INTRODUZIONE ALLE EQUAZIONI DIFFERENZIALI ALLE DERIVATE PARZIALI
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Partial exam 19.11.2013, solution to the exercises.

Question 1. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a harmonic function such that

$$\lim_{|x|\to+\infty} \frac{u(x)}{|x|} = 0.$$

Show that u is constant.

Solution. Choose $\varepsilon > 0$. Then there exists $M_\varepsilon > 0$ such that

$$|u(x)| \leq \varepsilon|x| \qquad \forall |x| \geq M_\varepsilon.$$

Define $K_\varepsilon = \max_{|x|\leq M_\varepsilon} |u(x)|$, then

$$|u(x)| \leq \varepsilon|x| + K_\varepsilon \qquad \forall\ x \in \mathbb{R}^n. \tag{1}$$

Fix $x_0 \in \mathbb{R}^n$. Then, by Cauchy estimates, there exists C depending only on n such that

$$|Du(x_0)| \leq \frac{C}{r} \sup_{B(x_0,r)} |u| \qquad \forall r > 0. \tag{2}$$

So, by (1) and (2), we get

$$|Du(x_0)| \leq \frac{C}{r}(\varepsilon r + \varepsilon|x_0| +\ K_\varepsilon) \leq C\varepsilon + \frac{C(|x_0| + K_\varepsilon)}{r} \qquad \forall r > 0.$$

Letting $r \to +\infty$, we get $|Du(x_0)| \leq C\varepsilon$ and we conclude by the arbitrariness of ε and of x_0 that $|Du| \equiv 0$. This implies that u is constant.

Question 2.

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$$B^+(0, R) = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 < R^2,\ x_n > 0\}.$$

Provide a representation formula for the solution to

$$\begin{cases} -\Delta u = 0 & x \in B^+(0, R), \\ u(x) = x_n^2 & x \in \partial B^+(0, R). \end{cases}$$

Solution. The Dirichlet problem in the half ball admits a unique solution $u \in \mathcal{C}^\infty(B^+(0,R)) \cap \mathcal{C}(\overline{B^+(0,R)})$ by Perron-Wiener theorem and weak maximum principle. Moreover $u = 0$ on the set $x_n = 0$. We consider the following (antysimmetric) extension of u to the ball $B(0,R)$

$$\tilde{u}(x) = \begin{cases} u(x_1, x_2, \dots, x_n) & x \in \overline{B(0,R)}\ x_n \geq 0 \\ -u(x_1, x_2, \dots, -x_n) & x \in \overline{B(0,R)}\ x_n < 0. \end{cases}$$

We claim that this function is harmonic in $B(0,R)$. We prove this claim (this is called **Schwartz reflection principle**). First of all observe that $\tilde{u}$ is continuous in $\overline{B}(0,R)$, harmonic in $B^+(0,R)$ and in $B^-(0,R)$. Moreover consider the following (little) extension of Koebe theorem:

Proposition. *Let A be a open set and $u \in \mathcal{C}(A)$. Assume that u satisfies* **locally** *the mean value property: that is for every $x \in A$ there exists $r_x > 0$ such that*

$$u(x) = \fint_{B(x,s)} u(y)dy \qquad \forall s \leq r_x.$$

Then $u \in \mathcal{C}^\infty(A)$ and $-\Delta u = 0$ in A.

The proof of this proposition follows exactly the same argument as the proof of Koebe theorem, since it is sufficient to show that for every x, there exists a neighbourhood of x where u coincides with the mollification u_ε. Note that $\tilde{u}$ satisfies the local mean value property at every point x with $x_n > 0$ or $x_n < 0$ (since it is $\mathcal{C}^2$ in these sets and harmonic, it is sufficient to choose $r_x = |x_n|$). Moreover at every point x such that $x_n = 0$ we get, for all $s \leq R$,

$$\tilde{u}(x) = 0 = \fint_{B(x,s)} \tilde{u}(y)dy = \frac{1}{\omega_n s^n}\left(\int_{B^+(x,s)} u(y_1, \dots, y_n)dy - \int_{B^-(x,s)} u(y_1, \dots, -y_n)dy\right).$$

So $\tilde{u}$ satisfies the local mean value property at every point and then $\tilde{u} \in \mathcal{C}^\infty(B(0,R)$ and it is harmonic.

Observe that the function $\tilde{u}$ at point $x \in \partial B(0,R)$ coincides with the (continuous) function $x_n|x_n|$.

So $\tilde{u}$ is the unique solution of the following Dirichlet problem

$$\begin{cases} -\Delta \tilde{u} = 0 & x \in B(0,R), \\ \tilde{u}(x) = x_n|x_n| & |x| = R. \end{cases}$$

By Poisson integral formula, taking $x \in B^+(0,R)$,

$$u(x) = \tilde{u}(x) = \frac{R^2 - |x|^2}{n\omega_n R}\int_{\partial B(0,R)} \frac{y_n|y_n|}{|x-y|^n} dS(y).$$