Removable singularities of harmonic functions.

**Definition.** Let $\Omega$ be a open set in $\mathbb{R}^n$ and let $x_0 \in \Omega$. Let $u : \Omega \setminus \{x_0\} \to \mathbb{R}$ be a harmonic function (in $\Omega \setminus \{x_0\}$). Then $x_0$ is a removable singularity of $u$ if there exists a harmonic function $\tilde{u}$ in $\Omega$ such that $u = \tilde{u}$ in $\Omega \setminus \{x_0\}$.

**Theorem 1.** Let $u$ be a harmonic function in $\Omega \setminus \{x_0\}$ and assume that $u = o(\Gamma(x-x_0))$ as $x \to x_0$ where $\Gamma$ is the fundamental solution of the Laplacian (with singularity at 0) that is

$$\lim_{x \to x_0} u(x)|x-x_0|^{n-2} = 0 \text{ if } n \geq 3 \quad \lim_{x \to x_0} \frac{u(x)}{\log |x-x_0|} = 0 \text{ if } n = 2.$$ 

Then $x_0$ is a removable singularity for $u$.

**Remark.** The condition is sharp, in fact $x_0$ is not removable for $\Gamma(x-x_0)$.

Note that if $u$ is bounded, then necessarily the condition in the theorem is satisfied.

**Remark.** This theorem is the analogous of Riemann theorem on removable singularities for holomorphic functions.

**Proof.** Let $r > 0$ such that $B(x_0, r) \subset \Omega$. Let $\tilde{u}$ be the solution of the Dirichlet problem

$$\begin{cases} -\Delta \tilde{u} = 0 & |x-x_0| < r \\ \tilde{u}(x) = u(x) & |x-x_0| = r. \end{cases}$$

Then $\tilde{u}$ is bounded in $B(x_0, r)$ (since by Maximum principle $|\tilde{u}| \leq \max_{|x-x_0|=r} u(x)$) and harmonic in $B(x_0, r)$. To conclude the proof it is enough to show that $\tilde{u} = u$ in $B(x_0, r)$.

Let $w = u - \tilde{u}$. Then $w$ is harmonic in $B(x_0, r) \setminus \{x_0\}$ and moreover (check it!)

$$\lim_{x \to x_0} \frac{w(x)}{r^{2-n} - |x-x_0|^{2-n}} = 0 \text{ if } n \geq 3 \quad \lim_{x \to x_0} \frac{w(x)}{\log r - \log |x-x_0|} = 0 \text{ if } n = 2.$$ 

So for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $x$ such that $|x-x_0| \leq \delta$,

$$|w(x)| \leq \varepsilon (r^{2-n} - |x-x_0|^{2-n}) \text{ if } n \geq 3 \quad |w(x)| \leq \varepsilon (\log r - \log |x-x_0|) \text{ if } n = 2.$$ 

Observe that $\varepsilon (r^{2-n} - |x-x_0|^{2-n})$ and $\varepsilon (\log r - \log |x-x_0|)$ are harmonic functions (resp. when $n \geq 3$ and $n = 2$) and are 0 on the set $|x-x_0| = r$. So, weak Maximum principle (applied in the set $\delta \leq |x-x_0| \leq r$) gives that

$$|w(x)| \leq \varepsilon (r^{2-n} - |x-x_0|^{2-n}) \text{ if } n \geq 3 \quad |w(x)| \leq \varepsilon (\log r - \log |x-x_0|) \text{ if } n = 2$$

for every $x$ such that $|x-x_0| \leq r$.

We conclude by the arbitrariness of $\varepsilon$ that $w = 0$. \qed