-1} \frac{q_{i_h,i_{h-1}}}{r_{i_h}+q_{i_h}} \right] \Bigg\} \mathbf{I}_{\{r_i=r_{i_k}\}}$$

$$= \sum_{\substack{i_0,\dots,i_k=1 \\ i_0\neq i_1,\dots,i_{k-1}\neq i_k}}^{N} w_{i_0} \left[\prod_{h=1}^{k} \frac{q_{i_h,i_{h-1}}}{r_{i_h}+q_{i_h}} \right] \mathbf{I}_{\{r_i=r_{i_k}\}} = \sum_{i_k=1}^{N} w^k_{i_k} \mathbf{I}_{\{r_i=r_{i_k}\}}.$$

Now, by using the representation in (79), we prove relation (23) by induction.

Base Case $k = 0$
By the Definition 2.9 we have

$$\theta_0(\underline{r}) = [w_1, \dots, w_N]' = \widetilde{Q}^0 \cdot \theta_0(\underline{r})$$

Inductive step
Observing that relation (23) is equivalent to

$$\theta_k(\underline{r}) = \widetilde{Q} \cdot \theta_{k-1}(\underline{r}), \; k > 0 \tag{85}$$

it is sufficient to prove (85) by using the induction hypothesis $\theta_{k-1}(\underline{r}) = \widetilde{Q}^{k-1} \cdot \theta_0(\underline{r})$.

Letting $D(\xi) = \{i_1 \neq i_2, \dots, i_{k-2} \neq \xi\}$ we have in fact

$$\theta_k(\underline{r}) = \begin{bmatrix} w_1^k \\ w_2^k \\ \vdots \\ w_N^k \end{bmatrix} \overset{(79)}{=} \begin{bmatrix} \sum\limits_{\substack{i_0,\dots,i_{k-1}=1 \\ i_0\neq i_1,\dots,i_{k-1}\neq 1}}^{N} w_{i_0} \frac{q_{1,i_{k-1}}}{r_1+q_1} \prod\limits_{h=1}^{k-1} \frac{q_{i_h,i_{h-1}}}{r_{i_h}+q_{i_h}} \\ \sum\limits_{\substack{i_0,\dots,i_{k-1}=1 \\ i_0\neq i_1,\dots,i_{k-1}\neq 2}}^{N} w_{i_0} \frac{q_{2,i_{k-1}}}{r_2+q_2} \prod\limits_{h=1}^{k-1} \frac{q_{i_h,i_{h-1}}}{r_{i_h}+q_{i_h}} \\ \vdots \\ \sum\limits_{\substack{i_0,\dots,i_{k-1}=1 \\ i_0\neq i_1,\dots,i_{k-1}\neq N}}^{N} w_{i_0} \frac{q_{N,i_{k-1}}}{r_N+q_N} \prod\limits_{h=1}^{k-1} \frac{q_{i_h,i_{h-1}}}{r_{i_h}+q_{i_h}} \end{bmatrix}$$

$$= \begin{bmatrix} \sum\limits_{\substack{i_{k-1}=1 \\ i_{k-1}\neq 1}}^{N} \frac{q_{1,i_{k-1}}}{r_1+q_1} \left\{ \sum\limits_{\substack{i_0,\dots,i_{k-2}=1 \\ i_0\neq i_1,\dots,i_{k-2}\neq i_{k-1}}}^{N} w_{i_0} \frac{q_{i_{k-1},i_{k-2}}}{r_{i_{k-1}}+q_{i_{k-1}}} \prod\limits_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r_{i_h}+q_{i_h}} \right\} \\ \sum\limits_{\substack{i_{k-1}=1 \\ i_{k-1}\neq 2}}^{N} \frac{q_{2,i_{k-1}}}{r_2+q_2} \left\{ \sum\limits_{\substack{i_0,\dots,i_{k-2}=1 \\ i_0\neq i_1,\dots,i_{k-2}\neq i_{k-1}}}^{N} w_{i_0} \frac{q_{i_{k-1},i_{k-2}}}{r_{i_{k-1}}+q_{i_{k-1}}} \prod\limits_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r_{i_h}+q_{i_h}} \right\} \\ \vdots \\ \sum\limits_{\substack{i_{k-1}=1 \\ i_{k-1}\neq N}}^{N} \frac{q_{N,i_{k-1}}}{r_N+q_N} \left\{ \sum\limits_{\substack{i_0,\dots,i_{k-2}=1 \\ i_0\neq i_1,\dots,i_{k-2}\neq i_{k-1}}}^{N} w_{i_0} \frac{q_{i_{k-1},i_{k-2}}}{r_{i_{k-1}}+q_{i_{k-1}}} \prod\limits_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r_{i_h}+q_{i_h}} \right\} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{q_{1,2}}{r_1+q_1} \Big\{ \sum\limits_{\substack{i_0,\dots,i_{k-2}=1 \\ D(2)}}^{N} w_{i_0} \frac{q_{2,i_{k-2}}}{r_2+q_2} \prod\limits_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r_{i_h}+q_{i_h}} \Big\} + \dots + \frac{q_{1,N}}{r_1+q_1} \Big\{ \sum\limits_{\substack{i_0,\dots,i_{k-2}=1 \\ D(N)}}^{N} w_{i_0} \frac{q_{N,i_{k-2}}}{r_N+q_N} \prod\limits_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r_{i_h}+q_{i_h}} \Big\} \\ \frac{q_{2,1}}{r^2+q_2} \Big\{ \sum\limits_{\substack{i_0,\dots,i_{k-2}=1 \\ D(1)}}^{N} w_{i_0} \frac{q_{1,i_{k-2}}}{r_1+q_1} \prod\limits_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r_{i_h}+q_{i_h}} \Big\} + \dots + \frac{q_{2,N}}{r_2+q_2} \Big\{ \sum\limits_{\substack{i_0,\dots,i_{k-2}=1 \\ D(N)}}^{N} w_{i_0} \frac{q_{N,i_{k-2}}}{r_N+q_N} \prod\limits_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r_{i_h}+q_{i_h}} \Big\} \\ \vdots \\ \frac{q_{N,1}}{r_N+q_N} \Big\{ \sum\limits_{\substack{i_0,\dots,i_{k-2}=1 \\ D(1)}}^{N} w_{i_0} \frac{q_{1,i_{k-2}}}{r_1+q_1} \prod\limits_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r_{i_h}+q_{i_h}} \Big\} + \dots + \frac{q_{N-1,N}}{r_{N-1}+q_{N-1}} \Big\{ \sum\limits_{\substack{i_0,\dots,i_{k-2}=1 \\ D(N-1)}}^{N} w_{i_0} \frac{q_{N-1,i_{k-2}}}{r_N+q_N} \prod\limits_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r_{i_h}+q_{i_h}} \Big\} \end{bmatrix}$$

$$= \widetilde{Q} \cdot \begin{bmatrix} \sum\limits_{\substack{i_0,\dots,i_{k-2} \\ D(1)}}^{N} w_{i_0} \frac{q_{1,i_{k-2}}}{r_1+q_1} \prod\limits_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r_{i_h}+q_{i_h}} \\ \sum\limits_{\substack{i_0,\dots,i_{k-2} \\ D(2)}}^{N} w_{i_0} \frac{q_{2,i_{k-2}}}{r_2+q_2} \prod\limits_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r_{i_h}+q_{i_h}} \\ \vdots \\ \sum\limits_{\substack{i_0,\dots,i_{k-2} \\ D(N)}}^{N} w_{i_0} \frac{q_{N,i_{k-2}}}{r_N+q_N} \prod\limits_{h=1}^{k-2} \frac{q_{i_h,i_{h-1}}}{r_{i_h}+q_{i_h}} \end{bmatrix} = \widetilde{Q} \cdot \theta_{k-1}(\underline{r})$$

$\square$

Proof of Proposition 2.12. The fact that $\widetilde{\mathbb{P}}(\nu_{t,T} = 0 | r_{\nu_t} = r_{i_0}) = e^{-q_{i_0}(T-t)}$ follows directly from properties of CTMCs. Let us suppose that $\nu_t = h \in \mathbb{N}$. At first we prove by induction that

$$\widetilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t} = r_{i_h}) = \sum_{\substack{i_{h+1},\dots,i_{h+k}=1 \\ i_{h+1}\neq i_h, i_{h+2}\neq i_{h+1},\dots,i_{h+k}\neq i_{h+k-1}}}^{N} e^{q_{i_h}t - q_{i_{h+k}}T} \varphi_{h,k}(Q) \cdot \Psi_{h,k}(t,T,Q) \tag{86}$$

where

$$\begin{cases} \Psi_{h,k}(t,T,Q) \triangleq \int_t^T e^{(q_{i_{h+1}} - q_{i_h})t_{h+1}} \int_{t_{h+1}}^T e^{(q_{i_{h+2}} - q_{i_{h+1}})t_{h+2}} \dots \\ \qquad\qquad \cdots \int_{t_{h+k-1}}^T e^{(q_{i_{h+k}} - q_{i_{h+k-1}})t_{h+k}} dt_{h+k} \dots dt_{h+2} dt_{h+1} \\ \varphi_{h,k}(Q) \triangleq q_{i_h,i_{h+1}} \cdot \ldots \cdot q_{i_{h+k-1},i_{h+k}}. \end{cases}$$

Base Case $(k = 1)$

$$\begin{aligned} \widetilde{\mathbb{P}}(\nu_{t,T} = 1 | r_{\nu_t} = r_{i_h}) &\overset{(25)}{=} \sum_{\substack{i_{h+1}=1 \\ i_{h+1}\neq i_h}}^{N} q_{i_h,i_{h+1}} \int_t^T e^{-q_{i_h}(t_{h+1}-t)} \widetilde{\mathbb{P}}(\nu_{t_{h+1},T} = 0 | r_{\nu_{t_{h+1}}} = r_{i_{h+1}}) dt_{h+1} \\ &= \sum_{\substack{i_{h+1}=1 \\ i_{h+1}\neq i_h}}^{N} q_{i_h,i_{h+1}} \int_t^T e^{-q_{i_h}(t_{h+1}-t)} e^{-q_{i_{h+1}}(T-t_{h+1})} dt_{h+1} \\ &= \sum_{\substack{i_{h+1}=1 \\ i_{h+1}\neq i_h}}^{N} q_{i_h,i_{h+1}} e^{(q_{i_h}t - q_{i_{h+1}}T)} \int_t^T e^{(q_{i_{h+1}} - q_{i_h})t_{h+1}} dt_{h+1} \end{aligned}$$

Inductive step:

$$\widetilde{\mathbb{P}}(\nu_{t,T} = k | r_{\nu_t} = r_{i_h}) \overset{(25)}{=} \sum_{\substack{i_{h+1}=1 \\ i_{h+1}\neq i_h}}^{N} q_{i_h,i_{h+1}} \int_t^T e^{-q_{i_h}(t_{h+1}-t)} \widetilde{\mathbb{P}}(\nu_{t_{h+1},T} = k-1 | r_{\nu_{t_{h+1}}} = r_{i_{h+1}}) dt_{h+1}$$

$$= \sum_{\substack{i_{h+1}=1 \\ i_{h+1}\neq i_h}}^{N} q_{i_h,i_{h+1}} \int_t^T e^{-q_{i_h}(t_{h+1}-t)} \cdot$$

$$\cdot \Bigg\{ \sum_{\substack{i_{h+2},\dots,i_{h+k}=1 \\ i_{h+2}\neq i_{h+1}, i_{h+3}\neq i_{h+2},\dots,i_{h+k}\neq i_{h+k-1}}}^{N} e^{q_{i_{h+1}}t_{h+1} - q_{i_{h+k}}T} \varphi_{h+1,k-1}(Q) \cdot \Psi_{h+1,k-1}(t_{h+1},T,Q) \Bigg\} dt_{h+1}$$

$$
\begin{aligned}
&= \sum_{\substack{i_{h+1},\dots,i_{h+k}=1 \\ i_{h+1}\neq i_h, i_{h+2}\neq i_{h+1},\dots,i_{h+k}\neq i_{h+k-1}}}^{N} q_{i_h,i_{h+1}} e^{q_{i_h}t - q_{i_{h+k}}T} \varphi_{h+1,k-1}(Q) \\
&\quad \int_t^T e^{(q_{i_{h+1}}-q_{i_h})t_{h+1}} \Psi_{h+1,k-1}(t_{h+1},T,Q)dt_{h+1} \\
&= \sum_{\substack{i_{h+1},\dots,i_{h+k}=1 \\ i_{h+1}\neq i_h, i_{h+2}\neq i_{h+1},\dots,i_{h+k}\neq i_{h+k-1}}}^{N} e^{q_{i_h}t - q_{i_{h+k}}T} \varphi_{h,k}(Q)\Psi_{h,k}(t,T,Q)dt_{h+1}
\end{aligned}
$$

where in the last passage we have used the fact that

$$
\varphi_{h+1,k-1} = q_{i_{h+1},i_{(h+1)+1}} \cdot \dots \cdot q_{i_{(h+1)+(k-1)-1},i_{(h+1)+(k-1)}} = \frac{\varphi_{h,k}}{q_{i_h,i_{h+1}}}
$$

and

$$
\begin{aligned}
\Psi_{h+1,k-1}(t_{h+1},T,Q) &= \int_{t_{h+1}}^T e^{(q_{i_{h+2}}-q_{i_{h+1}})t_{h+2}} \int_{t_{h+2}}^T e^{(q_{i_{h+3}}-q_{i_{h+2}})t_{h+3}} \dots \\
&\quad \dots \int_{t_{h+k-1}}^T e^{(q_{i_{h+k}}-q_{i_{h+k-1}})t_{h+k}} dt_{h+k}\dots dt_{h+3}dt_{h+2}
\end{aligned}
$$

Observe now that the probabilities $\widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t} = r_{i_h})$ do not depend on ν_t the number of jumps until t, so they can be represented as

$$
\widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t} = r_{i_0}) = \sum_{\substack{i_1,\dots,i_k=1 \\ i_1\neq i_0, i_2\neq i_1,\dots,i_k\neq i_{k-1}}}^{N} e^{q_{i_0}t - q_{i_k}T} \varphi_k(Q)\cdot\Psi_k(t,T,Q) \tag{87}
$$

where Ψ_k and φ_k are defined by (29) and (30) respectively.

Finally, recalling that the process r is a CTMC homogeneous w.r.t. the time, the random variable $\nu_{t,T}$ has the same distribution as $\nu_{0,T-t}$. In other terms we have

$$
\begin{aligned}
\widetilde{\mathbb{P}}(\nu_{t,T} = k|r_{\nu_t} = r_{i_0}) &= \widetilde{\mathbb{P}}(\nu_{0,T-t} = k|r(0) = r_{i_0}) \\
&= \sum_{\substack{i_1,\dots,i_k=1 \\ i_1\neq i_0, i_2\neq i_1,\dots,i_k\neq i_{k-1}}}^{N} e^{-q_{i_k}(T-t)} \varphi_k(Q)\cdot\Psi_k(0,T-t,Q).
\end{aligned} \tag{88}
$$

□

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