Stochastic Finance 2004
Autumn School & International Conference

Computing efficient hedging strategies in discontinuous market models

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Abstract. We consider the problem of finding efficient hedging strategies in market models where prices evolve along discontinuous trajectories as a random jump process. We base ourselves on results in [3], that are briefly summarized, and discuss relevant computational issues. Numerical results are also presented.

1. Introduction.

Our problem is of the following general form. Consider a market where agents may invest in a certain number N of (risky) assets, the prices of which we denote by the vector $S_t = (S_t^1, \cdots, S_t^N)$. We assume that S_t^i are already discounted with respect to a given non-risky asset (money market), thereby assuming implicitly that the short rate of interest is zero. We also assume that S_t^i admits a stochastic differential. We denote by $\xi_t = (\xi_t^1, \cdots, \xi_t^N)$ an investment strategy where ξ_t^i denotes the number of units of asset i, $(i = 1, \cdots, N)$ held in the portfolio at time t. Let V_t^ξ be the value, at time t, of the portfolio corresponding to a given strategy ξ that we assume to be self financing, i.e. such that

$$V_t^\xi = V_0^\xi + \int_0^t \xi_s dS_s \ , \quad V_0^\xi \quad \text{given} \tag{1}$$

For simplicity we do not consider transaction costs. Given a maturity T, the problem in its most general form consists in determining ξ such that

$$E\left\{\ell\left(F(S_T), V_T^\xi\right)\right\} \quad \to \quad \min \tag{2}$$

for a given function $F(\cdot)$ of the asset price vector at maturity and a given loss function $\ell(\cdot,\cdot)$. In particular, we are interested in the hedging of a given claim $F(S_T)$, for which we consider more specifically

$$\ell\left(F(S_T), V_T^\xi\right) = \ell\left(F(S_T) - V_T^\xi\right) \tag{3}$$

namely a loss function of the hedging error. We call *efficient* a strategy that achieves the min in (3).

Standard price evolution models are diffusion-type models. However, especially on small time scales, the price evolution exhibits a jumping behavior. This is also the case in other situations, where one does not necessarily consider small time scales, like e.g. in the case of default sensitive assets (see [2], [5]).

A possible model for such a jumping behavior is

$$S_t^i = S_0^i \exp\left[a^i N_t^+ - b^i N_t^-\right] \ , \quad i = 1, \cdots, N \tag{4}$$

where $a^i, b^i > 0$ and N_t^+, N_t^- are independent Poisson processes with intensities λ^+, λ^- respectively. A jump of N_t^+ causes an up-movement of the various S_t^i by a factor e^{a^i} and a jump of N_t^- a down-movement by the factor e^{-b^i}. The model thus generalizes the classical binomial market model by allowing the up- and down-movements to occur at random points in time. While the binomial model is complete, this one is incomplete. Corresponding to the multinomial generalization of the binomial model, here we could more generally consider

$$S_t^i = S_0^i \exp\left[\sum_{h=1}^{H} a^{i,h} N_t^{h,+} - \sum_{k=1}^{K} b^{i,k} N_t^{k,-}\right] \tag{5}$$

with $N_t^{h,+}, N_t^{k,-}$ independent Poisson jump processes.

We assume that the intensities λ^+, λ^- of N_t^+, N_t^- in (4) are constant over time. However, we allow them to be unknown and, taking the Bayesian point of view, we consider them as random variables, the distribution of which is continuously updated on the basis of the observed price movements. Therefore, while the intensities themselves are taken to be constant over time, their Bayesian updating gives them a dynamic aspect.

Standard approaches to solve the optimization problem (3) with (1) and (4) are based either on the method of Dynamic Programming (DP) or on the so-called *martingale method* (see e.g. a survey in [4]). Of the two, DP is inherently a dynamic approach. With uncertainty in the jump intensities and their dynamic Bayesian updating, DP thus turns out to be the more appropriate approach in our setting and this the more so if the purpose is to obtain quantitative results. For the standard diffusion-type price evolution models, DP leads to the solution of HJB-equations with the emphasis on finding explicit analytic solutions. In our context, DP leads to relations of the form as they appear in piecewise deterministic control problems (see e.g. [1]) and our purpose is to present a computationally feasible solution approach.

2. The specific problem.

We consider the case of a single risky asset so that (4) becomes

$$S_t = S_0 \exp\left[a\, N_t^+ - b\, N_t^-\right] \tag{6}$$

with $a, b > 0$ and N_t^+, N_t^- independent Poisson processes with intensities λ^+, λ^- respectively. We allow λ^+, λ^- to be unknown and, taking the Bayesian point of view, we consider them as random variables. Since, for given t, $N_t^i, i = +, -$ are Poisson with parameters $\lambda^i t$, a convenient distribution for λ^i as random variables is a Gamma distribution (*conjugate family*), i.e.

$$f(\lambda^i; \alpha^i, \beta) = \frac{\beta^{\alpha^i}}{\Gamma(\alpha^i)} (\lambda^i)^{\alpha^i - 1} e^{-\beta \lambda^i} \tag{7}$$

In fact, if the prior distribution for λ^i is Gamma, all updated distributions of λ^i are again Gamma : if λ^i has a prior with parameters (α_0^i, β_0) then, if at t one has observed N_t^+, the updated distribution is Gamma with parameters

$$\alpha_t^i = \alpha_0^i + N_t^i \ (i = +, -) \quad , \quad \beta_t = \beta_0 + t \tag{8}$$

Itô's formula implies that, according to (6), one has

$$dS_t = S_{t-}\left[(e^a - 1)dN_t^+ + (e^{-b} - 1)dN_t^-\right] \tag{9}$$

and the self financing property of the strategy ξ, expressed by (1), becomes

$$dV_t^\xi = \xi_t dS_t = \xi_t S_{t-}\left[(e^a - 1)dN_t^+ + (e^{-b} - 1)dN_t^-\right] \tag{10}$$

We assume V_0 to be given and, in what follows, we shall write V_t^ξ whenever we want to stress the dependence of the portfolio value on ξ. For observed portfolio values we shall simply write V_t.

We shall consider a strategy ξ to be admissible if it is predictable with respect to the filtration generated by S_t and such that $V_t^\xi \geq -c$ a.s. for a given $c > 0$. Given $T > 0$, the objective is to minimize

$$E\left\{\ell\left(F(S_T) - V_T^\xi\right)\right\} \quad \rightarrow \quad \min \tag{11}$$

where we suppose $F(\cdot)$ to be continuous and $\ell(\cdot)$ is considered to be a loss function, increasing, convex with $\ell(z) = 0$ for $z < 0$ (e.g. $\ell(z) = z^p$, $p \geq 1$, for $z > 0$). We shall assume that

$$E\left\{\ell\left(F(S_T) + c\right)\right\} < +\infty. \tag{12}$$

In view of the above, in particular the Bayesian updating, a sufficient statistic at the generic time t is the tuple

$$\big(V_t = v, N_t^+ = u, N_t^- = d, t\big) \tag{13}$$

where u, d are positive integers and $t \in [0, T]$. Concerning the range of values for v, let $C(u, d)$ denote the super-hedging capital that depends on the values of $N_t^+ = u, N_t^- = d$, but is independent of t (see [3]). It is the smallest initial capital, beyond which a given claim can always be perfectly hedged with a self financing portfolio. Our hedging problem thus looses its meaning for a value $V_t^\xi = v$ larger than $C(u, d)$. Consequently we shall consider $v \in [-c, C(u, d)]$.

We denote by $\mathcal{A}_{v,u,d,t}$ the class of admissible strategies over $[t, T]$, given the time $t-$statistic (v, u, d). Putting

$$\begin{cases} \tau_n := \inf\{t \geq 0 \,|\, N_t^+ + N_t^- = n\} \\ \hat{\tau}_n := \tau_n \wedge T \end{cases} \tag{14}$$

the admissibility condition $V_t^\xi \geq -c$ then implies

$$\xi_t \in I_{v,u,d} := \left[-\frac{c+v}{S_0 e^{au-bd}(e^a - 1)}, \frac{c+v}{S_0 e^{au-bd}(1 - e^{-b})} \right] \;, \quad t \in (\hat{\tau}_{u+d}, \hat{\tau}_{u+d+1}] \tag{15}$$

The optimal value function (*minimal expected risk*) in (v, u, d, t) is then

$$J^*(v, u, d, t) = \min_{\xi \in \mathcal{A}_{v,u,d,t}} E \left\{ \ell \left(F(S_T) - v - \int_t^T \xi_s dS_s \right) \mid N_t^+ = u, N_t^- = d \right\} \tag{16}$$

3. Solution approach. The solution approach is based on Dynamic Programming (DP). In the case of known intensities, putting $\lambda := \lambda^+ + \lambda^-$, it leads (see [3]) to the following relation for $J^*(v, u, d, t)$

$$\begin{aligned} &J^*(v, u, d, t) = (\mathcal{T} J^*)(v, u, d, t) := \\ &\int_0^{T-t} e^{-\lambda s} \min_{\zeta \in I_{v,u,d}} \left\{ \begin{array}{l} \lambda^+ J^*(v + \zeta S_0 e^{au-bd}(e^a - 1), u+1, d, t+s) \\ +\lambda^- J^*(v + \zeta S_0 e^{au-bd}(e^{-b} - 1), u, d+1, t+s) \end{array} \right\} ds \\ &\quad + e^{-\lambda(T-t)} \ell\big(F(S_0 e^{au-bd}) - v\big) \end{aligned} \tag{17}$$

Intuitively, in (17) $J^*(v, u, d, t)$ appears as the minimum over the investment decision of the "expectation" of the value of J^* at the next jump whereby one takes into account that, over the remaining time to maturity, there may be a next jump either upwards or downwards or no jump at all. The case of no further jump does not affect the minimization; on the other hand, even if the price S remains constant between two successive jumps, the horizon shrinks and so the strategy changes to take this into account. Finally, notice that

$$e^{\lambda(T-t)} = 1 - \int_0^{T-t} (\lambda^+ + \lambda^-) e^{-\lambda s} ds$$

The optimal investment decision at time $t \in (\tau_{u+d}, \tau_{u+d+1}]$ and with $v = V_{\tau_{u+d}}$ is then

$$\xi_t^* = arg \min_{\zeta \in I_{v,u,d}} \left\{ \begin{array}{l} \lambda^+ J^*(v + \zeta S_0 e^{au-bd}(e^a - 1), u+1, d, t) \\ +\lambda^- J^*(v + \zeta S_0 e^{au-bd}(e^{-b} - 1), u, d+1, t) \end{array} \right\} \tag{18}$$

When the intensities are unknown, the relation (17) becomes

$$\begin{aligned} &J^*(v, u, d, t) = (\mathcal{T} J^*)(v, u, d, t) := \\ &\int_0^{T-t} \min_{\zeta \in I_{v,u,d}} \left\{ \begin{array}{l} p^+(u, d, t, s)\, J^*(v + \zeta S_0 e^{au-bd}(e^a - 1), u+1, d, t+s) \\ +p^-(u, d, t, s)\, J^*(v + \zeta S_0 e^{au-bd}(e^{-b} - 1), u, d+1, t+s) \end{array} \right\} ds \\ &\quad + p^0(u + d, t)\, \ell\big(F(S_0 e^{au-bd}) - v\big) \end{aligned} \tag{19}$$

where $p^+(u,d,t,s), p^-(u,d,t,s)$ are the updated probabilities for an up- respectively down-jump at $t+s$, given that $N_t^+ = u, N_t^- = d$. Notice that, even if S remains constant from t to $t+s$, this still reveals additional information regarding the jump intensities. Furthermore, $p^0(u+d,t)$ is the updated probability that no more jumps occur in $[t,T]$, given $N_t^+ = u, N_t^- = d$.

In terms of the updated Gamma distributions for λ^+, λ^- we have (see [3])

$$\begin{cases} p^+(u,d,t,s) & = & \left(\frac{\beta_0+t}{\beta_0+t+s}\right)^{\alpha_0+u+d} \frac{\alpha_0^+ + u}{\beta_0+t+s} \\ p^-(u,d,t,s) & = & \left(\frac{\beta_0+t}{\beta_0+t+s}\right)^{\alpha_0+u+d} \frac{\alpha_0^- + d}{\beta_0+t+s} \end{cases} \tag{20}$$

where $\alpha_0 := \alpha_0^+ + \alpha_0^-$. Notice that both expressions have a common factor except for $\alpha_0^+ + u$ and $\alpha_0^- + d$ respectively. Furthermore,

$$p^0(u+d,t) = 1 - \int_0^{T-t} \left(p^+(u,d,t,s) + p^-(u,d,t,s)\right) ds = \left(\frac{\beta_0 + t}{\beta_0 + T}\right)^{\alpha_0+u+d} \tag{21}$$

For what concerns the optimal investment decision, its value at time t has an expression that is analogous to the case of known intensities (for explicitly computable expressions see S.i)-S.iii), respectively S.i')-S.iii') below).

4. Computational aspects.

From the previous section it follows that, if one is able to compute the solution $J^*(v,u,d,t)$ of (17) respectively (19) for all tuples (v,u,d,t), then one can compute also the optimal strategy and the given problem is completely solved. A direct solution of (17) resp. (19) is difficult, if not impossible to obtain and so in this section we present, extending some of the results in [3], a computationally feasible approximation approach, structured along two levels :

i) *successive iterations*

ii) *quantization coupled with interpolation*

4.1. Successive iterations. The operator $\mathcal{T}$, defined in (17) for the case of known intensities, is a contraction operator with contraction constant $1 - e^{-\lambda T} < 1$ so that the solution J^* of (17) can be obtained in the limit of successive iterations of this same operator $\mathcal{T}$. However, in the case of unknown intensities, the operator $\mathcal{T}$ in (19) contracts with factor $1 - p^0(u+d,t)$ that, see (21), tends to 1 (no contraction) in the limit when the total number $u+d$ of observed jumps tends to ∞.

This situation can be circumvented as follows. Let J^n be the $n-$th iterate of $\mathcal{T}$, both for known and unknown intensities, according to

$$J^0 \equiv 0 \quad \text{and, for } h \leq n, \quad J^h = \mathcal{T} J^{h-1} \tag{22}$$

It can be shown that, see [3],

$$J^n(v,u,d,t) = \min_{\xi \in \mathcal{A}_{v,u,d,t}} E\left\{\ell\left(F(S_T) - V_T^\xi\right), \tau_{u+d+n} > T \mid N_t^+ = u, N_t^- = d\right\} \tag{23}$$

i.e. the $n-$th iterate can be interpreted as minimal risk in (v,u,d,t) if at most n jumps occur in the remaining interval $[t,T]$. It follows that if one fixes a priori a maximum number n of jumps then, both for known and unknown intensities, the $n-$th iterate of $\mathcal{T}$ suffices to obtain the optimal value under this restriction on the number of jumps.

One can then easily see that, for all (v,u,d,t),

$$J^n(\cdot) \leq J^*(\cdot) \leq J^n(\cdot) + E\left\{\ell\left(F(S_T) + c\right), \tau_{u+d+n} \leq T \mid N_t^+ = u, N_t^- = d\right\} \tag{24}$$

Having made the assumption that $E\{\ell(F(S_T) + c)\} < +\infty$, it then follows that

$$J^n(v,u,d,t) \overset{n\to\infty}{\longrightarrow} J^*(v,u,d,t) \tag{25}$$

uniformly in (v,t) for all (u,d).

Consider now the optimal strategy under the restriction of at most n jumps that we denote by ξ_t^n. It is computed as follows, where we consider now only the case of unknown intensities and where we take into account the fact that, by (20), $p^+(\cdot)$ and $p^-(\cdot)$ have a common factor and distinguish themselves only by the factors $\alpha_0^+ + u$ and $\alpha_0^- + d$ respectively :

S.i) for $t \in [0, \hat{\tau}_1]$ and $v = V_0$ put

$$\xi_t^n = arg \min_{\zeta \in I_{v,0,0}} \left\{ \begin{array}{l} \alpha_0^+ J^{n-1}(v + \zeta S_0 e^{au-bd}(e^a - 1), 1, 0, t) \\ \\ +\alpha_0^- J^{n-1}(v + \zeta S_0 e^{au-bd}(e^{-b} - 1), 0, 1, t) \end{array} \right\}$$

S.ii) for $t \in (\hat{\tau}_h, \hat{\tau}_{h+1}]$, $(1 \le h < n-2)$, having observed $N_{\hat{\tau}_h}^+ = u, N_{\hat{\tau}_h}^- = d$, $(u+d=h)$, $V_{\hat{\tau}_h} = v$, put :

$$\xi_t^n = arg \min_{\zeta \in I_{v,u,d}} \left\{ \begin{array}{l} (\alpha_0^+ + u)\, J^{n-h-1}(v + \zeta S_0 e^{au-bd}(e^a - 1), u+1, d, t) \\ \\ +(\alpha_0^- + d)\, J^{n-h-1}(v + \zeta S_0 e^{au-bd}(e^{-b} - 1), u, d+1, t) \end{array} \right\}$$

S.iii) if $\tau_{n-1} < T$ then, for $t \in (\hat{\tau}_{n-1}, T]$, put :

$$\xi_t^n \equiv 0 \quad \textit{(i.e. transfer all funds to the money account)}$$

Let V_t^n be the wealth process associated with ξ_t^n, i.e.

$$V_t^n = V_0 + \int_0^t \xi_s^n dS_s \tag{26}$$

As with (24), one can easily see that, in particular at the initial time i.e. for $(v,u,d,t) = (V_0,0,0,0)$, one has

$$\left\{ \begin{array}{llll} J^n(\cdot) & \le E\left\{\ell\left(F(S_T) - V_T^n\right)\right\} \le & J^n(\cdot) & + E\left\{\ell\left(F(S_T)+c\right), \tau_n \le T\right\} \\ \\ \downarrow\ n \to \infty & & \downarrow\ n \to \infty & \qquad \downarrow\ n \to \infty \\ \\ J^*(\cdot) & & J^*(\cdot) & \qquad 0 \end{array} \right. \tag{27}$$

which is a relation that specifies the sense in which the performance of the strategy ξ^n is suboptimal. In particular, (27) shows that, for $n \to \infty$, the performance of the strategy ξ^n tends to that of the optimal one ξ^*.

Concluding this subsection we have found that, by iterating the operator $\mathcal{T}$ (both for known as well as unknown intensities) a sufficiently large number n of times, one can approximate the optimal value and the performance of the optimal strategy as closely as possible. It remains to actually compute the iterations with the operator $\mathcal{T}$ corresponding to the various possible tuples (v,u,d,t). This is the subject of the next subsection.

4.2. Computation by quantization. The iterations with the operator $\mathcal{T}$ in (17) respectively (19) have to be computed for all possible tuples (v,u,d,t). Given a maximum number n of jumps, one has $u+d \le n$ and so the pair (u,d) takes only a finite number of possible values. The pair (v,t) however takes a continuum of possible values with

$$v \in [-c, C(n)] \quad , \quad t \in [0,T] \tag{28}$$

where, given n,

$$C(n) := \max\{C(u,d) \,|\, u+d \le n\} \tag{29}$$

and where $C(u,d)$ is the super-hedging capital introduced after (13). To make the iteration in (22) computable, we have thus to discretize the possible values of (v,t) and we do this by *quantization* (computation over a grid of values) followed by an interpolation, in the same variables, of the computed values.

More precisely, given n, consider a finite grid G

$$G \subset D := [-c, C(n)] \times [0, T] \tag{30}$$

containing the extremal points of D. Define

$$(\mathcal{T}_G J)(v, u, d, t) := \begin{cases} (\mathcal{T} J)(v, u, d, t) & , \quad \text{if } (v, t) \in G \\ \text{cadlag interpolation} & , \quad \text{else} \end{cases} \tag{31}$$

where by *cadlag interpolation* we mean a right-continuous, piecewise constant interpolation.

Let $J^{n,G}$ denote the $n-$th iterate of $\mathcal{T}_G$ according to

$$J^{0,G} \equiv 0 \quad \text{and, for } h \leq n, \quad J^{h,G} = \mathcal{T}_G J^{h-1,G} \tag{32}$$

More specifically, denote by v_j, $(j = 0, 1, \cdots, J)$ and t_i, $(i = 0, 1, \cdots, I)$ the points in $[-c, C(n)]$ and $[0, T]$ respectively that define the grid $G \subset D$ and let $V_j := [v_j, v_{j+1})$, $(j = 0, \cdots, J-1)$. Taking into account the definition of the operator $\mathcal{T}$ in (19) and the expressions (20), (21), the computation of the recursions in (32) can then be performed according to the formula (see also (55) in [3])

$$\begin{aligned} J^{h,G}(v_j, u, d, t_i) &= J^{1,G}(v_j, u, d, t_i) + \sum_{l=0}^{I-1} 1_{\{t_i \leq t_l\}} \\ &\min_{\zeta \in I_{v_j,u,d}} \left\{ \begin{array}{l} \gamma_{i,l}^u \sum_{m=0}^{J-1} 1_{\{V_m\}} \left(v_j + \zeta S_0 e^{au-bd}(e^a - 1) \right) J^{h-1,G}(v_m, u+1, d, t_l) \\ + \gamma_{i,l}^d \sum_{m=0}^{J-1} 1_{\{V_m\}} \left(v_j + \zeta S_0 e^{au-bd}(e^{-b} - 1) \right) J^{h-1,G}(v_m, u, d+1, t_l) \end{array} \right\} \end{aligned} \tag{33}$$

where

$$\begin{cases} \gamma_{i,l}^u & := \frac{(\beta_0+t_i)^{\alpha_0+u+d}(\alpha_0^+ + u)}{\alpha_0+u+d} \left[(\beta_0 + t_l)^{-(\alpha_0+u+d)} - (\beta_0 + t_{l+1})^{-(\alpha_0+u+d)} \right] \\ \gamma_{i,l}^d & := \gamma_{i,l}^u \frac{(\alpha_0^- + d)}{(\alpha_0^+ + u)} \end{cases} \tag{34}$$

and where (see again (19))

$$\begin{aligned} J^{1,G}(v_j, u, d, t_i) &= p^0(u+d, t_i)\, \ell \left(F(S_0 e^{au-bd}) - v_j \right) \\ &= \left(\frac{\beta_0+t_i}{\beta_0+T} \right)^{\alpha_0+u+d} \ell \left(F(S_0 e^{au-bd}) - v_j \right) \end{aligned} \tag{35}$$

Denote by $\xi^{n,G}$ the strategy, defined by analogy to ξ^n, but corresponding to the iterations of $\mathcal{T}_G$ and let $V_t^{n,G}$ be the associated wealth process. More precisely, recalling from (20) that $p^+(\cdot)$ and $p^-(\cdot)$ have a common factor and distinguish themselves only by the terms $(\alpha_0^+ + u)$ and $(\alpha_0^- + d)$ respectively, we have (compare with (S.i-S.iii):

S.i') for $t \in [0, \hat{\tau}_1]$ and $v = V_0$ put

$$\xi_t^{n,G} = arg \min_{\zeta \in I_{v,0,0}} \left\{ \begin{array}{l} \alpha_0^+ \sum_{m=0}^{J-1} 1_{\{V_m\}}(v + \zeta S_0 e^{au-bd}(e^a - 1))\, J^{n-1,G}(v_m, 1, 0, t) \\ + \alpha_0^- \sum_{m=0}^{J-1} 1_{\{V_m\}}(v + \zeta S_0 e^{au-bd}(e^{-b} - 1))\, J^{n-1,G}(v_m, 0, 1, t) \end{array} \right\}$$

S.ii') for $t \in (\hat{\tau}_h, \hat{\tau}_{h+1}]$, $(1 \leq h < n-2)$, having observed $N_{\hat{\tau}_h}^+ = u, N_{\hat{\tau}_h}^- = d$, $(u+d = h)$, $V_{\hat{\tau}_h} = v$, put :

$$\xi_t^{n,G} = arg \min_{\zeta \in I_{v,u,d}} \left\{ \begin{array}{l} (\alpha_0^+ + u) \sum_{m=0}^{J-1} 1_{\{V_m\}}(v + \zeta S_0 e^{au-bd}(e^a - 1))\, J^{n-h-1,G}(v_m, u+1, d, t) \\ + (\alpha_0^- + d) \sum_{m=0}^{J-1} 1_{\{V_m\}}(v + \zeta S_0 e^{au-bd}(e^{-b} - 1))\, J^{n-h-1,G}(v_m, u, d+1, t) \end{array} \right\}$$

S.iii') if $\tau_{n-1} < T$ then, for $t \in (\hat{\tau}_{n-1}, T]$, put :

$$\xi_t^{n,G} \equiv 0 \quad \textit{(i.e. transfer all funds to the money account)}$$

With respect to S.i)-S.iii), here we have emphasized the fact that, as a function of v, $J^{n,G}(v,u,d,t)$ remains constant for all values of v between two grid points. Notice furthermore that, also as a function of t, $J^{n,G}(v,u,d,t)$ remains constant between two grid points and so $\xi_t^{n,G}$ changes only at a jump time as indicated in S.i')-S.iii') or, within a same interval $(\hat{\tau}_h, \hat{\tau}_{h+1}]$, when t crosses a grid point (see the numerical results below).

Since, with $(v,t) \in G$, the tuples (v,u,d,t) are now finite in number, the values $J^{n,G}$ and the strategies $\xi^{n,G}$ can actually be computed.

We next discuss the goodness of the approximation introduced above.

4.3. Bounds and convergence. Given n, let

$$E^n := \{(v,u,d,t) \mid v \geq -c,\, u+d \leq n,\, t \in [0,T]\} \tag{36}$$

and denote by $\mathcal{D}(E^n)$ the space of cadlag functions on E^n endowed with the sup-norm $||\cdot||_{E^n}$.

We have now two facts. The first one follows straightforwardly from the continuity of $J^*(v,u,d,t)$ (recall that we had assumed $F(\cdot)$ continuous and $\ell(\cdot)$ convex), namely :

$$\begin{cases} \varepsilon(G) := ||\, J^* - \mathcal{T}_G J^*||_{E^n} \;\to\; 0 \\ \text{for } \delta^G := \sup\limits_{(v,t)\in D,\,(v',t')\in G} [|v-v'| + |t-t'|] \;\to\; 0 \end{cases} \tag{37}$$

The second follows from results in [3], in particular Corollary 4.3 and section 5 (notice that, in the notation of [3], we have $H_n^* = H_n^n$ in section 5 there), namely :

$$||\, J^* - J^{n,G}||_{E^n} \leq \frac{\varepsilon(G)}{p^0(n,0)} \tag{38}$$

Combining (37) with (38) we have that, for given n, the upper bound in (38) tends to zero for $\delta^G \to 0$, i.e. for the grid G becoming finer and finer. Notice however that, since $\lim_{n\to\infty} p^0(n,0) = 0$, the convergence to zero of this upper bound becomes slower as n increases.

To evaluate the goodness of the approximation introduced by the computable quantities $J^{n,G}$ and $\xi^{n,G}$ that depend on the choice of n and G, notice that by analogy to (27) we have at the initial time $t = 0$

$$J^{n,G}(\cdot) \leq E\left\{\ell\left(F(S_T) - V_T^{n,G}\right)\right\} \leq J^{n,G}(\cdot) + E\left\{\ell\left(F(S_T)+c\right),\, \tau_n \leq T\right\} \tag{39}$$

which, combined with (38) and the fact that the rightmost term tends to zero for $n \to \infty$, specifies the sub-optimality of the performance of the computable strategy $\xi^{n,G}$: having chosen n sufficiently large so that the rightmost term is small enough, choose the grid G sufficiently fine so that $J^{n,G}(\cdot)$ is close enough to $J^*(\cdot)$ in the sense of (38). By our approach one can thus approximate the optimal value and the performance of the optimal strategy as closely as possible.

5. Example and numerical results.

5.1. Description of the example. We consider here an example corresponding to example 5.1 in [3]. More precisely, we consider the geometric Poisson price model (6) with $S_0 = 1$ and a, b such that $e^a = 2, e^{-b} = 1/2$. Assume the intensities λ^+, λ^- unknown and having as prior distribution a Gamma with parameters $(\alpha_0^+ = 1, \beta_0 = 1)$ and $(\alpha_0^- = 1, \beta_0 = 1)$ respectively. The claim is supposed to be a European call, namely $F(S_T) = (S_T - 1)^+$ and as loss function take a quadratic, namely $\ell(z) = [\max(z,0)]^2$. Finally, take a horizon of $T = 2$ and let the lower bound for the portfolio value correspond to $c = 0.5$. In the given situation the value of $C(n)$ in (29) is bounded from above by $C(n) \leq 2^{n-2}$. As domain D for the pair (v,t) we therefore take the rectangle $D(n) = [-0.5, 2^{n-2}] \times [0,2]$.

5.2. Numerical results. For the given example we report here numerical results for portfolio values and strategies when the maximum number of jumps is supposed to be either $n = 5$ or $n = 8$ so that some comparison can be made. The portfolio values are reported also for $n = 6$ and $n = 7$. We recall from section 4.1 that, if the actual number of jumps turns out to be larger than the given n, then we put $\xi_t^{n,G} \equiv 0$ for $t \in (\hat{\tau}_{n-1}, T]$, i.e. we transfer all funds to the money account.

5.2.1. Case of $n = 5$ and quantization given by $(v_0, \cdots, v_3) = (-\frac{1}{2}, 0, 1, 2, 4, 8)$, $(t_0, \cdots, t_3) = (0, \frac{1}{2}, 1, 2)$. The strategy $\xi_t^{5,G}$ is described in the following table, where an interval in the first column means that the strategy can be assigned any value within that interval and where the values for $V_{\hat{\tau}_k}$ $(k = 1, 2, 3)$ are the values on the grid that correspond to the left end point of the interval that contains the actual value of $V_{\hat{\tau}_k}$.

$$\xi_s^{5,G} = \begin{cases} 1 & \text{for } s \in [0, \hat{\tau}_1] \cap [0,2],\ V_0 = 0 \\ [1,2] & \text{for } s \in [0, \hat{\tau}_1] \cap [0,2],\ V_0 = 1 \\ [0,2] & \text{for } s \in [0, \hat{\tau}_1] \cap [0,2],\ V_0 = 2 \\ [-2,4] & \text{for } s \in [0, \hat{\tau}_1] \cap [0,2],\ V_0 = 4 \\ 0 & \text{for } s \in [0, \hat{\tau}_1] \cap [0,2],\ V_0 = 8 \\ \frac{3}{2} & \text{for } s \in (\hat{\tau}_1, \hat{\tau}_2] \cap [0,2] \ \text{ with } u = 1, d = 0,\ V_{\hat{\tau}_1} = 1 \\ 1 & \text{for } s \in (\hat{\tau}_1, \hat{\tau}_2] \cap [0,2] \ \text{ with } u = 1, d = 0,\ V_{\hat{\tau}_1} = 2 \\ 0 & \text{for } s \in (\hat{\tau}_1, \hat{\tau}_2] \cap [0,2] \ \text{ with } u = 1, d = 0,\ V_{\hat{\tau}_1} = 8 \\ 0 & \text{for } s \in (\hat{\tau}_1, \hat{\tau}_2] \cap [0,2] \ \text{ with } u = 0, d = 1,\ V_{\hat{\tau}_1} \in \{-\frac{1}{2}, 0, 8\} \\ [-2,8] & \text{for } s \in (\hat{\tau}_1, \hat{\tau}_2] \cap [0,2] \ \text{ with } u = 0, d = 1,\ V_{\hat{\tau}_1} = 2 \\ [-6,8] & \text{for } s \in (\hat{\tau}_1, \hat{\tau}_2] \cap [0,2] \ \text{ with } u = 0, d = 1,\ V_{\hat{\tau}_1} = 4 \\ 0 & \text{for } s \in (\hat{\tau}_2, \hat{\tau}_3] \cap [0,2] \ \text{ with } u = 1, d = 1,\ V_{\hat{\tau}_2} \in \{-\frac{1}{2}, 8\} \\ 1 & \text{for } s \in (\hat{\tau}_2, \hat{\tau}_3] \cap [0,2] \ \text{ with } u = 1, d = 1,\ V_{\hat{\tau}_2} = 0 \\ [1,2] & \text{for } s \in (\hat{\tau}_2, \hat{\tau}_3] \cap [0,2] \ \text{ with } u = 1, d = 1,\ V_{\hat{\tau}_2} = 1 \\ [0,4] & \text{for } s \in (\hat{\tau}_2, \hat{\tau}_3] \cap [0,2] \ \text{ with } u = 1, d = 1,\ V_{\hat{\tau}_2} = 2 \\ [-2,4] & \text{for } s \in (\hat{\tau}_2, \hat{\tau}_3] \cap [0,2] \ \text{ with } u = 1, d = 1,\ V_{\hat{\tau}_2} = 4 \\ 0 & \text{for } s \in (\hat{\tau}_2, \hat{\tau}_3] \cap [1,2] \ \text{ with } u = 2, d = 0,\ V_{\hat{\tau}_2} = 8 \\ 1 & \text{for } s \in (\hat{\tau}_2, \hat{\tau}_3] \cap [1,2] \ \text{ with } u = 2, d = 0,\ V_{\hat{\tau}_2} = 4 \\ 0 & \text{for } s \in (\hat{\tau}_2, \hat{\tau}_3] \cap [0,2] \ \text{ with } u = 0, d = 2,\ V_{\hat{\tau}_2} \in \{-\frac{1}{2}, 0, 8\} \\ [-8,16] & \text{for } s \in (\hat{\tau}_2, \hat{\tau}_3] \cap [0,2] \ \text{ with } u = 0, d = 2,\ V_{\hat{\tau}_2} = 2 \\ [-16,16] & \text{for } s \in (\hat{\tau}_2, \hat{\tau}_3] \cap [0,2] \ \text{ with } u = 0, d = 2,\ V_{\hat{\tau}_2} = 4 \\ 0 & \text{for } s \in (\hat{\tau}_3, \hat{\tau}_4] \cap [0,2] \ \text{ with } u = 2, d = 1,\ V_{\hat{\tau}_3} \in \{-\frac{1}{2}, 8\} \\ \frac{3}{2} & \text{for } s \in (\hat{\tau}_3, \hat{\tau}_4] \cap [0,2] \ \text{ with } u = 2, d = 1,\ V_{\hat{\tau}_3} = 1 \\ 1 & \text{for } s \in (\hat{\tau}_3, \hat{\tau}_4] \cap [0,2] \ \text{ with } u = 2, d = 1,\ V_{\hat{\tau}_3} = 2 \\ 0 & \text{for } s \in (\hat{\tau}_3, \hat{\tau}_4] \cap [0,2] \ \text{ with } u = 1, d = 2,\ V_{\hat{\tau}_3} \in \{-\frac{1}{2}, 0, 8\} \\ [-4,8] & \text{for } s \in (\hat{\tau}_3, \hat{\tau}_4] \cap [0,2] \ \text{ with } u = 1, d = 2,\ V_{\hat{\tau}_3} = 2 \\ [-8,8] & \text{for } s \in (\hat{\tau}_3, \hat{\tau}_4] \cap [0,2] \ \text{ with } u = 1, d = 2,\ V_{\hat{\tau}_3} = 4 \\ 0 & \text{for } s \in (\hat{\tau}_3, \hat{\tau}_4] \cap [0,2] \ \text{ with } u = 3, d = 0,\ V_{\hat{\tau}_3} = 8 \\ 0 & \text{for } s \in (\hat{\tau}_3, \hat{\tau}_4] \cap [0,2] \ \text{ with } u = 0, d = 3,\ V_{\hat{\tau}_3} \in \{-\frac{1}{2}, 0, 8\} \\ [-16,32] & \text{for } s \in (\hat{\tau}_3, \hat{\tau}_4] \cap [0,2] \ \text{ with } u = 0, d = 3,\ V_{\hat{\tau}_3} = 2 \\ [-32,32] & \text{for } s \in (\hat{\tau}_3, \hat{\tau}_4] \cap [0,2] \ \text{ with } u = 0, d = 3,\ V_{\hat{\tau}_3} = 4 \end{cases}$$

The values of $J^{5,G}$ for the various initial conditions $(v, 0, 0, 0)$ corresponding to the 5 non-negative grid-values of v are shown on the next table. As expected, they are decreasing in v.

$$\begin{cases} J^{5,G}(0,0,0,0) &= 0.8366 \\ J^{5,G}(1,0,0,0) &= 0.7023 \\ J^{5,G}(2,0,0,0) &= 0.6994 \\ J^{5,G}(4,0,0,0) &= 0.6994 \\ J^{5,G}(8,0,0,0) &= 0.6994 \end{cases}$$

5.2.2. Case of $n = 8$ and quantization given by $(v_0, \cdots, v_9) = (-\frac{1}{2}, 0, 1, 2, 4, 8, 16, 32, 64)$, $(t_0, \cdots, t_3) = (0, \frac{1}{2}, 1, 2)$**.** Corresponding to the previous subsection, in the next table we describe the strategy $\xi_t^{8,G}$ that now is naturally more complex.

$$\xi_s^{8,G} = \begin{cases}
1 & \text{for } s \in [0, \hat\tau_1] \cap [0,2],\ V_0 = 0 \\
[1,2] & \text{for } s \in [0, \hat\tau_1] \cap [0,2],\ V_0 = 1 \\
\{0,2\} & \text{for } s \in [0, \hat\tau_1] \cap [0,2],\ V_0 = 2 \\
\{0,4\} & \text{for } s \in [0, \hat\tau_1] \cap [0,2],\ V_0 = 4 \\
[0,8] & \text{for } s \in [0, \hat\tau_1] \cap [0,2],\ V_0 = 8 \\
[-8,24] & \text{for } s \in [0, \hat\tau_1] \cap [0,2],\ V_0 = 16 \\
[-24,32] & \text{for } s \in [0, \hat\tau_1] \cap [0,2],\ V_0 = 32 \\
0 & \text{for } s \in [0, \hat\tau_1] \cap [0,2],\ V_0 = 64 \\
0 & \text{for } s \in (\hat\tau_1, \hat\tau_2] \cap [0,2] \ \text{ with } u = 1, d = 0,\ V_{\hat\tau_1} = 64 \\
\frac{3}{2} & \text{for } s \in (\hat\tau_1, \hat\tau_2] \cap [0,2] \ \text{ with } u = 1, d = 0,\ V_{\hat\tau_1} = 1 \\
1 & \text{for } s \in (\hat\tau_1, \hat\tau_2] \cap [0,2] \ \text{ with } u = 1, d = 0,\ V_{\hat\tau_1} = 2 \\
\{0,2\} & \text{for } s \in (\hat\tau_1, \hat\tau_2] \cap [0,2] \ \text{ with } u = 1, d = 0,\ V_{\hat\tau_1} = 4 \\
[0,4] & \text{for } s \in (\hat\tau_1, \hat\tau_2] \cap [0,2] \ \text{ with } u = 1, d = 0,\ V_{\hat\tau_1} = 8 \\
0 & \text{for } s \in (\hat\tau_1, \hat\tau_2] \cap [0,2] \ \text{ with } u = 0, d = 1,\ V_{\hat\tau_1} \in \{-\frac{1}{2}, 64\} \\
2 & \text{for } s \in (\hat\tau_1, \hat\tau_2] \cap [0,1] \ \text{ with } u = 0, d = 1,\ V_{\hat\tau_1} = 0 \\
0 & \text{for } s \in (\hat\tau_1, \hat\tau_2] \cap [1,2] \ \text{ with } u = 0, d = 1,\ V_{\hat\tau_1} = 0 \\
\frac{1}{2} & \text{for } s \in (\hat\tau_1, \hat\tau_2] \cap [0,2] \ \text{ with } u = 0, d = 1,\ V_{\hat\tau_1} = 2 \\
1 & \text{for } s \in (\hat\tau_1, \hat\tau_2] \cap [0,2] \ \text{ with } u = 0, d = 1,\ V_{\hat\tau_1} = 4 \\
[0,2] & \text{for } s \in (\hat\tau_1, \hat\tau_2] \cap [0,2] \ \text{ with } u = 0, d = 1,\ V_{\hat\tau_1} = 8 \\
[-2,6] & \text{for } s \in (\hat\tau_1, \hat\tau_2] \cap [0,2] \ \text{ with } u = 0, d = 1,\ V_{\hat\tau_1} = 16 \\
[-6,8] & \text{for } s \in (\hat\tau_1, \hat\tau_2] \cap [0,2] \ \text{ with } u = 0, d = 1,\ V_{\hat\tau_1} = 32 \\
0 & \text{for } s \in (\hat\tau_2, \hat\tau_3] \cap [0,2] \ \text{ with } u = 1, d = 1,\ V_{\hat\tau_2} \in \{-\frac{1}{2}, 64\} \\
1 & \text{for } s \in (\hat\tau_2, \hat\tau_3] \cap [1,2] \ \text{ with } u = 1, d = 1,\ V_{\hat\tau_2} = 0 \\
[1,2] & \text{for } s \in (\hat\tau_2, \hat\tau_3] \cap [0,2] \ \text{ with } u = 1, d = 1,\ V_{\hat\tau_2} = 1 \\
\{0,2\} & \text{for } s \in (\hat\tau_2, \hat\tau_3] \cap [0,2] \ \text{ with } u = 1, d = 1,\ V_{\hat\tau_2} = 2 \\
[0,4] & \text{for } s \in (\hat\tau_2, \hat\tau_3] \cap [0,2] \ \text{ with } u = 1, d = 1,\ V_{\hat\tau_2} = 4 \\
[-4,12] & \text{for } s \in (\hat\tau_2, \hat\tau_3] \cap [0,2] \ \text{ with } u = 1, d = 1,\ V_{\hat\tau_2} = 8 \\
[-12,24] & \text{for } s \in (\hat\tau_2, \hat\tau_3] \cap [0,2] \ \text{ with } u = 1, d = 1,\ V_{\hat\tau_2} = 16 \\
0 & \text{for } s \in (\hat\tau_2, \hat\tau_3] \cap [1,2] \ \text{ with } u = 2, d = 0,\ V_{\hat\tau_2} = 64 \\
1 & \text{for } s \in (\hat\tau_2, \hat\tau_3] \cap [1,2] \ \text{ with } u = 2, d = 0,\ V_{\hat\tau_2} = 4 \\
[0,2] & \text{for } s \in (\hat\tau_2, \hat\tau_3] \cap [1,2] \ \text{ with } u = 2, d = 0,\ V_{\hat\tau_2} = 8 \\
0 & \text{for } s \in (\hat\tau_2, \hat\tau_3] \cap [0,2] \ \text{ with } u = 0, d = 2,\ V_{\hat\tau_2} \in \{-\frac{1}{2}, 0, 64\} \\
[4,8] & \text{for } s \in (\hat\tau_2, \hat\tau_3] \cap [0,2] \ \text{ with } u = 0, d = 2,\ V_{\hat\tau_2} = 1 \\
[0,8] & \text{for } s \in (\hat\tau_2, \hat\tau_3] \cap [0,2] \ \text{ with } u = 0, d = 2,\ V_{\hat\tau_2} = 2 \\
[-24,56] & \text{for } s \in (\hat\tau_2, \hat\tau_3] \cap [0,2] \ \text{ with } u = 0, d = 2,\ V_{\hat\tau_2} = 8 \\
[-56,120] & \text{for } s \in (\hat\tau_2, \hat\tau_3] \cap [0,2] \ \text{ with } u = 0, d = 2,\ V_{\hat\tau_2} = 16 \\
[-120,128] & \text{for } s \in (\hat\tau_2, \hat\tau_3] \cap [0,2] \ \text{ with } u = 0, d = 2,\ V_{\hat\tau_2} = 32 \\
0 & \text{for } s \in (\hat\tau_3, \hat\tau_4] \cap [0,2] \ \text{ with } u = 2, d = 1,\ V_{\hat\tau_3} = \{-\frac{1}{2}, 64\} \\
\frac{3}{2} & \text{for } s \in (\hat\tau_3, \hat\tau_4] \cap [0,2] \ \text{ with } u = 2, d = 1,\ V_{\hat\tau_3} = 1 \\
1 & \text{for } s \in (\hat\tau_3, \hat\tau_4] \cap [0,2] \ \text{ with } u = 2, d = 1,\ V_{\hat\tau_3} = 2 \\
[0,2] & \text{for } s \in (\hat\tau_3, \hat\tau_4] \cap [0,2] \ \text{ with } u = 2, d = 1,\ V_{\hat\tau_3} = 4 \\
[-2,6] & \text{for } s \in (\hat\tau_3, \hat\tau_4] \cap [0,2] \ \text{ with } u = 2, d = 1,\ V_{\hat\tau_3} = 8 \\
0 & \text{for } s \in (\hat\tau_3, \hat\tau_4] \cap [0,2] \ \text{ with } u = 1, d = 2,\ V_{\hat\tau_3} \in \{-\frac{1}{2}, 0, 64\} \\
[0,4] & \text{for } s \in (\hat\tau_3, \hat\tau_4] \cap [0,2] \ \text{ with } u = 1, d = 2,\ V_{\hat\tau_3} = 2 \\
[-4,12] & \text{for } s \in (\hat\tau_3, \hat\tau_4] \cap [0,2] \ \text{ with } u = 1, d = 2,\ V_{\hat\tau_3} = 4 \\
[-12,28] & \text{for } s \in (\hat\tau_3, \hat\tau_4] \cap [0,2] \ \text{ with } u = 1, d = 2,\ V_{\hat\tau_3} = 8 \\
[-28,60] & \text{for } s \in (\hat\tau_3, \hat\tau_4] \cap [0,2] \ \text{ with } u = 1, d = 2,\ V_{\hat\tau_3} = 16 \\
0 & \text{for } s \in (\hat\tau_3, \hat\tau_4] \cap [0,2] \ \text{ with } u = 3, d = 0,\ V_{\hat\tau_3} = 64 \\
1 & \text{for } s \in (\hat\tau_3, \hat\tau_4] \cap [0,2] \ \text{ with } u = 3, d = 0,\ V_{\hat\tau_3} = 8
\end{cases}$$

$$\xi_s^{8,G} = \begin{cases} 0 & \text{for } s \in (\hat\tau_3, \hat\tau_4] \cap [0,2] \text{ with } u=0, d=3,\ V_{\hat\tau_3} \in \{-\frac{1}{2}, 0, 64\} \\ \text{[-8,32]} & \text{for } s \in (\hat\tau_3, \hat\tau_4] \cap [0,2] \text{ with } u=0, d=3,\ V_{\hat\tau_3} = 2 \\ \text{[-56,128]} & \text{for } s \in (\hat\tau_3, \hat\tau_4] \cap [0,2] \text{ with } u=0, d=3,\ V_{\hat\tau_3} = 8 \\ \text{[-120,256]} & \text{for } s \in (\hat\tau_3, \hat\tau_4] \cap [0,2] \text{ with } u=0, d=3,\ V_{\hat\tau_3} = 16 \\ \text{[-248,256]} & \text{for } s \in (\hat\tau_3, \hat\tau_4] \cap [0,2] \text{ with } u=0, d=3,\ V_{\hat\tau_3} = 32 \\ 0 & \text{for } s \in (\hat\tau_4, \hat\tau_5] \cap [0,2] \text{ with } u=4, d=0,\ V_{\hat\tau_4} = 64 \\ 1 & \text{for } s \in (\hat\tau_4, \hat\tau_5] \cap [0,2] \text{ with } u=4, d=0,\ V_{\hat\tau_4} = 16 \\ 0 & \text{for } s \in (\hat\tau_4, \hat\tau_5] \cap [0,2] \text{ with } u=0, d=4,\ V_{\hat\tau_4} \in \{-\frac{1}{2}, 0, 64\} \\ \text{[-32,64]} & \text{for } s \in (\hat\tau_4, \hat\tau_5] \cap [0,2] \text{ with } u=0, d=4,\ V_{\hat\tau_4} = 2 \\ \text{[-128,256]} & \text{for } s \in (\hat\tau_4, \hat\tau_5] \cap [0,2] \text{ with } u=0, d=4,\ V_{\hat\tau_4} = 8 \\ \text{[-256,512]} & \text{for } s \in (\hat\tau_4, \hat\tau_5] \cap [0,2] \text{ with } u=0, d=4,\ V_{\hat\tau_4} = 16 \\ \text{[-512,512]} & \text{for } s \in (\hat\tau_4, \hat\tau_5] \cap [0,2] \text{ with } u=0, d=4,\ V_{\hat\tau_4} = 32 \\ 0 & \text{for } s \in (\hat\tau_4, \hat\tau_5] \cap [0,2] \text{ with } u=2, d=2,\ V_{\hat\tau_4} \in \{-\frac{1}{2}, 64\} \\ \text{[1,2]} & \text{for } s \in (\hat\tau_4, \hat\tau_5] \cap [0,2] \text{ with } u=2, d=2,\ V_{\hat\tau_4} = 1 \\ 1 & \text{for } s \in (\hat\tau_4, \hat\tau_5] \cap [0,2] \text{ with } u=2, d=2,\ V_{\hat\tau_4} = 0 \\ \text{[0,2]} & \text{for } s \in (\hat\tau_4, \hat\tau_5] \cap [0,2] \text{ with } u=2, d=2,\ V_{\hat\tau_4} = 2 \\ \text{[-2,6]} & \text{for } s \in (\hat\tau_4, \hat\tau_5] \cap [0,2] \text{ with } u=2, d=2,\ V_{\hat\tau_4} = 4 \\ 0 & \text{for } s \in (\hat\tau_4, \hat\tau_5] \cap [0,2] \text{ with } u=3, d=1,\ V_{\hat\tau_4} \in \{-\frac{1}{2}, 64\} \\ 1 & \text{for } s \in (\hat\tau_4, \hat\tau_5] \cap [0,2] \text{ with } u=3, d=1,\ V_{\hat\tau_4} = 4 \\ 0 & \text{for } s \in (\hat\tau_4, \hat\tau_5] \cap [0,2] \text{ with } u=1, d=3,\ V_{\hat\tau_4} \in \{-\frac{1}{2}, 0, 64\} \\ \text{[0,8]} & \text{for } s \in (\hat\tau_4, \hat\tau_5] \cap [0,2] \text{ with } u=1, d=3,\ V_{\hat\tau_4} = 1 \\ \text{[-4,16]} & \text{for } s \in (\hat\tau_4, \hat\tau_5] \cap [0,2] \text{ with } u=1, d=3,\ V_{\hat\tau_4} = 2 \\ \text{[-12,32]} & \text{for } s \in (\hat\tau_4, \hat\tau_5] \cap [0,2] \text{ with } u=1, d=3,\ V_{\hat\tau_4} = 4 \\ \text{[-28,64]} & \text{for } s \in (\hat\tau_4, \hat\tau_5] \cap [0,2] \text{ with } u=1, d=3,\ V_{\hat\tau_4} = 8 \\ \text{[-60,128]} & \text{for } s \in (\hat\tau_4, \hat\tau_5] \cap [0,2] \text{ with } u=1, d=3,\ V_{\hat\tau_4} = 16 \\ 0 & \text{for } s \in (\hat\tau_5, \hat\tau_6] \cap [0,2] \text{ with } u=5, d=0,\ V_{\hat\tau_5} = 64 \\ 1 & \text{for } s \in (\hat\tau_5, \hat\tau_6] \cap [0,2] \text{ with } u=5, d=0,\ V_{\hat\tau_5} = 32 \\ 0 & \text{for } s \in (\hat\tau_5, \hat\tau_6] \cap [0,2] \text{ with } u=0, d=5,\ V_{\hat\tau_5} \in \{-\frac{1}{2}, 0, 64\} \\ \text{[-64,128]} & \text{for } s \in (\hat\tau_5, \hat\tau_6] \cap [0,2] \text{ with } u=0, d=5,\ V_{\hat\tau_5} = 2 \\ \text{[-256,512]} & \text{for } s \in (\hat\tau_5, \hat\tau_6] \cap [0,2] \text{ with } u=0, d=5,\ V_{\hat\tau_5} = 8 \\ \text{[-512,1024]} & \text{for } s \in (\hat\tau_5, \hat\tau_6] \cap [0,2] \text{ with } u=0, d=5,\ V_{\hat\tau_5} = 16 \\ \text{[-1024,1024]} & \text{for } s \in (\hat\tau_5, \hat\tau_6] \cap [0,2] \text{ with } u=0, d=5,\ V_{\hat\tau_5} = 32 \\ 0 & \text{for } s \in (\hat\tau_5, \hat\tau_6] \cap [0,2] \text{ with } u=4, d=1,\ V_{\hat\tau_5} \in \{-\frac{1}{2}, 64\} \\ 1 & \text{for } s \in (\hat\tau_5, \hat\tau_6] \cap [0,2] \text{ with } u=4, d=1,\ V_{\hat\tau_5} = 8 \\ 0 & \text{for } s \in (\hat\tau_5, \hat\tau_6] \cap [0,2] \text{ with } u=1, d=4,\ V_{\hat\tau_5} \in \{-\frac{1}{2}, 0, 64\} \\ \text{[-8,16]} & \text{for } s \in (\hat\tau_5, \hat\tau_6] \cap [0,2] \text{ with } u=1, d=4,\ V_{\hat\tau_5} = 1 \\ \text{[-16,32]} & \text{for } s \in (\hat\tau_5, \hat\tau_6] \cap [0,2] \text{ with } u=1, d=4,\ V_{\hat\tau_5} = 2 \\ \text{[-32,64]} & \text{for } s \in (\hat\tau_5, \hat\tau_6] \cap [0,2] \text{ with } u=1, d=4,\ V_{\hat\tau_5} = 4 \\ \text{[-64,128]} & \text{for } s \in (\hat\tau_5, \hat\tau_6] \cap [0,2] \text{ with } u=1, d=4,\ V_{\hat\tau_5} = 8 \\ \text{[-128,256]} & \text{for } s \in (\hat\tau_5, \hat\tau_6] \cap [0,2] \text{ with } u=1, d=4,\ V_{\hat\tau_5} = 16 \\ 0 & \text{for } s \in (\hat\tau_5, \hat\tau_6] \cap [0,2] \text{ with } u=3, d=2,\ V_{\hat\tau_5} \in \{-\frac{1}{2}, 64\} \\ \frac{3}{2} & \text{for } s \in (\hat\tau_5, \hat\tau_6] \cap [0,2] \text{ with } u=3, d=2,\ V_{\hat\tau_5} = 1 \\ 1 & \text{for } s \in (\hat\tau_5, \hat\tau_6] \cap [0,2] \text{ with } u=3, d=2,\ V_{\hat\tau_5} = 2 \\ 0 & \text{for } s \in (\hat\tau_5, \hat\tau_6] \cap [0,2] \text{ with } u=2, d=3,\ V_{\hat\tau_5} \in \{-\frac{1}{2}, 0, 64\} \\ \text{[0,4]} & \text{for } s \in (\hat\tau_5, \hat\tau_6] \cap [0,2] \text{ with } u=2, d=3,\ V_{\hat\tau_5} = 1 \\ \text{[-2,8]} & \text{for } s \in (\hat\tau_5, \hat\tau_6] \cap [0,2] \text{ with } u=2, d=3,\ V_{\hat\tau_5} = 2 \\ \text{[-6,16]} & \text{for } s \in (\hat\tau_5, \hat\tau_6] \cap [0,2] \text{ with } u=2, d=3,\ V_{\hat\tau_5} = 4 \end{cases}$$

$$\xi_s^{8,G} = \begin{cases} 0 & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=6, d=0,\ V_{\hat\tau_6}=64 \\ \frac{1}{2} & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=6, d=0,\ V_{\hat\tau_6}=32 \\ 0 & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=0, d=6,\ V_{\hat\tau_6}\in\{-\frac{1}{2},0,64\} \\ [-256,512] & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=0, d=6,\ V_{\hat\tau_6}=4 \\ [-1024,2048] & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=0, d=6,\ V_{\hat\tau_6}=16 \\ [-2048,2048] & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=0, d=6,\ V_{\hat\tau_6}=32 \\ 0 & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=1, d=5,\ V_{\hat\tau_6}\in\{-\frac{1}{2},0,64\} \\ [-16,32] & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=1, d=5,\ V_{\hat\tau_6}=1 \\ [-32,64] & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=1, d=5,\ V_{\hat\tau_6}=2 \\ [-64,128] & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=1, d=5,\ V_{\hat\tau_6}=4 \\ [-128,256] & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=1, d=5,\ V_{\hat\tau_6}=8 \\ [-256,512] & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=1, d=5,\ V_{\hat\tau_6}=16 \\ 0 & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=5, d=1,\ V_{\hat\tau_6}=64 \\ 1 & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=5, d=1,\ V_{\hat\tau_6}=16 \\ 0 & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=3, d=3,\ V_{\hat\tau_6}\in\{-\frac{1}{2},64\} \\ 1 & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=3, d=3,\ V_{\hat\tau_6}=0 \\ [0,2] & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=3, d=3,\ V_{\hat\tau_6}=1 \\ 0 & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=2, d=4,\ V_{\hat\tau_6}\in\{-\frac{1}{2},0,64\} \\ [-4,8] & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=2, d=4,\ V_{\hat\tau_6}=1 \\ [-8,16] & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=2, d=4,\ V_{\hat\tau_6}=2 \\ [-16,32] & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=2, d=4,\ V_{\hat\tau_6}=4 \\ 0 & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=4, d=2,\ V_{\hat\tau_6}=64 \\ [1,\frac{3}{2}] & \text{for } s \in (\hat\tau_6, \hat\tau_7] \cap [0,2] & \text{with } u=4, d=2,\ V_{\hat\tau_6}=4 \end{cases}$$

In the next table we also show the values of $J^{8,G}$ for the various initial conditions $(v,0,0,0)$ corresponding to the 8 non-negative grid-values of v. Since more jumps imply a riskier context, the values of $J^{8,G}$ are naturally larger than the corresponding values for $n=5$.

$$\begin{cases} J^{8,G}(0,0,0,0) &= 16.8681 \\ J^{8,G}(1,0,0,0) &= 15.388 \\ J^{8,G}(2,0,0,0) &= 15.3612 \\ J^{8,G}(4,0,0,0) &= 15.3604 \\ J^{8,G}(8,0,0,0) &= 15.3603 \\ J^{8,G}(16,0,0,0) &= 15.3603 \\ J^{8,G}(32,0,0,0) &= 15.3603 \\ J^{8,G}(64,0,0,0) &= 15.3603 \end{cases}$$

Finally, without reporting also the strategies for the intermediate cases of $n=6$ and $n=7$, in the next two tables we also show the values of $J^{n,G}$ for these cases and for the various initial conditions $(v,0,0,0)$ corresponding to the n non-negative grid-values of v. From these values and those shown above for $n=5$ and $n=8$ one can get a feeling for the increase of the minimal expected risk corresponding to an increase in the number of jumps, i.e. to an increase of the riskiness of the situation (for one more jump the minimal expected risk increases roughly by a factor of 3).

$$\begin{cases} J^{6,G}(0,0,0,0) &= 2.3432 \\ J^{6,G}(1,0,0,0) &= 2.050 \\ J^{6,G}(2,0,0,0) &= 2.0437 \\ J^{6,G}(4,0,0,0) &= 2.0435 \\ J^{6,G}(8,0,0,0) &= 2.0435 \\ J^{6,G}(16,0,0,0) &= 2.0435 \end{cases}$$

$$\begin{cases} J^{7,G}(0,0,0,0) & = & 6.3283 \\ J^{7,G}(1,0,0,0) & = & 5.677 \\ J^{7,G}(2,0,0,0) & = & 5.663 \\ J^{7,G}(4,0,0,0) & = & 5.6632 \\ J^{7,G}(8,0,0,0) & = & 5.6632 \\ J^{7,G}(16,0,0,0) & = & 5.6632 \\ J^{7,G}(32,0,0,0) & = & 5.6632 \end{cases}$$

5.3. Conclusions. As can be guessed from the numerical results reported above, the calculations become increasingly heavier with increasing values of n. The grid on the other hand influences less markedly the computational complexity. One may thus conclude that the algorithm described in the paper performs sufficiently well in situations where one does not expect too many jumps to happen as in the case of default sensitive assets. If there are many jumps such as in situations of high frequency data and small time scales, then it may be advisable to model the price evolution by means of continuous trajectories (approximating the de facto discontinuous trajectories by continuous ones) and use an algorithm tailored to this latter situation.

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