# Estimation via stochastic filtering in financial market models

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Abstract. When specifying a financial market model one needs to specify also the model coefficients. The latter may be only partially known and so, in order to solve problems related to financial markets, it becomes important to exploit all the information coming from the market itself in order to continuously update the knowledge of the not fully known quantities and this is where stochastic filtering becomes useful. The information from the market is not only given by the prices of the underlying primary assets, but also by the prices of the liquidly traded derivatives. A major problem in this context is that derivative prices are given as expectations under a martingale measure, while the actual observations take place under the real world probability measure. In the paper we discuss various ways to deal with this problem.

## 1. Introduction

When specifying a financial market model, one has also to specify the model coefficients. The latter may however be only partially known or depend on stochastic factors that in turn may not be fully observable. When solving problems related to financial markets, like in portfolio optimization or derivative pricing and hedging, it is therefore appropriate to exploit all the information coming from the market itself to continuously update the knowledge of the not fully known quantities in the model and this is where stochastic filtering proves itself as a useful technique. In fact, in stochastic filtering, which can be viewed as a dynamic extension of Bayesian statistics, all not fully known quantities are considered as random variables or stochastic processes and their distribution is continuously updated on the basis of the currently available information.

The main actors in a financial market are the various assets that may be classified into two main categories : primary or underlying assets and derivative assets, where the prices of the latter are "derived" from the prices of the primary assets and can be expressed as expectations under a so-called *martingale measure* (MM). In a complete market there exists only one MM and so all prices are fully specified within the model. If however the market is incomplete, and this corresponds to essentially all practical situations, then there exist more MM's and so, in order to

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perform the pricing of derivatives that are not already traded on the market, one has first to infer the prevailing martingale measure or, equivalently, the so called *market price of risk*. This market price of risk cannot be directly observed on the market so that, again, filtering techniques may be used to continuously update its knowledge. It may be remarked in this context that the pricing of derivatives in an incomplete market can also be accomplished be the method of pricing by utility maximization or by "indifference pricing" (see e.g. [**D**], [**F**], [**HKS**], [**K**], [**RE**]), where the martingale measure is linked to a utility function. Apart from the fact that we need a dynamic representation over time of the observed derivative prices, our aim here is a fully data-oriented approach, whereby the prevailing martingale measure is continuously updated on the basis of the observed market data via the market price of risk. In this sense our approach is more statistical and, if linked to a portfolio optimization problem, it corresponds to approaches in stochastic control under partial information.

The prices of the primary assets as well as those of the derivative assets that are liquidly traded constitute the main information available on a given market and thus also the basic ingredient of filtering. In this context, the fact that the prices of the derivative assets, also of those that are liquidly traded, are specified as expectations under a martingale measure becomes a major problem since the actual observations take place under the real world probability measure, under which the dynamics of the observables in a stochastic dynamic filtering model have thus to be specified.

The purpose of this paper is to present some approaches to deal with this major problem for different types of market models. More precisely, in section 2 we shall consider a standard market model of the Black and Scholes type where the coefficients are not completely known. In section 3 we shall then consider the case when those coefficients depend furthermore on a not fully observed, exogenous stochastic factor process so that the market will be inherently incomplete. For the case of such incomplete markets, in section 4 we shall then consider a general setup for the problem of derivative pricing; its practical implementability however depends heavily on the specific problem at hand. In the last section 5 we shall consider the problem of derivative pricing for the case of factor models for which it is possible to impose conditions such that the real world probability measure is itself a MM thus avoiding the basic problem mentioned above.

The various approaches discussed in the paper are related to research performed by the author in collaboration with various colleagues. We shall refer to these papers during the discussion of the individual techniques, in particular in relation to computational implementation.

## 2. The case when the underlyings have a market and their prices are Markovian

We consider here a standard multivariate Black and Scholes market model for the underlying assets, namely, given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$,

$$dS_t = diag(S_t)A_t dt + diag(S_t)\Sigma_t dw_t \tag{2.1}$$

where $S_t = [S_t^1, \cdots, S_t^N]'$ and $w_t = [w_t^1, \cdots, w_t^M]'$ is a $(P, \mathcal{F}_t)-$Wiener with $M \geq N$.

Considering a claim $H(S_T)$ that for simplicity we assume to depend only on $S_T$, its $t-$price ($t \leq T$) is

$$\begin{aligned} \Pi(t, S_t) &= e^{-\int_t^T r_s ds} E^Q\{H(S_T) \,|\, \mathcal{F}_t\} \\ &= e^{-\int_t^T r_s ds} E^Q\{H(S_T) \,|\, S_t\} \end{aligned} \tag{2.2}$$

where $r_t$ is assumed to be deterministically given and, under the measure $Q$,

$$dS_t = diag(S_t) r_t \underline{1} dt + diag(S_t) \Sigma_t dw_t^Q \tag{2.3}$$

with

$$dw_t^Q = dw_t + \theta_t dt \tag{2.4}$$

where $\theta_t = [\theta_t^1, \cdots, \theta_t^M]'$ is the *market price of risk* that satisfies

$$A_t - r_t \underline{1} = \Sigma_t \theta_t \tag{2.5}$$

Since we assumed $M \geq N$, there exists always a $\theta_t$ satisfying (2.5) and thus also a martingale measure $Q$.

**Remarks :**

- If the market is complete then, for the only purpose of derivative pricing, the knowledge of $\Sigma_t$ suffices, $A_t$ is not needed. For other purposes, such as portfolio optimization that is performed under the real world probability measure, one needs also the knowledge of $A_t$. Once $\Sigma_t$ is given, $A_t$ follows from $\theta_t$ via (2.5).
- If $A_t$ and $\Sigma_t$ and thus also $\theta_t$ are unobserved then, borrowing ideas from Bayesian statistics, we may more generally assume them to be stochastic processes that could also be adapted to a filtration larger than $\mathcal{F}_t^w$, i.e. they may also be affected by some exogenous randomness. In this way the market becomes incomplete and the estimation of $\theta$ becomes important also for derivative pricing. To formalize the dependence of $\theta_t$ on exogenous randomness we assume that, under the real world measure $P$, we have

$$d\theta_t = \kappa(\bar{\theta} - \theta_t) dt + \rho^w dw_t + \rho^v dv_t \tag{2.6}$$

  where $\kappa$ is a diagonal matrix, $\bar{\theta} = [\bar{\theta}^1, \cdots, \bar{\theta}^M]'$, $\rho^w$, $\rho^v$ are matrices of appropriate dimensions, $v_t$ is a (multivariate) Wiener process independent of $w_t$, and $\pi_0(\theta_0)$ a given Gaussian initial distribution. With model (2.6), which is a mean-reverting model, we assume that the evolution of $\theta_t$ is affected by that of the underlyings (driving noise $w_t$) and also of exogenous factors (noise $v_t$).
- While $\Sigma_t$ may be estimated either as implied or historical volatility, we shall use a stochastic filtering approach to estimate $\theta_t$, and thus also $A_t$, on the basis of observed prices of the underlyings and their derivatives. In this context we recall that, while the observations take place under the real world measure, derivative prices are specified as expectations under a martingale measure. Notice also that, for an incomplete market, inference of $\theta_t$ allows not only to infer $A_t$ but also the prevailing martingale measure $Q$.

**2.1. Estimation of $\theta_t$ via filtering.** Assume that on the market one can observe, in addition to the asset prices $S_t$, $K$ derivative prices $\Pi_i^*(t)$ , $i = 1, \cdots, K$ so that the observation filtration becomes

$$\mathcal{F}_t^O = \sigma\{S_u,\, \Pi_i^*(u)\,;\; u \le t\,;\; i = 1, \cdots, K\} \tag{2.7}$$

The filtering problem consists in computing recursively the conditional (filter) distribution

$$\pi_t\left(\theta_t \,|\, \mathcal{F}_t^O\right) \tag{2.8}$$

starting from a given $\pi_0(\theta_0)$ and with $\theta_t$ evolving according to (2.6). In this way we obtain not only a point estimate, but an entire continuously updated distribution for $\theta$.

Having assigned the dynamics for $\theta_t$, we next derive the dynamics, under $P$, of the observed prices. For this purpose, according to (2.2), let the (theoretical) prices at time $t \le T$ of the $K$ derivatives on the market be denoted by $\Pi^i(t, S_t)$ , $i = 1, \cdots, K$ and let the corresponding claims be $H^i(S_T)$.

Putting, for $i = 1, \cdots, K$,

$$F^i(t, S_t) = e^{\int_t^T r_s ds}\Pi^i(t, S_t) = E^Q\{H^i(S_T) \,|\, \mathcal{F}_t\} \tag{2.9}$$

by Itô's formula one has

$$\begin{array}{rcl} dF^i(t, S_t) & = & \left[F_t^i(\cdot) + F_S^i(\cdot) diag(S_t) r_t \underline{1} + \frac{1}{2} tr[\Sigma_t'\, diag(s) F_{SS}^i(\cdot) diag(s) \Sigma_t]\right] dt \\ & + & F_S^i(\cdot) diag(S_t) \Sigma_t dw_t^Q \end{array} \tag{2.10}$$

Since, by definition, $F^i(t, S_t)$ are martingales under $Q$, the finite variation terms in (2.10) have to vanish, which implies for $F^i(t, s)$

$$\left\{ \begin{array}{l} F_t^i(t,s) + F_S^i(t,s) diag(s) r_t \underline{1} + \frac{1}{2} tr[\Sigma_t'\, diag(s) F_{SS}^i(t,s) diag(s) \Sigma_t] = 0 \\ F^i(T,s) = H^i(s) \end{array} \right. \tag{2.11}$$

Taking into account (2.4), the dynamics of $S_t$ and $Y_t^i := F^i(t, S_t)$ under $P$ are now

$$\left\{ \begin{array}{rcl} dS_t & = & diag(S_t)[r_t \underline{1} + \Sigma_t \theta_t] dt + diag(S_t) \Sigma_t dw_t \\ dY_t^i & = & F_S^i(t, S_t) diag(S_t) \Sigma_t \theta_t + F_S^i(t, S_t) diag(S_t) \Sigma_t dw_t \end{array} \right. \tag{2.12}$$

The complete filter model can then be synthesized as

$$\left\{ \begin{array}{rcl} d\theta_t & = & \kappa(\bar\theta - \theta_t) dt + \rho^w dw_t + \rho^v dv_t \\ dS_t & = & diag(S_t)[r_t \underline{1} + \Sigma_t \theta_t] dt + diag(S_t) \Sigma_t dw_t \\ dY_t^i & = & F_S^i(t, S_t) diag(S_t) \Sigma_t \theta_t dt + F_S^i(t, S_t) diag(S_t) \Sigma_t dw_t \end{array} \right. \tag{2.13}$$

where we suppose $r_t$ to be deterministically given and also $\Sigma_t$ to be given as an observed quantity either through the quadratic variation or as implied volatility $\hat\Sigma_t = \Sigma_t(S_t, Y_t^i)$.

Model (2.13) is of the conditionally Gaussian type (recall that $\pi_0(\theta_0)$ was assumed to be Gaussian) to which the Kalman filter can be applied to obtain the solution (2.8) (see [**LS**]). The parameters $(\kappa, \bar\theta, \rho^w, \rho^v)$ may be estimated by maximizing the likelihood of the innovations (see [**H**]). An approach along the lines of this section may be found in [**BCR**].

## 3. The case when the underlyings are not Markovian (Factor models)

Instead of (2.1) we consider now the following *stochastic factor model*

$$\begin{cases} dS_t = diag(S_t)A_t(Z_t)dt + diag(S_t)\Sigma_t(Z_t)dw_t \\ \\ dZ_t = b_t(Z_t)dt + \gamma_t(Z_t)dv_t \end{cases} \tag{3.1}$$

where the Wiener processes $w_t$ and $v_t$ are assumed to be independent and the stochastic factor $Z_t$ may be a hidden or not fully observed process. The coefficients $A_t(Z_t), \Sigma_t(Z_t)$ are random processes that depend on $Z_t$ but that, in analogy to the previous section, may possess also additional randomness, this time adapted to $\mathcal{F}_t^w$.

The market is now incomplete also without the additional randomness assumption on $A_t(Z_t), \Sigma_t(Z_t)$ and so the estimation of $Z_t$, which again will be based on observations of $S_t$ as well as of their derivatives, becomes necessary also for derivative pricing.

Considering again a claim $H(S_T)$, its $t-$price ($t \leq T$) is now of the form

$$\Pi(t, S_t, Z_t) = e^{-\int_t^T r_s ds} E^Q\{H(S_T) \,|\, \mathcal{F}_t\} \tag{3.2}$$

where we assume again $r_t$ to be deterministically given. Under $Q$

$$dS_t = diag(S_t) r_t \underline{1} dt + diag(S_t)\Sigma_t(Z_t)dw_t^Q \tag{3.3}$$

with

$$dw_t^Q = dw_t + \theta_t dt \tag{3.4}$$

and the (unitary) market price of risk $\theta_t$ satisfies

$$A_t(Z_t) - r_t \underline{1} = \Sigma_t(Z_t)\theta_t \tag{3.5}$$

so that it may be considered as a function $\theta_t = \theta(t, Z_t)$. Furthermore, having assumed $A_t(Z_t)$ and $\Sigma_t(Z_t)$ to possess additional randomness adapted to $\mathcal{F}_t^w$, the same holds for $\theta_t$.

Considering for the moment just a relation of the form $\theta_t = \theta(t, Z_t)$, applying Itô's rule one obtains

$$\begin{aligned} & d\theta_t = d\theta(t, Z_t) \\ & = \left[ \frac{\partial}{\partial t}\theta(t, Z_t) + \frac{\partial}{\partial Z}\theta(t, Z_t) b_t(Z_t) + \frac{1}{2} tr \left( \gamma_t'(Z_t) \frac{\partial^2}{(\partial Z)^2}\theta(t, Z_t)\gamma_t(Z_t) \right) \right] dt \\ & \;\; + \frac{\partial}{\partial Z}\theta(t, Z_t)\gamma_t(Z_t) dv_t \\ & := \Theta_t(Z_t)dt + \Psi_t(Z_t)dv_t \end{aligned} \tag{3.6}$$

thereby defining implicitly the functions $\Theta_t(Z_t)$ an $\Psi_t(Z_t)$. We shall now formalize the additionally assumed $\mathcal{F}_t^w-$adapted randomness in $\theta_t$ by postulating for $\theta_t$ the following dynamics under $P$

$$d\theta_t = \Theta_t(Z_t)dt + \rho^w dw_t + \Psi_t(Z_t)dv_t \tag{3.7}$$

**3.1. Estimation of $\theta_t$ and $Z_t$ via stochastic filtering.** Given an observation filtration $\mathcal{F}_t^O$ as in (2.7), here the filtering problem consists of computing recursively the filter distribution $\pi_t(\theta_t, Z_t | \, \mathcal{F}_t^O)$ starting from a given $\pi_0(Z_0, \theta_0)$ and with $\theta_t$ evolving according to (3.7).

To derive the dynamics, under $P$, of the observed prices, in analogy to section 2 put

$$F^i(t, S_t, Z_t) := e^{\int_t^T r_s ds}\Pi^i(t, S_t, Z_t) \tag{3.8}$$

The martingality of $F^i(t, S_t, Z_t)$ under $Q$ implies

(3.9)
$$\begin{cases} F^i_t(t,s,z) + F^i_S(t,s,z)diag(s)r_t\underline{1} + \frac{1}{2}tr[\Sigma'_t(z)\,diag(s)F^i_{SS}(t,s,z)diag(s)\Sigma_t(z)] \\ \qquad\qquad +F^i_Z(t,s,z)b_t(z) + \frac{1}{2}tr[\gamma'_t(z)F_{ZZ}(t,s,z)\gamma_t(z)] = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\,(t,s,z);\; i=1,\cdots,K \\ F^i(T,s,z) = H^i(s) \end{cases}$$

Synthesizing, we obtain now the following filtering model (under $P$)

$$\begin{cases} d\theta_t &=& \Theta_t(Z_t)dt + \rho^w dw_t + \Psi_t(Z_t)dv_t \\ dZ_t &=& b_t(Z_t)dt + \gamma_t(Z_t)dv_t \\ dS_t &=& diag(S_t)[r_t\underline{1} + \Sigma_t(Z_t)\theta_t]dt + diag(S_t)\Sigma_t(Z_t)dw_t \\ dY^i_t &=& [F^i_S(t,S_t,Z_t)diag(S_t)\Sigma_t(Z_t)\theta_t]dt \\ &+& F^i_S(t,S_t,Z_t)diag(S_t)\Sigma_t(Z_t)dw_t + F^i_Z(t,S_t,Z_t)\gamma_t(Z_t)dv_t \\ && \qquad\qquad\qquad\qquad\qquad\qquad i=1,\cdots,K \end{cases} \tag{3.10}$$

where the only parameter is now $\rho^w$ that may be estimated either via a combined filtering and parameter estimation by computing $\pi_t(Z_t, \theta_t, \rho^w \,|\, \mathcal{F}^O_t)$ or via calibration by matching theoretical with observed prices.

The main difference with the previous model (2.13) is that (3.10) is a nonlinear filtering model where, in addition, the observation diffusion coefficients depend on unobserved quantities causing the filtering problem to degenerate. This latter difficulty can be overcome by considering the observable prices to be observable only in additional noise, which can be justified by bid-ask spread, mispricing, a-synchronicity, etc. We shall also assume this additional noise to be sufficiently small to prevent substantial arbitrage opportunities.

Putting then

$$\bar{Y}^i_t = \begin{cases} S^i_t & , \quad i = 1,\cdots,N \\ Y^{i-N}_t & , \quad i = N+1,\cdots,N+K \end{cases} \tag{3.11}$$

and denoting by $\eta^i_t$ the (cumulative) noisy observations, we let

$$d\eta^i_t = \bar{Y}^i_t dt + d\beta^i_t\,;\;\; i = 1,\cdots,N+K \tag{3.12}$$

where $\beta_t = (\beta^1_t, \cdots, \beta^{N+K}_t)'$ is an independent observation noise and $\bar{Y}^i_0$ is assumed to be observed without noise. This approach corresponds to the one in [**BCR**] and has been applied in the context of bond markets in [**CPR**].

## 4. Filtering for derivative pricing under partial information (general setup)

Consider again the stochastic factor model (3.1) of the previous section 3 and put $\mathcal{F}_t = \sigma\{w_s, v_s;\, s \le t\}$ so that, with $\mathcal{F}^O_t$ as in (2.7), one has $\mathcal{F}^O_t \subset \mathcal{F}$. Assume $r_t$ to be deterministically given.

Given a claim $H(S_T)$ and a martingale measure $Q$, define the $t-$price ($t \le T$) of $H(S_T)$ with respect to the information $\mathcal{F}^O_t$ as

$$\tilde{\Pi}(t) = E^Q\left\{ e^{-\int_t^T r_s ds} H(S_T) \,|\, \mathcal{F}^O_t \right\} \tag{4.1}$$

It is an arbitrage-free price with respect to the information represented by $\mathcal{F}^O_t$ in the sense that

$$\frac{\tilde{\Pi}(t)}{B_t} = E^Q\left\{ \frac{\tilde{\Pi}(T)}{B_T} \;|\; \mathcal{F}^O_t \right\} \tag{4.2}$$

where $B_t = B_0 \exp\left\{\int_0^t r_s ds\right\}$. (For additional justification of the definition in (4.1) see [**BGJR**]).

To perform pricing of illiquid (OTC) derivatives on the basis of the information $\mathcal{F}_t^O$, one has thus to compute expectations of the form

$$E^Q\{H(S_T) \,|\, \mathcal{F}_t^O\} \tag{4.3}$$

where, under $Q$, $S_t$ satisfies (3.3) with $w_t^Q$ according to (3.4). Furthermore, one has $\frac{dQ}{dP}_{|\mathcal{F}_T} = L_T$ with

$$dL_t = -L_t\theta_t dw_t \,;\ L_0 = 1 \quad (\text{under } P) \tag{4.4}$$

By Bayes' rule

$$E^Q\{H(S_T) \,|\, \mathcal{F}_t^O\} = \frac{E^P\{L_T H(S_T) \,|\, \mathcal{F}_t^O\}}{E^P\{L_T \,|\, \mathcal{F}_t^O\}} \tag{4.5}$$

so that, to compute (4.3), it suffices to obtain an explicit expression for the numerator in the right hand side of (4.5) (the denominator is of the same form with $H(\cdot) \equiv 1$).

Since for the case of the given model (3.1) with $\theta_t$ as in (3.7) and $L_t$ as in (4.4), the tuple $(S_t, Z_t, \theta_t, L_t)$ is Markov under $P$, one may write

$$\begin{aligned} E^P\{L_T H(S_T) \,|\, \mathcal{F}_t^O\} &= E^P\{E^P\{L_T H(S_T) \,|\, \mathcal{F}_t\} \,|\, \mathcal{F}_t^O\} \\ &= E^P\{\Psi^H(t, S_t, Z_t, \theta_t, L_t) \,|\, \mathcal{F}_t^O\} \end{aligned} \tag{4.6}$$

for a suitable function $\Psi^H(\cdot)$.

To compute the quantities of interest in (4.3) it is therefore useful to be able to obtain the filter distribution

$$\pi_t(Z_t, \theta_t, L_t | \,\mathcal{F}_t^O) \tag{4.7}$$

corresponding to the filter model

$$\left\{ \begin{array}{lcl} d\theta_t &=& \Theta_t(Z_t)dt + \rho^w dw_t + \Psi_t(Z_t)dv_t \\ dL_t &=& -L_t\theta_t dw_t \\ dZ_t &=& b_t(Z_t)dt + \gamma_t(Z_t)dv_t \\ dS_t &=& diag(S_t)[r_t\underline{1} + \Sigma_t(Z_t)\theta_t]dt + diag(S_t)\Sigma_t(Z_t)dw_t \\ dY_t^i &=& [F_S^i(t, S_t, Z_t)diag(S_t)\Sigma_t(Z_t)\theta_t]dt \\ && + \; F_S^i(t, S_t, Z_t)diag(S_t)\Sigma_t(Z_t)dw_t + F_Z^i(t, S_t, Z_t)\gamma_t(Z_t)dv_t \\ && \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = 1, \cdots, K) \\ d\eta_t^i &=& \bar{Y}_t^i dt + d\beta_t^i \;;\; i = 1, \cdots, N+K \end{array} \right. \tag{4.8}$$

$$\text{where} \quad \bar{Y}_t^i = \left\{ \begin{array}{lc} S_t^i & , \quad i = 1, \cdots, N \\ Y_t^{i-N} & , \quad i = N+1, \cdots, N+K \end{array} \right.$$

the functions $F^i(t, s, z)$ satisfy (3.9), and the initial distribution of $(\theta_0, L_0, Z_0, S_0, Y_0^i)$ is characterized by $\pi_0(\theta_0, Z_0)$, $L_0 = 1$ and $S_0, Y_0^i$ deterministically given (observed without noise).

The approach described in this section is a very general approach. In exchange for its generality its practical implementation requires however the explicit determination of the function $\Psi^H(\cdot)$ in (4.6), of $F(\cdot)$ according to (3.9) and the solution of the nonlinear filtering problem corresponding to (4.8). The feasibility of these computations depends heavily on the specific problem at hand. For a more specific setup, in the next section we shall describe a filtering approach that, although possessing some of the features of the approach of this section, has in particular cases led to explicitly computable solutions.

## 5. Filtering for pricing under partial observation in factor models

Let $Z_t$ denote a Markovian factor process with general state space, where some of the components may be unobservable, but others may be observable asset prices. Note that we assume $Z_t$ to be globally Markov in the sense that the evolution of each component may depend on the entire vector $Z$.

Assume that there exists a time instant $T > 0$ that without loss of generality we may consider to be the same for all assets, at which the price of any asset can be expressed as a known function of $Z_T$. We assume thus that for each asset there exists a corresponding $H(\cdot)$ such that

$$\Pi^H(T) = H(T, Z_T) \tag{5.1}$$

where by $\Pi^H(T)$ we have denoted the price, at $T > 0$, of the given asset. Notice that for the components of $Z$ that are asset prices themselves, the function $H(\cdot)$ is simply the projection onto the corresponding component. Concerning the short rate $r_t$ we assume, as before, that it is given as a deterministic time function.

Analogously to what one does in general asset pricing theory, given the Markovianity of the process $Z_t$, we make now the further assumption that, for $t \neq T$, the asset prices may be expressed as

$$\Pi^H(t) = F^H(t; Z_t) \tag{5.2}$$

for a suitable function $F^H(\cdot)$.

On a given filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ we may then consider the factor market model

$$\begin{cases} dZ_t = b_t(Z_t)dt + \gamma_t(Z_t)dw_t \\ \Pi^H(t) = F^H(t, Z_t) \end{cases} \tag{5.3}$$

where, for $t = T$, $\Pi^H(T) = H(T, Z_T)$ with $H(\cdot)$ a known function. To prevent arbitrage, the function $F^H(t, Z_t)$ cannot be chosen arbitrarily and so one may ask what are the conditions on $F^H(\cdot)$ to have absence of arbitrage. Furthermore, since derivative prices are expressed as expectations under a martingale measure while the filter dynamics have to be defined under the real world probability measure, one may more specifically ask what are the conditions on $F^H(\cdot)$ so that $P$ itself becomes a martingale measure.

The required conditions follow from imposing that the discounted values of $\Pi^H(t) = F^H(t, Z_t)$ are $(P, \mathcal{F}_t)-$martingales. Applying Itô's rule to $F^H(t, Z_t)$ and putting, because of the martingale property, the finite variation terms equal to zero, one then obtains

$$\begin{cases} F^H_t(t,z) + F^H_Z(t,z)b_t(z) \\ \quad +\frac{1}{2}tr[\gamma_t'(z)F^H_{ZZ}(t,z)\gamma_t(z)] - r_tF^H(t,z) = 0\,,\ \forall(t,z) \\ \\ F^H(T,z) = H(T,z). \end{cases} \tag{5.4}$$

Note that, for particular families of $H(\cdot)$, this condition may take specific and computationally convenient forms. An example are the exponentially affine models for the bond market where the above conditions reduce to a system of ODE's (see e.g. Section 17.3 in [**B**]). Due to the Feynman-Kac formula, from (5.4) one also has

$$F^H(t,z) = E_{t,z}\left\{e^{-\int_t^T r_s ds} H(T, Z_T)\right\} \tag{5.5}$$

Let

$$\mathcal{F}_t^O = \sigma\left\{\Pi^{H_i}(s)\,;\; s \le t\,;\; i = 1, \cdots, K\right\} \tag{5.6}$$

represent the information coming from market data. We are interested in the pricing of illiquid (OTC) derivatives on the basis of the information represented by $\mathcal{F}^O$. Given a maturity $\tau$, we consider simple claims of the form $\Phi(F^H(\tau, Z_\tau))$ on an underlying asset with price $\Pi^H(\tau) = F^H(\tau, Z_\tau)$. In line with (4.1) we then want to compute, for $t \le \tau$, an expression of the form

$$\begin{aligned} &E\left\{e^{-\int_t^T r_s ds}\Phi(F^H(\tau, Z_\tau)) \mid \mathcal{F}_t^O\right\} \\ &= E\left\{E\left\{e^{-\int_t^T r_s ds}\Phi(F^H(\tau, Z_\tau)) \mid \mathcal{F}_t\right\} \mid \mathcal{F}_t^O\right\} \\ &= E\{\Psi(t, Z_t) \mid \mathcal{F}_t^O\} \end{aligned} \tag{5.7}$$

where we implicitly define the function $\Psi^H(\cdot)$.

To compute the rightmost expression in (5.7) we need the filter distribution $\pi_t(Z_t | \mathcal{F}_t^O)$ corresponding to the model

(5.8)
$$\begin{cases} dZ_t = b_t(Z_t)dt + \gamma_t(Z_t)dw_t \\ dY_t^i = r_t F^{H_i}(t, Z_t)dt + F_Z^{H_i}(t, Z_t)\gamma_t(Z_t)dw_t \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, \cdots, K \end{cases}$$

where $Z_t$ contains unobserved components, while all $Y_t^i$ $(i = 1, \cdots, K)$ are for the moment considered as observables.

In this filtering model the observation diffusion term depends in general again on $Z_t$ so that, as in the previous section, we introduce a further observation noise $\beta_t = (\beta_t^1, \cdots, \beta_t^K)$ thereby considering $Z_t$ and $Y_t^i$ as only partially observed with observations $\eta_t^i$ satisfying

$$d\eta_t^i = Y_t^i dt + d\beta_t^i \;; \quad i = 1, \cdots, K. \tag{5.9}$$

In the context of bond markets this approach has been explicitly implemented in [**GR**] without the need of a further observation noise and using the Kalman filter.

**5.1. An alternative approach.** As already pointed out, one of the main problems for applying filtering techniques when the observations contain also derivative asset prices, is that the derivative prices are specified as expectations under a martingale measure, while the filter dynamics have to be specified under the real world probability measure. To deal with this issue, the previous approach was based on the assumption that $\Pi^H(t) = F^H(t, Z_t)$ and on the conditions imposed on $F^H(\cdot)$ for $P$ to be a martingale measure.

An alternative for having $P$ itself a martingale measure can be based on the techniques of change of numeraire and one may ask the question whether there is a numeraire (numeraire portfolio) for which the real world measure becomes a martingale measure. The answer is positive in the sense that such a numeraire is given by the *growth optimal portfolio* (GOP), which (see e.g. [**L**]) is a self financing portfolio that achieves maximum expected logarithmic utility from terminal wealth.

Such an approach is described in [**PR**] and it turns out that, when one performs pricing with the GOP as numeraire, then in the case of complete markets the prices obviously coincide with those computed as expectations with respect to the unique martingale measure, while in incomplete markets they correspond to the prices computed under the minimal martingale measure.

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