PDE approach to utility maximization for market models with hidden Markov factors

Hideo Nagai and Wolfgang J. Runggaldier

Abstract. We consider the problem of maximizing expected utility from terminal wealth for a power utility of the risk-averse type assuming that the dynamics of the risky assets are affected by hidden "economic factors" that evolve as a finite-state Markov process. For this partially observable stochastic control problem we determine a corresponding complete observation problem that turns out to be of the risk sensitive type and for which the Dynamic programming approach leads to a nonlinear PDE that, via a suitable transformation, can be made linear. By means of a probabilistic representation we obtain a unique viscosity solution to the latter PDE that induces a unique viscosity solution to the former. This probabilistic representation allows to obtain, on one hand regularity results, on the other a computational approach based on Monte Carlo simulation.

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1. Introduction

We consider a market model with one locally riskless security and a certain number of risky securities. The goal is to find an admissible self-financing investment strategy that maximizes the expected utility from terminal wealth at a given maturity and with a power utility function of the risk-averse type.

We assume that the dynamics of the risky assets are affected by exogenous "economic factors" that evolve as a finite-state Markov process. We allow these economic factors to be hidden, i.e. they may not be observed directly. Information about these factors can therefore be obtained only by observing the prices of the risky assets.

Our problem is thus of the type of a partially observed stochastic control problem and we shall determine its solution by solving a corresponding complete observation control problem. After discussing some problems that arise for a complete observation problem based on unnormalized filter values, we construct an equivalent complete observation control problem, where the new state is given by the pair (p_t, Y_t) consisting of the conditional state probability vector (normalized filter) p_t for the hidden factor process and of the log-asset prices Y_t. This pair forms a Markov process also in our more general setup where the coefficients in the security price dynamics are nonlinearly dependent upon the factors. The equivalent complete observation control problem turns out to be of the type of a risk sensitive stochastic control problem. It is approached by the method of Dynamic Programming (DP) that leads to a nonlinear HJB equation. Applying a transformation that is by now rather classical, this nonlinear HJB equation is transformed into a linear one. By means of a probabilistic representation as expectation of a suitable function of the underlying Markov process (p_t, Y_t), we obtain a unique viscosity solution to the latter PDE that induces a unique viscosity solution to the former. This probabilistic representation allows to obtain, on one hand, regularity results on the basis of classical results on expectations of functions of diffusion processes; on the other hand it allows to obtain a computational approach based on Monte Carlo simulation. This latter computational approach is important since, as we shall show, an explicit analytic solution is very difficult to obtain in the given setup.

Portfolio optimization problems under partial information are becoming more and more popular, also because of their practical interest. They have been studied using both major portfolio optimization methodologies, namely Dynamic Programming (DP) and the "Martingale Method"(MM). While DP has a longer tradition in general, also MM has been applied already since some time for the cases when the drift/appreciation rate in a diffusion-type market model is supposed to be an unknown constant, a hidden finite-state Markov process, or a linear-Gaussian factor process. Along this line are the papers [9], [10], [8], [22] and, more recently, [5] and [20]. The case when the volatility is driven by a hidden process is studied in [16]. After the early paper [3], a DP-approach for a finite-horizon linear-Gaussian model with one unobserved factor that is independent of the risky asset has been used in [18]. In this latter paper the author also ends up with a nonlinear PDE. However, instead of using a transformation to reduce the equation to a linear one, the author introduces an auxiliary problem of the linear-quadratic type and obtains from the latter the solution of the former problem. When investment decisions are modelled to take place in discrete time, the entire portfolio optimization problem reduces to one in discrete time and here a DP-approach under partial information can be found in [19]. A risk-sensitive finite horizon control problem under partial information for a general linear-Gaussian model has been considered in [13] where, by solving two kinds of Riccati differential equations, it was possible to construct an optimal strategy. The results are extended to the case of infinite time horizon in [15] by studying the asymptotics of the solutions of inhomogeneous (time dependent) Riccati differential equations as the time horizon goes to infinity.

In relation to the literature as described above, in the present paper we consider the portfolio maximization problem under a hidden Markov setting, where the coefficients of the security prices are nonlinearly dependent on economic factors that evolve as a $k-$state Markov chain (Section 2). The problem is reformulated in Section 3 as a risk-sensitive stochastic control problem under complete observation and in Section 4 an optimal strategy is constructed from the solution of the corresponding HJB-equation.

2. Problem setup

Let us consider a market model with $N+1$ securities $(S_t^0, S_t) := (S_t^0, S_t^1, ..., S_t^N)^*$, were S^* stands for the transpose of the matrix S, and an economic factor process X_t, which is supposed to be a finite state Markov chain taking its values in the set of the unit vectors $E = \{e_1, e_2, ..., e_k\}$ in R^k. The bond price S_t^0 is assumed to satisfy the ordinary differential equation:

$$dS_t^0 = r(t, S_t) S_t^0 dt, \quad S_0^0 = s^0, \tag{2.1}$$

where $r(t, S)$ is a nonnegative, bounded and locally Lipschitz continuous function in $S \in R_+^N = \{(x^1, ..., x^N); x^i \geq 0, \ i = 1, 2, .., N\}$. The other security prices $S_t^i, \ i = 1, 2, ..., N$, are assumed to be governed by the following stochastic differential equations:

$$\begin{aligned} &dS_t^i = S_t^i \{a^i(t, X_t, S_t) dt + \textstyle\sum_{j=1}^N b_j^i(t, S_t) dW_t^j\}, \\ &S_0^i = s^i, \ \ i = 1, ..., N \end{aligned} \tag{2.2}$$

where $a^i(t, X, S)$ and $b_j^i(t, S)$ are bounded and, for each t and X, locally Lipschitz continuous functions in S, b is uniformly non degenerate, i.e. $z^* b b^* z \geq c|z|^2$, $\forall z \in R^N$, $\exists c > 0$ and $W_t = (W_t^j)_{j=1,..,N}$ is an N - dimensional standard Brownian motion process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ and is independent of X_t. The Markov chain X_t can be expressed in terms of a martingale M_t of the pure jump type, namely

$$\begin{aligned} &dX_t = \Lambda(t) X_t dt + dM_t, \\ &X_0 = \xi, \end{aligned} \tag{2.3}$$

where $\Lambda(t)$ is the Q matrix (transition intensity matrix) of the Markov chain and ξ is a random variable taking its values in E. Set

$$\mathcal{G}_t = \sigma(S_u; u \leq t)$$

and let us denote by h_t^i, $(i = 0, 1, ..., N)$ the portfolio proportion of the amount invested in the i-th security relative to the total wealth V_t that the investor possesses. It is defined as follows :

Definition 2.1. $(h_t^0, h_t) \equiv (h_t^0, h_t^1, h_t^2, ..., h_t^N)^*$ is said to be an investment strategy if the following conditions are satisfied

i) h_t is an R^N valued $\mathcal{G}_t$ - progressively measurable stochastic process such that

$$\sum_{i=1}^{N} h_t^i + h_t^0 = 1$$

ii)
$$P(\int_0^T |h_s|^2 ds < \infty) = 1.$$

The set of all investment strategies will be denoted by $\mathcal{H}(T)$. When $(h_t^0, h_t^*)_{0 \le t \le T} \in \mathcal{H}(T)$ we shall often write $h \in \mathcal{H}(T)$ for simplicity.

For given $h \in \mathcal{H}(T)$, and under the assumption of self-financing, the wealth process $V_t = V_t(h)$ satisfies

$$\begin{cases} \dfrac{dV_t}{V_t} &= \sum_{i=0}^N h_t^i \dfrac{dS_t^i}{S_t^i} \\ &= h_t^0 r(t, S_t)dt + \sum_{i=1}^m h_t^i \{a^i(t, X_t, S_t)dt + \sum_{j=1}^N b_j^i(t, S_t)dW_t^j\} \\ V_0 &= v \end{cases}$$

Taking into account $i)$ above, V_t turns out to be the solution of

$$\begin{cases} \dfrac{dV_t}{V_t} &= r(t, S_t)dt + h_t^*(a(t, X_t, S_t) - r(t, S_t)\mathbf{1})dt + h_t^* b(t, S_t)dW_t, \\ V_0 &= v, \end{cases}$$

where $\mathbf{1} = (1, 1, ..., 1)^*$.

Our problem is the following. For a given constant $\mu < 1,\ \mu \neq 0$ maximize the expected (power) utility of terminal wealth up to the time horizon T, namely

$$J(v; h; T) = \frac{1}{\mu} E[V_T(h)^\mu] = \frac{1}{\mu} E[e^{\mu \log V_T(h)}], \tag{2.4}$$

where h ranges over the set $\mathcal{A}(0, T)$ of all admissible strategies that will be defined below in (3.17).

We consider here the maximization problem with partial information, since the economic factors X_t are in general not directly observable and so one has to select the strategies only on the basis of past information of the security prices.

3. Reduction to risk-sensitive stochastic control under complete information

There are a priori more possible approaches to determine an equivalent complete observation control problem. One may base it on a Zakai-type equation for an unnormalized filter. One may however also base it on normalized filters. Each approach has its advantages and disadvantages, the major advantage for the Zakai-type approach being that the dynamics are linear. In subsection 3.1 we first discuss such an approach in a form related to [13] and show that, in our setting, an explicit solution is difficult to obtain despite the linearity of the dynamics for the unnormalized filter. Although we therefore abandon this approach in favour of one based on normalized filter values, we still wanted to discuss it here because it forms a basis for the other approach that will be derived in subsection 3.2 and that is related to [13] and [15]. We want to point out that, in the given setup, the standard approach leading to the so-called “separated problem” fails because of questions of measurability with respect to the full and the observation filtrations and the fact that in a crucial expectation there appears the product of the function of interest with a Radon-Nikodym derivative (see (3.7) and the comment preceding (3.8)).

Before discussing the individual approaches, let us introduce some notation and expressions that will be used in the sequel.

Let us set

$$Y_t^i = \log S_t^i, \quad i = 0, 1, 2, ..., N,$$

with $Y_t = (Y_t^1, Y_t^2, ..., Y_t^N)^*$ and $\mathbf{e}^Y = (e^{Y^1}, ..., e^{Y^N})^*$. Then

$$dY_t^0 = R(t, Y_t)dt$$

and

$$dY_t = \bar{A}(t, X_t, Y_t)dt + B(t, Y_t)dW_t, \tag{3.1}$$

where

$$\begin{aligned} \bar{A}^i(t, x, y) &= a^i(t, x, \mathbf{e}^y) - \tfrac{1}{2}(bb^*)^{ii}(t, \mathbf{e}^y), \\ B_j^i(t, y) &= b_j^i(t, \mathbf{e}^y), \quad R(t, y) = r(t, \mathbf{e}^y) \end{aligned}$$

Putting

$$\eta(t, x, y, h) := \frac{1-\mu}{2} h^* BB^*(t, y)h - R(t, y) - h^*(A(t, x, y) - R(t, y)\mathbf{1}), \tag{3.2}$$

with

$$A^i(t, x, y) = a^i(t, x, \mathbf{e}^y),$$

by Itô’s formula we see that

$$dV_t^\mu = V_t^\mu\{-\mu\eta(t, X_t, Y_t, h_t)dt + \mu h_t^* B(t, Y_t)dW_t\}, \quad V_0 = v^\mu \tag{3.3}$$

and so

$$\begin{aligned} V_t^\mu = v^\mu \exp\{-\mu {\textstyle\int_0^t} \eta(s, X_s, Y_s, h_s)ds \\ +\mu {\textstyle\int_0^t} h_s^* B(s, Y_s)dW_s - \tfrac{\mu^2}{2} {\textstyle\int_0^t} h_s^* BB^*(s, Y_s)h_s ds\}. \end{aligned} \tag{3.4}$$

3.1. Approach via a Zakai-type equation

Given our assumptions on the boundedness of the coefficients, let us introduce a new probability measure $\hat{P}$ on $(\Omega, \mathcal{F})$ defined by

$$\left.\frac{d\hat{P}}{dP}\right|_{\mathcal{F}_T} = \rho_T,$$

where

$$\rho_T = e^{-\int_0^T \bar{A}^*(t,X_t,Y_t)(BB^*)^{-1}B(t,Y_t)dW_t - \frac{1}{2}\int_0^T \bar{A}^*(BB^*)^{-1}\bar{A}(t,X_t,Y_t)dt} \tag{3.5}$$

Under the probability measure $\hat{P}$

$$\hat{W}_t = W_t + \int_0^t B^*(BB^*)^{-1}(s,Y_s)\bar{A}(s,X_s,Y_s)ds$$

is a Brownian motion process and Y_t satisfies

$$dY_t = B(t,Y_t)d\hat{W}_t. \tag{3.6}$$

The criterion (2.4) can be rewritten under the new probability measure as

$$\begin{aligned}
&\tfrac{1}{\mu}E[V_T^\mu] \\
&= \tfrac{1}{\mu}v^\mu \hat{E}[e^{-\mu\int_0^T \eta(s,X_s,Y_s,h_s)ds + \mu\int_0^T h_s^* B(s,Y_s)dW_s - \frac{\mu^2}{2}\int_0^T h_s^* BB^*(s,Y_s)h_s ds}\rho_T^{-1}] \\
&= \tfrac{1}{\mu}v^\mu \hat{E}[e^{-\mu\int_0^T \eta(s,X_s,Y_s,h_s)ds + \int_0^T Q^*(s,X_s,Y_s,h_s)dY_s - \frac{1}{2}\int_0^T Q^* BB^* Q(s,X_s,Y_s,h_s)ds}]
\end{aligned} \tag{3.7}$$

where

$$Q(t,X_t,Y_t,h_t) = (BB^*(t,Y_t))^{-1}\bar{A}(t,X_t,Y_t) + \mu h_t.$$

Since the argument of the expectation in (3.7) is of the form of a Radon Nikodym derivative multiplied with the function of interest, we shall treat it as a whole considering the following process

$$\begin{aligned}
H_t = \exp\{&-\mu\textstyle\int_0^t \eta(s,X_s,Y_s,h_s)ds + \int_0^t Q^*(s,X_s,Y_s,h_s)dY_s \\
&-\tfrac{1}{2}\textstyle\int_0^t Q^*BB^*(s,Y_s)Q(s,X_s,Y_s,h_s)ds\}
\end{aligned} \tag{3.8}$$

and

$$q_t^i = \hat{E}[H_t X_t^i | \mathcal{G}_t],$$

where $X_t^i = \mathbf{1}_{\{e_i\}}(X_t)$. Then

$$E\{V_T^\mu\} = v^\mu \hat{E}\{\hat{E}[H_T|\mathcal{G}_T]\} = v^\mu \sum_{i=1}^k \hat{E}\{\hat{E}[H_T X_T^i|\mathcal{G}_T]\} = v^\mu \hat{E}\{\sum_{i=1}^k q_T^i\} \tag{3.9}$$

where (see Corollary 3.3 in [1]; see also section 7.3 in [4]) q_t^i satisfy

$$\begin{aligned}
dq_t^i &= (\Lambda(t)q_t)^i dt - \mu\eta(t,e_i,Y_t,h_t)q_t^i dt + q_t^i Q^*(t,e_i,Y_t,h_t)dY_t, \\
q_0^i &= p_0^i \equiv P(\xi = e_i), \quad i = 1,2,...,k
\end{aligned} \tag{3.10}$$

Next we give some arguments to show that, as mentioned in the Introduction, an explicit solution to the problem (3.9) and (3.10) is difficult to obtain.

Set $q_t = (q_t^i)$. Then (q_t, Y_t) can be regarded as the controlled process for the stochastic control problem of maximizing the criterion

$$J = v^\mu \hat{E}\{\sum_{i=1}^k q_T^i\}.$$

Let us introduce the value function

$$w(t,q,y) = \sup_{h\in\mathcal{A}(t,T)} \hat{E}\{\sum_{i=1}^k q_T^i(t)\}$$

where, analogously to $\mathcal{A}(0,T)$, $\mathcal{A}(t,T)$ denotes the admissible strategies over the interval $[t,T]$, $q_s^i(t),\ t \le s \le T$ is a solution of (3.10) with the initial condition $q_t^i(t) = q^i$ and $Y_s,\ (t \le s \le T)$ is solution of (3.6) with initial condition $Y_t = y$.

The Bellman equation for w then becomes

$$\begin{cases} \frac{\partial w}{\partial s} + \sup_h L_s(h)w = 0\,, \qquad t \le s \le T, \quad (q,y) \in [0,\infty)^k \times R^N \\ \\ w(T,q,y) = \sum_{i=1}^k q^i, \end{cases}$$

where

$$\begin{cases} L_s(h) = \frac{1}{2}\sum_{i,j}[BB^*(s,y)]^{ij}\frac{\partial^2}{\partial y^i \partial y^j} + \sum_{i,j} q^i[Q^*(s,e_i,y,h)B(s,y)]^j\frac{\partial^2}{\partial q^i \partial y^j} \\ \\ \qquad\qquad +\frac{1}{2}\sum_{i,j} q^i Q^*(s,e_i,y,h)BB^*Q(s,e_j,y,h)q^j\frac{\partial^2}{\partial q^i \partial q^j} \\ \\ \qquad\qquad +\sum_i\{[q^*\Lambda(s)^*]^i - \mu\eta(s,e_i,y,h)q^i\}\frac{\partial}{\partial q^i} \end{cases}$$

As can now be easily seen, an explicit solution of this Bellman equation is rather difficult to obtain and so we abandon this approach in favour of one based on the normalized filter that will however continue the main line of the arguments of the present section.

3.2. Approach based on the normalized filter

In order to derive the corresponding full information control problem we put

$$p_t^i = P(X_t = e_i|\mathcal{G}_t), \quad i = 1,..,k, \tag{3.11}$$

and use the notation

$$f(s,p_s,y,h) = \sum_{i=1}^k f(s,e_i,y,h)p_s^i, \tag{3.12}$$

for a given function $f(s,x,y,h)$ on $[0,T]\times E\times R^N\times R^N$, while the defined function is on $[0,T]\times\Delta_{k-1}\times R^N\times R^N$ with Δ_{k-1} the $k-1$ dimensional simplex

$$\Delta_{k-1} = \{(d_1,d_2,..,d_k); d_1+d_2+...+d_k = 1,\ \ 0 \le d_i \le 1,\ i = 1,..,k\}.$$

It is known that these (normalized) conditional probabilities p_t^i, $i = 1, 2, .., k$, satisfy the following equation ("Wonham filter", see [11], [21])

$$dp_t^i = (\Lambda(t)p_t)^i dt + p_t^i[\bar{A}^*(t, e_i, Y_t) - \bar{A}^*(t, p_t, Y_t)] \cdot [BB^*(t, Y_t)]^{-1}[dY_t - \bar{A}(t, p_t, Y_t)dt],$$

namely

$$dp_t = \Lambda(t)p_t dt + D(p_t)[\bar{A}^*(t, Y_t) - \mathbf{1}\bar{A}^*(t, p_t, Y_t)] \cdot [BB^*(t, Y_t)]^{-1}[dY_t - \bar{A}(t, p_t, Y_t)dt], \tag{3.13}$$

where $\bar{A}(t, Y)$ is an $N \times k$ matrix defined by $\bar{A}(t, Y) = (\bar{A}^i(t, e_j, Y))$ and $D(p)$ is a diagonal matrix of which the component in position ii is p^i.

In full analogy with (3.8) we now define

$$\hat{H}_t = \exp\{-\mu \int_0^t \eta(s, p_s, Y_s, h_s)ds + \int_0^t Q^*(s, p_s, Y_s, h_s)dY_s - \frac{1}{2}\int_0^t Q^*BB^*(s, Y_s)Q(s, p_s, Y_s, h_s)ds\}, \tag{3.14}$$

We then have

$$\begin{aligned} d(\hat{H}_t p_t^i) &= \hat{H}_t dp_t^i + p_t^i d\hat{H}_t + d\langle \hat{H}, p^i\rangle_t \\ &= \hat{H}_t(\Lambda(t)p_t)^i dt \\ &+ \hat{H}_t p_t^i[\bar{A}^*(t, e_i, Y_t) - \bar{A}^*(t, p_t, Y_t)][BB^*(t, Y_t)]^{-1}[dY_t - \bar{A}(t, p_t, Y_t)dt] \\ &- \mu\hat{H}_t p_t^i \eta(t, p_t, Y_t, h_t)dt + \hat{H}_t p_t^i Q^*(t, p_t, Y_t, h_t)dY_t + d\langle \hat{H}, p^i\rangle_t \\ &= (\Lambda(t)\hat{H}_t p_t)^i dt - \mu\eta(t, e_i, Y_t, h_t)\hat{H}_t p_t^i dt + \hat{H}_t p_t^i Q^*(t, e_i, Y_t, h_t)dY_t, \end{aligned} \tag{3.15}$$

where the last equality is obtained from noticing that, given the previous definitions, the following three equalities hold

$$d\langle \hat{H}, p^i\rangle_t = \hat{H}_t p_t^i[\bar{A}(t, e_i, Y_t)^* - \bar{A}(t, p_t, Y_t)^*][BB^*]^{-1}\bar{A}(t, p_t, Y_t)dt + \hat{H}_t p_t^i \mu h_t^*[\bar{A}(t, e_i, Y_t)^* - \bar{A}(t, p_t, Y_t)^*]dt;$$

$$-\mu\eta(t, e_i, Y_t, h_t)\hat{H}_t p_t^i dt + \mu\eta(t, p_t, Y_t, h_t)\hat{H}_t p_t^i dt = \hat{H}_t p_t^i \mu h_t^*[\bar{A}(t, e_i, Y_t)^* - \bar{A}(t, p_t, Y_t)^*]dt;$$

$$\begin{aligned} &\hat{H}_t p_t^i[\bar{A}(t, e_i, Y_t)^* - \bar{A}(t, p_t, Y_t)^*][BB^*]^{-1}[dY_t - \bar{A}(t, p_t, Y_t)dt] + \hat{H}_t p_t^i Q(t, p_t, Y_t, h_t)^* dY_t \\ &= \hat{H}_t p_t^i Q^*(t, e_i, Y_t, h_t)dY_t - \hat{H}_t p_t^i[\bar{A}(t, e_i, Y_t)^* - \bar{A}(t, p_t, Y_t)^*][BB^*]^{-1}\bar{A}(t, p_t, Y_t). \end{aligned}$$

Therefore, we see that $q_t^i = \hat{H}_t p_t^i$ thus showing that q_t^i are indeed un-normalized conditional probabilities and

$$\hat{E}[H_T|\mathcal{G}_T] = \sum_{i=1}^{k} q_T^i = \hat{H}_T.$$

We have thus proved the following Proposition, which establishes the equivalence of the original incomplete information control problem with the present corresponding complete one. The latter has as state variable process the (finite-dimensional) Markovian pair (p_t, Y_t) satisfying (3.13) and (3.6) respectively, and as objective function $\frac{1}{\mu} v^\mu \hat{E}[\hat{H}_T]$, where $\hat{H}_T$ depends, see (3.14), on the chosen strategy h_t.

Proposition 3.1. *The criterion (2.4) can be expressed equivalently as follows*

$$J(v;h;T) \equiv \frac{1}{\mu} E[V_T^\mu] = \frac{1}{\mu} v^\mu \hat{E}[H_T] = \frac{1}{\mu} v^\mu \hat{E}[\hat{H}_T].$$

Notice that, for Markovianity, we have to consider as state variables in the complete observation problem the pair (p_t, Y_t) and not just p_t alone, because in our original problem the coefficients depend on S_t and therefore on Y_t. Notice also that the state-variable pair (p_t, Y_t) is finite-dimensional.

The criterion expressed in the rightmost equivalent form above can be shown to be of the form of a risk-sensitive stochastic control problem in finite dimension. To this effect let us introduce another change of measure with the Girsanov density defined by

$$\begin{aligned} \left.\frac{d\tilde{P}}{d\hat{P}}\right|_{\mathcal{G}_T} = \zeta_T &= e^{\int_0^T Q^*(s,p_s,Y_s,h_s)dY_s - \frac{1}{2}\int_0^T Q^* B B^* Q(s,p_s,Y_s,h_s)ds} \\ &= e^{\int_0^T Q^*(s,p_s,Y_s,h_s)B(s,Y_s)d\hat{W}_s - \frac{1}{2}\int_0^T Q^* B B^* Q(s,p_s,Y_s,h_s)ds}. \end{aligned} \tag{3.16}$$

Notice that the new probability measure $\tilde{P}$ depends, through ζ_T, on the chosen strategy h_t. In order that $\tilde{P}$ is a probability measure we have to require that the set $\mathcal{A}(0,T)$ of admissible strategies is given by

$$\mathcal{A}(0,T) = \left\{ h \in \mathcal{H}(T) \mid \hat{E}\{\zeta_T\} = E\{\rho_T \zeta_T\} = 1 \right\} \tag{3.17}$$

Under the probability measure $\tilde{P}$ we now have that

$$\tilde{W}_t = \int_0^t B^{-1}(s,Y_s)dY_s - \int_0^t B^*(s,Y_s)Q(s,p_s,Y_s,h_s)ds \tag{3.18}$$

is a standard $\mathcal{G}_t$ - Brownian motion process and we have

$$\begin{aligned} dY_t &= B(t,Y_t)d\tilde{W}_t + BB^*(t,Y_t)Q(t,p_t,Y_t,h_t)dt \\ &= B(t,Y_t)d\tilde{W}_t + \{\bar{A}(t,p_t,Y_t) + \mu BB^*(t,Y_t)h_t\}dt \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} dp_t &= D(p_t)[\bar{A}^*(t,Y_t) - \mathbf{1}\bar{A}^*(t,p_t,Y_t)][BB^*(t,Y_t)]^{-1}B(t,Y_t)d\tilde{W}_t \\ &\quad +\{\Lambda(t)p_t + \mu D(p_t)[\bar{A}^*(t,Y_t) - \mathbf{1}\bar{A}^*(t,p_t,Y_t)]h_t\}dt. \end{aligned} \tag{3.20}$$

Since

$$\frac{1}{\mu}v^\mu \hat{E}[\hat{H}_T] = \frac{1}{\mu}v^\mu \tilde{E}[\exp\{-\mu \int_0^T \eta(s,p_s,Y_s,h_s)ds\}]$$

we are reduced to considering the risk-sensitive stochastic control problem that consists in maximizing

$$\frac{1}{\mu}v^\mu \tilde{E}[\exp\{-\mu \int_0^T \eta(s,p_s,Y_s,h_s)ds\}] \tag{3.21}$$

subject to the controlled process (p_t, Y_t) on $\Delta_{k-1} \times R^N$ being governed by the controlled stochastic differential equations (3.20) and (3.19) defined on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{G}_t, \tilde{P})$.

The solution to this latter complete observation problem forms the subject of the next section 4.

4. HJB-equation

For ease of notation, given $t \in [0,T]$, let us now introduce for $s \in [t,T]$ the vector process

$$Z_s := [p_s, Y_s]^*, \quad p_s \in \Delta_{k-1},\ Y_t \in R^N$$

so that, putting

$$\beta(s,Z_s) := \begin{bmatrix} \Lambda(s)p_s \\ \bar{A}(s,p_s,Y_s) \end{bmatrix}, \quad \text{a } (k+N)-\text{vector}$$

$$\alpha(s,Z_s) := \begin{bmatrix} D(p_s)[\bar{A}^*(s,Y_s) - \mathbf{1}\bar{A}^*(s,p_s,Y_s)](BB^*)^{-1}B(s,Y_s) \\ B(s,Y_s) \end{bmatrix} \tag{4.1}$$

which is a $(k+N)\times N-$ matrix and

$$\beta_\mu(s,Z_s;h_s) := \beta(s,Z_s) + \mu\alpha(s,Z_s)B^*(s,Y_s)h_s, \quad \text{a } (k+N)-\text{vector}, \tag{4.2}$$

from (3.20) and (3.19) the dynamics of Z_s on $(\Omega, \mathcal{F}, \mathcal{G}_s, \tilde{P})$ and for $s \in [t,T]$ become

$$\begin{cases} dZ_s = \beta_\mu(s,Z_s;h_s)ds + \alpha(s,Z_s)d\tilde{W}_s \\ Z_t = z \end{cases} \tag{4.3}$$

where the strategy h_s affects the evolution of Z_s directly through the drift β_μ and, recalling the comment before (3.17), indirectly also through the measure $\tilde{P}$, i.e. through $\tilde{W}_s$.

Recall now the objective function (2.4) and its representation in Proposition 3.1 and in (3.21) that are all defined for the initial time $t = 0$. For a generic t with $0 \leq t \leq T$ and for $V_t = v$, $Z_t = z$, put

$$J(t; v; z, h; T) = \frac{1}{\mu} v^{\mu} G(t, z, h)$$

where, letting with some abuse of notation $\eta(s, Z_s, h_s) := \eta(s, p_s, Y_s, h_s)$ with $\eta(s, p_s, Y_s, h_s)$ as in (3.2) and with the notation as in (3.12), we define

$$G(t, z, h) = \tilde{E}_{t,z} \left\{ \exp \left[-\mu \int_t^T \eta(s, Z_s, h_s)\, ds \right] \right\} \tag{4.4}$$

In view of the HJB equation put now

$$w(t, z) := \sup_{h \in \mathcal{A}(t,T)} \log G(t, z, h) \tag{4.5}$$

so that

$$\sup_{h \in \mathcal{A}(0,T)} J(v; h; T) = \frac{1}{\mu} v^{\mu} e^{w(0, Z_0)} \tag{4.6}$$

Based on the definition of $\eta(t, z, h)$ and the dynamics of Z in (4.3) with drift β_μ as in (4.2), we may now formally write for $w(t, z)$ in (4.5) the following Bellman equation of the Dynamic programming approach

$$\begin{cases} \frac{\partial w}{\partial t} + \frac{1}{2} tr[\alpha \alpha^* D^2 w] + \frac{1}{2} (\nabla w)^* \alpha \alpha^* \nabla w \\ \quad + \sup_h \left[\beta_\mu(t, z, h)^* \nabla w + \mu \gamma^*(t, z) h - \frac{1}{2} \mu (1 - \mu) h^* B B^* h \right] \\ \quad + \mu R(t, z) = 0 \\ w(T, z) = 0 \end{cases} \tag{4.7}$$

where

$$\gamma(t, z) = A(t, p, Y) - R(t, z) \mathbf{1} \tag{4.8}$$

Given our assumptions that b is uniformly non degenerate, the maximizing $\hat{h}$ in (4.7) is

$$\hat{h} = \hat{h}(t, z) = \frac{1}{1 - \mu} (BB^*)^{-1}(t, z) \left[B(t, z) \alpha^*(t, z) \nabla w(t, z) + \gamma(t, z) \right] \tag{4.9}$$

and (4.7) itself becomes

$$\begin{cases} \frac{\partial w}{\partial t} + \frac{1}{2} tr[\alpha \alpha^* D^2 w] + \frac{1}{2(1-\mu)} (\nabla w)^* \alpha \alpha^* \nabla w + \Phi^* \nabla w + \Psi = 0 \\ w(T, z) = 0 \end{cases} \tag{4.10}$$

where, for simplicity of notation, we have put

$$\begin{aligned} \Phi(t, z) &:= \beta(t, z) + \frac{\mu}{1-\mu} \alpha(t, z) B^{-1}(t, z) \gamma(t, z) \\ \Psi(t, z) &:= \mu R(t, z) + \frac{\mu}{2(1-\mu)} \gamma^*(t, z) (BB^*)^{-1}(t, z) \gamma(t, z) \end{aligned} \tag{4.11}$$

and which is a nonlinear 2nd order PDE. We shall now transform (4.10) into a linear PDE by following a by now classical procedure (see e.g. [6], [7]) and according to which we put

$$v(t,z) = e^{\frac{1}{1-\mu}\, w(t,z)} \tag{4.12}$$

With this transformation (4.10) becomes now

$$\begin{cases} \frac{\partial v}{\partial t} + \frac{1}{2} tr[\alpha\alpha^* D^2 v] + \Phi^*(t,z)\nabla v + \frac{\Psi(t,z)}{1-\mu} v = 0 \\ v(T,z) = 1 \end{cases} \tag{4.13}$$

It can now be easily seen that $v(t,z)$ is a viscosity solution for (4.13) if and only if $w = (1-\mu)\log v$ is a viscosity solution for (4.10).

Notice that, in spite of the linearity of the PDE in (4.13), an explicit analytic solution is very difficult to obtain in our setting (to this effect see also the Remark 4.2 at the end of this section). However, the linearity of the PDE leads to a Feynman-Kac representation of the solution, which makes it then possible to compute it numerically by simulation as we shall mention also below. Set then

$$\bar{v}(t,z) = E_{t,z}\left\{ \exp\left[\frac{1}{1-\mu} \int_t^T \Psi(s, Z_s)\, ds \right] \right\} \tag{4.14}$$

where Z_s now satisfies, instead of (4.3), the following

$$\begin{cases} dZ_s &= \Phi(s, Z_s)dt + \alpha(s, Z_s)dW_s \\ Z_t &= z \end{cases} \tag{4.15}$$

where W_s is a Wiener process and which, given our assumptions of bounded and locally Lipschitz continuous coefficients with b uniformly non degenerate, admits a unique strong/pathwise solution. A solution to this equation can rather easily be simulated for the purpose of calculating then numerically the value of $\bar{v}(t,z)$.

Finally, using also the boundedness of $\bar{v}$, from Theorem 4.4.3 and Appendix 7.7.2 in [14] it follows that $\bar{v}(t,z)$ is the unique viscosity solution for (4.13) and, consequently, $\bar{w} = (1-\mu)\log \bar{v}$ is the unique viscosity solution for (4.10). Thus we have the following proposition.

Proposition 4.1. *Under the assumptions in section* 2 *equation (*4.10*) has a unique viscosity solution w and it is expressed as $w(t,z) = (1-\mu)\log \bar{v}$, where $\bar{v}$ is the function defined by (*4.14*).*

Under stronger assumptions on r, a^i, b^i_j such that they are $\mathcal{C}^2$ functions with derivatives of polynomial growth we have by Theorem 5.5 in [2] that $\bar{v}(t,z)$, and therefore also $\bar{w}(t,z)$, are of class $\mathcal{C}^2$ and with derivatives of polynomial growth. The formal Bellman equation (4.7) becomes thus an equation having a classical solution and the function $\hat{h}$ in (4.9) exists and $\hat{h}(t, Z_t)$ is thus an optimal control.

We close this section with the following Remark that is intended to better explain why an explicit analytic solution to (4.13) is difficult to obtain.

Remark 4.2. We show here the expressions for the coefficients of the HJB equation (4.13) in the simplest case when the coefficients in the asset price dynamics (2.2) are autonomous and do not depend on the asset price itself and the factor process X_t is a two-state homogeneous Markov process with $Q-$matrix

$$\Lambda^* = \begin{pmatrix} \lambda_1 & -\lambda_2 \\ -\lambda_1 & \lambda_2 \end{pmatrix}.$$

Denote by p_t the conditional state probability for state 1 in the generic period t, i.e.

$$p_t = P\{X_t = e_1 | \mathcal{G}_t\}$$

We have now

$$\alpha\alpha^*(p) = \begin{pmatrix} p^2(1-p^2)(a(e_1)-a(e_2))^2 B^{-2} & -p^2(1-p^2)(a(e_1)-a(e_2))^2 B^{-2} & p(1-p)(a(e_1)-a(e_2)) \\ -p^2(1-p^2)(a(e_1)-a(e_2))^2 B^{-2} & p^2(1-p^2)(a(e_1)-a(e_2))^2 B^{-2} & p(1-p)(a(e_2)-a(e_1)) \\ (a(e_1)-a(e_2))p(1-p) & (a(e_2)-a(e_1))p(1-p) & B^2 \end{pmatrix}$$

$$\Phi(p) = \begin{pmatrix} \lambda_1 p - \lambda_2(1-p) \\ -\lambda_1 p + \lambda_2(1-p) \\ a(e_1)p + a(e_2)(1-p) - \frac{1}{2}B^2 \end{pmatrix} + \frac{\mu}{1-\mu} \begin{pmatrix} p(1-p)(a(e_1)-a(e_2))B^{-2}(a(e_1)p+a(e_2)(1-p)-R) \\ p(1-p)(a(e_2)-a(e_1))B^{-2}(a(e_1)p+a(e_2)(1-p)-R) \\ a(e_1)p+a(e_2)(1-p)-R \end{pmatrix}$$

$$\Psi(p) = \mu R + \frac{\mu}{1-\mu}\left[a(e_1)p + a(e_2)(1-p) - R\right]^2 B^{-2}$$

and from here it can be seen that, even in this simple case, an explicit solution of the HJB equation (4.13) is difficult to obtain.

5. Conclusions and computational remarks

Given our expected utility maximization problem for a power utility of the risk averse type, where the coefficients in the asset price dynamics are driven by a hidden finite state Markov process representing "economic factors", we have first discussed a corresponding complete observation control problem based on unnormalized conditional probabilities (unnormalized filter) satisfying a linear Zakai-type equation and shown that for this problem it is difficult to obtain an explicit solution. We have then studied an equivalent complete observation problem based on normalized filter values. For this problem we have studied the corresponding HJB equation that has been shown to admit a unique viscosity solution that can be computed as an expectation according to (4.14) and (4.15). Under sufficient regularity assumptions this solution has enough regularity so that an optimal investment strategy exists and can be computed from the solution of the HJB equation according to (4.9). This strategy is a function of the process $Z_s = [p_s, Y_s]^*$ formed by the pair consisting of the filter p_s in (3.11) for the unobserved factor process X_s and the log-prices Y_s, all of which are accessible to the economic agent.

Since a solution can be obtained in the form of an expectation according to (4.14) and (4.15), it can in general be computed by Monte Carlo simulation. This is important since, as discussed in section 4, also for the complete observation problem based on normalized filter values an analytic solution is very difficult to obtain.

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Hideo Nagai
Department of Mathematical Science for Social Systems, Graduate School of Engineering Science
Osaka University, Toyonaka, 560-8531
Japan
e-mail: `nagai@sigmath.es.osaka-u.ac.jp`

Wolfgang J. Runggaldier
Dipartimento di Matematica Pura ed Applicata
Universita degli Studi di Padova
Via Belzoni, 7, 35131 Padova
Italy
e-mail: `runggal@math.unipd.it`