INTRODUZIONE ALLE EQUAZIONI DIFFERENZIALI ALLE DERIVATE PARZIALI
Laurea Magistrale in Matematica- docente A. Cesaroni

Exam 06.02.2014, solution to the exercises.

Question 3.

Consider the following quasilinear problem

$$(Q)\begin{cases} u_t - u_{xx} + (u_x)^2 = 0 & x \in \mathbb{R}, t > 0 \\ u(x,0) = \log(\frac{2+x^2}{1+x^2}) & x \in \mathbb{R}. \end{cases}$$

1. Find (using an appropriate transformation and a representation formula) a solution to (Q).
2. Prove that there exists at most one bounded solution $u \in \mathcal{C}^{2,1}(\mathbb{R} \times (0,+\infty)) \cap \mathcal{C}(\mathbb{R} \times [0,+\infty))$ to (Q).
3. Study the behaviour of the bounded solution $u(x,t)$ to Q as $t \to +\infty$.

Solution.

1. Let assume that a solution u to the problem exist. We define $v(x,t) = e^{-u(x,t)}$. So $v_t = -u_t e^{-u}$, $v_{xx} = -u_{xx}e^{-u} + (u_x)^2 e^{-u}$. So v satisfies

$$(N)\begin{cases} v_t - v_{xx} = 0 & x \in \mathbb{R}, t > 0 \\ u(v,0) = \frac{1+x^2}{2+x^2} & x \in \mathbb{R}. \end{cases}$$

   The initial data is bounded, positive and regular, so the problem (N) admits a unique bounded solution

$$v(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} \frac{1+y^2}{2+y^2} e^{-\frac{(x-y)^2}{4t}} dy.$$

   Observe that by the properties of the heat kernel, $v \in \mathcal{C}^{\infty}(\mathbb{R} \times (0,+\infty)) \cap \mathcal{C}(\mathbb{R} \times [0,+\infty))$ moreover $v > 0$, v is bounded, in particular $\frac{1}{2} \leq v(x,t) \leq 1$.

   So, we can define $u(x,t) = -\log v(x,t)$, and $u \in \mathcal{C}^{\infty}(\mathbb{R} \times (0,+\infty)) \cap \mathcal{C}(\mathbb{R} \times [0,+\infty))$ is a bounded solution to (Q).

2. Let u_1, u_2 be two bounded solutions to (Q), then $v_1 = e^{-u_1}$ e $v_2 = e^{-u_2}$ are two bounded solutions to (N). By the comparison principle between sub and supersolutions with exponenatial growth of the heat equation in $\mathbb{R} \times (0,+\infty)$, we obtain $v_1 = v_2$. This implies that $u_1 = u_2$.

3. By 1. we get that the unique bounded solution to (Q) is $u(x,t) = -\log v(x,t)$. Moreover $\lim_{t\to+\infty} v(x,t) = 1$ locally uniformly in $\mathbb{R}$. Then $\lim_{t\to+\infty} u(x,t) = 0$ locally uniformly in $\mathbb{R}$.

Question 4.

1. State and prove d'Alembert formula for the solution of the wave equation in dimension 1.

2. Find the solution to the following Cauchy Dirichlet problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & x > 0, t > 0 \\ u(0,t) = 0 & t > 0 \\ u(x,0) = (1 - \cos x)^2 & x \geq 0 \\ u_t(x,0) = x \sin x & x \geq 0. \end{cases}$$

Solution. 2 We solve the problem with the reflection method. We consider the antisymmetric extension to the initial data on the whole $\mathbb{R}$ as follows. Let $u_0(x) = (1 - \cos x)^2$ and $v_0(x) = x \sin x$, then we define

$$\tilde{u_0}(x) = \begin{cases} (1 - \cos x)^2 & x \geq 0 \\ -(1 - \cos x)^2 & x < 0 \end{cases} \qquad \tilde{v_0}(x) = \begin{cases} x \sin x & x \geq 0 \\ -x \sin x & x < 0 \end{cases}.$$

Then $\tilde{u_0} \in \mathcal{C}^2(\mathbb{R})$ (it is also more regular...) and $\tilde{v_0} \in \mathcal{C}^1(\mathbb{R})$. We consider the Cauchy problem

$$(C) \begin{cases} u_{tt} - c^2 u_{xx} = 0 & x \in \mathbb{R}, t > 0 \\ u(x,0) = \tilde{u_0}(x) & x \in \mathbb{R} \\ u_t(x,0) = \tilde{v_0}(x)x & x \in \mathbb{R}. \end{cases}$$

The solution of (C) is given by

$$\tilde{u}(x,t) = \frac{1}{2}(\tilde{u_0}(x+ct) + \tilde{u_0}(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{v_0}(s)ds.$$

The restriction of $\tilde{u}$ to the half line $x > 0$ gives the solution to the Cauchy-Dirichlet problem:

$$u(x,t) = \begin{cases} \frac{(1-\cos(x+ct))^2 + (1-\cos(x-ct))^2}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} s \sin s ds & x > ct \\ \frac{(1-\cos(x+ct))^2 - (1-\cos(x-ct))^2}{2} + \frac{1}{2c}\int_{ct-x}^{x+ct} s \sin s ds & x \leq ct. \end{cases}$$