ONE-DIMENSIONAL SYMMETRY FOR SEMILINEAR EQUATIONS WITH UNBOUNDED DRIFT

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Abstract. We consider semilinear equations with unbounded drift in the whole of \( \mathbb{R}^n \) and we show that monotone solutions with finite energy are one-dimensional.

1. Introduction. In the paper [9] E. De Giorgi formulated the celebrated conjecture that bounded monotone solutions to the Allen-Cahn equation
\[
\Delta u = u^3 - u
\]
are necessarily one-dimensional (in the sense that the level sets are hyperplanes) at least if \( n \leq 8 \). This conjecture has been proved by Ghoussoub and Gui in [17] (see also [3]) in dimension \( n = 2 \), and by Ambrosio and Cabré [2] in dimension \( n = 3 \) (see also [1]), and a counterexample has been given by del Pino, Kowalczyk and Wei in [10] for \( n \geq 9 \). Under the additional assumption that \( u \) connects \(-1\) to \(1\), a proof has been presented by Savin [22] in dimension \( n \leq 8 \).

In this paper we consider the semilinear elliptic equation
\[
\Delta u + c(z)u_z + (\nabla_y g(y), \nabla_y u) + f(u) = 0,
\]
where we write \( x = (y, z) \in \mathbb{R}^{n-1} \times \mathbb{R} \). A solution \( u \) of (2) of the form
\[
u(x) = u_0(\omega, x) \quad \forall x \in \mathbb{R}^n,
\]
where \( u_0 : \mathbb{R} \rightarrow \mathbb{R} \) and \( \omega \in \mathbb{R}^n \) with \( |\omega| = 1 \) will be called one-dimensional.

We are interested in symmetry results for solutions \( u \) which are monotone in the \( z \)-variable, i.e. satisfy
\[
u_z(x) > 0 \quad \forall x \in \mathbb{R}^n.
\]
In particular, we will show that, under suitable assumptions, monotone solutions to (2) are necessarily one-dimensional (see Theorem 1.1).

Our methods rely on the geometric approach developed in [13] (see also [6, 7, 11, 12, 16, 23]), and our computations follow those in [14, 15], where the authors prove Liouville type results for stable solutions to elliptic equations in complete Riemannian manifolds with nonnegative Ricci curvature.

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1.1. Main result. Let us state the main result of this paper:

**Theorem 1.1.** Assume that \( f : \mathbb{R} \to \mathbb{R} \) is locally Lipschitz, \( g \in C^2(\mathbb{R}^{n-1}) \), \( c \in C^1(\mathbb{R}) \) and that

\[
\frac{c'(z)}{n} \mathbf{I}_{n-1} \geq \nabla_y^2 g(y) \quad \text{for every } (y, z) \in \mathbb{R}^{n-1} \times \mathbb{R},
\]

(4)

where \( \mathbf{I}_{n-1} \) denotes the identity matrix on \( \mathbb{R}^{n-1} \). Let \( C \in C^2(\mathbb{R}) \) be a primitive of \( c \), and let \( u \) be a solution to (2) satisfying (3) and one of the following conditions:

a) \[
\int_{\mathbb{R}^n} |\nabla u|^2 e^{g(y)+C(z)} \, dzdy < +\infty;
\]

b) For all \( z \in \mathbb{R} \)

\[
\int_{\mathbb{R}^{n-1}} |\nabla u|^2 e^{g(y)+C(z)} \, dy \leq K \quad \text{for some } K > 0;
\]

c) \( n = 2 \) and for all \( (y, z) \in \mathbb{R}^n \)

\[
|\nabla u|^2 e^{g(y)+C(z)} \leq K \quad \text{for some } K > 0;
\]

(7)

then \( u \) is one-dimensional, and

\[
\langle (c'(z) \mathbf{I}_{n-1} - \nabla_y^2 g(y)) \nabla_y u, \nabla_y u \rangle = 0.
\]

(8)

In particular, if the strict inequality holds in (4) for some \( (y, z) \) then \( u \) depends only on \( z \).

From Theorem 1.1 we get the following corollaries which extend a result in [6], valid for the Ornstein-Uhlenbeck case \( C(z) = -z^2/2 \), \( g(y) = -|y|^2/2 \).

**Corollary 1.** Let \( C, g \) bounded above and satisfying (4). Assume also that \( n = 2 \) or

\[
\int_{\mathbb{R}^{n-1}} e^{g(y)} \, dy < +\infty.
\]

(9)

Let \( u \in W^{1,\infty}(\mathbb{R}^n) \) be a solution to (2) satisfying (3), then \( u \) is one-dimensional.

**Proof.** If \( C, g \) are bounded above and \( u \in W^{1,\infty}(\mathbb{R}^n) \), then (7) holds, and moreover condition (9) implies (6). The thesis then follows directly from Theorem 1.1. \( \square \)

**Remark 1.** From [19, Th. 2.4 and Rem. 2.5] it follows that, if \( \nabla_y^2 g(y) \) and \( c'(z) \) are uniformly bounded below, every bounded solution to (2) belongs to \( W^{1,\infty}(\mathbb{R}^n) \).

**Corollary 2.** Let \( C, g \) be concave, satisfying (4) and \(-C, -g\) coercive. Let \( u \in L^\infty(\mathbb{R}^n) \) be a solution to (2) satisfying (3), then \( u \) is one-dimensional.

**Proof.** In [8, Thm 2.5, Cor. 4.3] it is proved that if \(-C, -g\) are convex and coercive then any (weak) solution to (2) such that

\[
\int_{\mathbb{R}^n} u^2 e^{g(y)+C(z)} \, dzdy < +\infty
\]

also satisfies (5) (see Remark 3). In particular, any bounded solution to (2) satisfies (5), and we can conclude by Theorem 1.1. \( \square \)
When \( c(z) \equiv c \in \mathbb{R} \), solutions to (2) correspond to traveling (or standing if \( c = 0 \)) wave solutions to the reaction-diffusion equation:

\[
v_t = \Delta v + (\nabla_y g, \nabla_y v) + f(v) \quad \text{in } \mathbb{R}^n \times (0, +\infty). \tag{10}
\]

A traveling wave solution is a particular solution \( v \) to (10), uniformly translating in the \( z \)-direction at constant speed \( c \), of the form

\[
v(t, x) = u(y, z - ct).
\]

We refer to [24, 27] and references therein for classical results about existence and uniqueness of traveling waves in infinite cylinders.

**Corollary 3.** Let \( g \) be concave and let \( v(t, x) = u(y, z - ct) \) be a traveling or a standing wave solution to (10). If \( u \) satisfies one of the three conditions of Theorem 1.1, then \( u \) is one-dimensional. Moreover \( u \) depends only on \( z \) unless \( n = 2 \), \( g \) is constant and (7) holds.

Conditions (5), (6), (7) are quite restrictive. However, traveling wave solutions satisfying these conditions are relevant to propagation and are sometimes called variational traveling waves. We refer to [20, 21] for a general analysis of such solutions, including necessary and sufficient conditions for existence.

If these conditions are not satisfied, equation (10) admits in general traveling waves which are not one-dimensional even in the case \( n = 2 \) and \( g = 0 \) (see [4, 5, 18]).

2. **A Ornstein-Uhlenbeck type equations.** More generally, we shall consider the following equation of Ornstein-Uhlenbeck type:

\[
\Delta u + \langle \nabla G(x), \nabla u \rangle + f(u) = 0 \quad x \in \mathbb{R}^n, \tag{11}
\]

where \( f : \mathbb{R} \to \mathbb{R} \) is a locally Lipschitz function and \( G \in C^2(\mathbb{R}^n) \).

Notice that solutions to (11) are critical point of the functional

\[
I(u) := \int_{\mathbb{R}^n} \left( \frac{\|\nabla u\|^2}{2} + F(u) \right) e^{G(x)} dx, \tag{12}
\]

where \( F'(t) = -f(t) \). We define the function \( \lambda_G \in C^0(\mathbb{R}^n) \) as

\[
\lambda_G(x) := \text{maximal eigenvalue of } \nabla^2 G(x). \tag{13}
\]

Observe that, if \( G(x) := g(y) + C(z) \), then (11) reduces to (2), and \( \lambda_G(x) \geq C''(z) \) for every \( x \in \mathbb{R}^n \).

2.1. **h-stable solutions.** We denote by \( \mu \) the measure on \( \mathbb{R}^n \) with density \( e^{G(x)} \) w.r.t. the Lebesgue measure, and we let \( W^{k,p}_\mu(\mathbb{R}^n) \subset W^{k,p}_{loc}(\mathbb{R}^n) \), for \( k, p \in \mathbb{N} \), be the corresponding Sobolev spaces.

Notice that, if \( G \) is concave, then \( \mu \) is a finite measure iff

\[
\lim_{|x| \to +\infty} G(x) = -\infty.
\]

We introduce now the notion of \( h \)-stability for solutions to (11).

**Definition 2.1.** Let \( h : \mathbb{R}^n \to \mathbb{R} \) be a measurable function. A solution \( u \) to (11) is \( h \)-stable if

\[
\int_{\mathbb{R}^n} \left( \|\nabla \varphi \|^2 - f'(u) \varphi^2 \right) d\mu \geq \int_{\mathbb{R}^n} h(x) \varphi^2 d\mu \quad \forall \ \varphi \in C^1_c(\mathbb{R}^n).
\]

If \( h \equiv 0 \), then \( u \) is said to be stable.
We recall that a function \( u \in W^{1,2}_{\text{loc}}(\mathbb{R}^n) \) is a weak solution to (11) if
\[
\int_{\mathbb{R}^n} \left( \langle \nabla u, \nabla \varphi \rangle - f(u) \varphi \right) d\mu = 0 \quad \forall \ \varphi \in C_c^1(\mathbb{R}^n).
\]
Note that every critical point of the functional \( I \) in (12) is a weak solution to (11).

By classical elliptic regularity theory, if \( u \) is a weak solution then \( u \in C^{2,\alpha}(\mathbb{R}^n) \) for all \( \alpha < 1 \), in particular it is also a classical solution to (11).

**Remark 2.** The function \( u \in W^{1,2}_{\text{loc}}(\mathbb{R}^n) \) is a local minimizer of the functional \( I \) in (12) if \( I \) does not decrease under compactly supported perturbations, i.e.
\[
I(u) \leq I(v) \quad \text{whenever} \ v \in W^{1,2}_{\text{loc}}(\mathbb{R}^n) \ \text{and} \ \{u \neq v\} \subset K \subset \subset \mathbb{R}^n.
\]
Every local minimizer of \( I \) is a stable weak solution to (11).

In [6] the authors show that, when \( G(x) = -|x|^2/2 \), monotone solutions to (11) are \(-1\)-stable (i.e. stable with respect to the constant function \( h \equiv -1 \)).

In the following we will consider \( h \)-stable solutions to (11) which have *finite energy*, in the sense that
\[
|\nabla u| \in L^2_\mu(\mathbb{R}^n). \quad (14)
\]
Note that if \( G(y,z) = g(y) + C(z) \), this condition reduces to (5). When \( n = 2 \), we can substitute this condition with
\[
|\nabla u|^2 e^G \in L^\infty(\mathbb{R}^n). \quad (15)
\]

**Remark 3.** If the measure \( \mu \) is finite then \( L^\infty(\mathbb{R}^n) \subset L^2_\mu(\mathbb{R}^n) \). If the function \( G \) is concave, by [8, Thm 2.5, Cor. 4.3] this implies that every bounded solution to (11) belongs to \( W^{2,2}_\mu(\mathbb{R}^n) \) and hence satisfies (14).

On the other hand, assumption (14) can be satisfied also when \( \mu \) is not finite: for instance, if \( G(x) = g(y) \) is such that (9) holds and \( f(s) = s - s^3 \), the function
\[
u(z) = \tanh \left( \frac{z}{\sqrt{2}} \right)
\]
is a monotone stable solution to (11) with finite energy.

3. \( \lambda_G \)-stability and finite energy imply one-dimensional symmetry. We now show that \( \lambda_G \)-stable solutions to (11), where \( \lambda_G \) is defined in (13), which satisfy (14) or (15) are one-dimensional. Similar results for stable solutions have been obtained in the setting of Riemannian manifolds with nonnegative Ricci curvature in [14, 15].

Given a differentiable function \( v : \mathbb{R}^n \to \mathbb{R} \), we set \( v_i := \partial_i v \) for all \( i = 1, \ldots, n \).

**Lemma 3.1.** Let \( u \in W^{1,2}_{\text{loc}}(\mathbb{R}^n) \) be a weak solution to (11). Then for any \( i = 1, \ldots, n \) and \( \varphi \in C_c^1(\mathbb{R}^n) \) we have
\[
\int_{\mathbb{R}^n} \left( \langle \nabla u_i, \nabla \varphi \rangle - \langle \nabla u, \nabla G_i \rangle \varphi - f'(u) u_i \varphi \right) d\mu(x) = 0.
\]
Recalling that

Let Proposition 1.

Proof. It suffices to prove (3.1) for \( \varphi \in C_c^\infty(\mathbb{R}^n) \). From (2.1), applied with \( \varphi \) replaced by \( \varphi_i \), we get

\[
0 = \int_{\mathbb{R}^n} \langle \nabla u, \nabla \varphi_i \rangle - f(u)\varphi_i \, d\mu(x)
= \int_{\mathbb{R}^n} - \langle \nabla u_i, \nabla \varphi \rangle - \langle \nabla u, \nabla \varphi \rangle G_i + f'(u)u_i\varphi + f(u)\varphi G_i \, d\mu(x)
= \int_{\mathbb{R}^n} - \langle \nabla u_i, \nabla \varphi \rangle + f'(u)u_i\varphi + f(u)\varphi G_i \, d\mu(x) + \\
+ \int_{\mathbb{R}^n} - \langle \nabla u, \nabla (\varphi G_i) \rangle + \langle \nabla u, \nabla G_i \rangle \varphi \, d\mu(x).
\]

Recalling (2.1), applied with \( \varphi \) replaced by \( \varphi G_i \), we obtain the thesis. \( \square \)

**Proposition 1.** Let \( h \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( u \) be a \( h \)-stable solution to (11). Then for every \( \varphi \in C_c^1(\mathbb{R}^n) \) we have

\[
\int_{\mathbb{R}^n} \left( |\nabla^2 u|^2 - |\nabla u|^2 \right)^2 + \langle (h(x)I_n - \nabla^2 G(x)) \nabla u, \nabla u \rangle \varphi^2 \, d\mu(x)
\leq \int_{\mathbb{R}^n} |\nabla u|^2 |\nabla \varphi|^2 \, d\mu(x).
\]

Proof. Let \( \varphi \in C_c^1(\mathbb{R}^n) \). Using (3.1) with test function \( u_i\varphi^2 \) we obtain

\[
\int_{\mathbb{R}^n} \langle \nabla u_i, \nabla(u_i\varphi^2) \rangle - f'(u)u_i^2\varphi^2 \, d\mu(x) = \int_{\mathbb{R}^n} \langle \nabla u, \nabla G_i \rangle u_i\varphi^2 \, d\mu(x).
\]

Summing over \( i \), (17) gives

\[
\int_{\mathbb{R}^n} |\nabla^2 u|^2 \varphi^2 + \frac{1}{2} \langle \nabla |\nabla u|^2, \nabla \varphi^2 \rangle - f'(u)|\nabla u|^2 \varphi^2 \, d\mu(x)
= \int_{\mathbb{R}^n} \langle \nabla^2 G(x) \nabla u, \nabla u \rangle \varphi^2 \, d\mu(x).
\]

Using (2.1) with test function \( |\nabla u|\varphi \) we then get

\[
\int_{\mathbb{R}^n} h(x)|\nabla u|^2 \varphi^2 \, d\mu(x) \leq \int_{\mathbb{R}^n} (|\nabla u|^2 - |\nabla u|)^2 \, d\mu(x)
\]

\[
= \int_{\mathbb{R}^n} \varphi^2 |\nabla |\nabla u|^2|^2 + |\nabla u|^2 |\nabla \varphi|^2 + \frac{1}{2} \langle \nabla |\nabla u|^2, \nabla \varphi^2 \rangle - f'(u)|\nabla u|^2 \varphi^2 \, d\mu(x).
\]

Substituting (18) in (19) we get the result. \( \square \)

**Corollary 4.** Recalling that \( |\nabla^2 u|^2 - |\nabla |\nabla u|^2|^2 \geq 0 \) (see Remark 4), from (16) it follows

\[
\int_{\mathbb{R}^n} \langle (h(x)I_n - \nabla^2 G(x)) \nabla u, \nabla u \rangle \varphi^2 \, d\mu(x) \leq \int_{\mathbb{R}^n} |\nabla u|^2 |\nabla \varphi|^2 \, d\mu(x).
\]

If \( h \geq \lambda_G \), from (16) and the definition of \( \lambda_G \) in (13) it follows

\[
\int_{\mathbb{R}^n} \left( |\nabla^2 u|^2 - |\nabla |\nabla u|^2|^2 \right) \varphi^2 \, d\mu(x) \leq \int_{\mathbb{R}^n} |\nabla u|^2 |\nabla \varphi|^2 \, d\mu(x).
\]

**Remark 4.** The Poincaré type formula (21) was first obtained by Sternberg and Zumbrun [26]. Notice that the quantity \( |\nabla^2 u|^2 - |\nabla |\nabla u|^2|^2 \) has a geometric interpretation, in the sense that it can be expressed in terms of the principal curvatures of level sets of \( u \).
we refer to [26] and [13] for more details. Then (21) becomes

\[ |\nabla^2 u|^2 - |\nabla \nabla u|^2 = |\nabla_T \nabla u|^2 + |\nabla u|^2 \sum_{j=1}^{n-1} k_j^2 \]

on \( L_{u,x} \cap \{ \nabla u \neq 0 \} \), so that (21) becomes

\[
\int_{\{\nabla u \neq 0\}} (|\nabla u|^2 \mathcal{K}^2 + |\nabla_T \nabla u|^2) \varphi^2 \, d\mu(x) + \int_{\{\nabla u = 0\}} (|\nabla^2 u|^2 - |\nabla \nabla u|^2) \varphi^2 \, d\mu(x) \\
\leq \int_{\mathbb{R}^n} |\nabla u|^2 |\nabla \varphi|^2 \, d\mu(x).
\]

where \( \mathcal{K} := \sum_{j=1}^{n-1} k_j^2 \). By Stampacchia’s Theorem, since \( \mu << \mathcal{L}^n \), we get

\[
\nabla \nabla u(x) = 0 \quad \mu\text{-a.e } x \in \{ |\nabla u| = 0 \} \\
\nabla u_j(x) = 0 \quad \mu\text{-a.e } x \in \{ |\nabla u| = 0 \} \subseteq \{ u_j = 0 \}
\]

for any \( j = 1, \ldots, n \). Hence the previous inequality becomes

\[
\int_{\{\nabla u \neq 0\}} (|\nabla u|^2 \mathcal{K}^2 + |\nabla_T \nabla u|^2) \varphi^2 \, d\mu(x) \leq \int_{\mathbb{R}^n} |\nabla u|^2 |\nabla \varphi|^2 \, d\mu(x).
\]

We refer to [26] and [13] for more details.

We now state the main result of this section.

**Theorem 3.2.** Assume that \( G \in C^2(\mathbb{R}^n) \) and \( h \in L^1_{\text{loc}}(\mathbb{R}^n) \) with \( h \geq \lambda_G \). Let \( u \) be a \( h \)-stable solution to (11) such that one of the following conditions hold:

i) \( u \) satisfies (14);

ii) \( n = 2 \) and \( u \) satisfies (15).

Then \( u \) is one-dimensional, i.e. there exists \( \omega \in \mathbb{S}^{n-1} \) and \( u_0 : \mathbb{R} \rightarrow \mathbb{R} \) such that

\[ u(x) = u_0(\langle \omega, x \rangle) \quad \forall x \in \mathbb{R}^n. \]

Moreover,

\[ \langle (h(x)I_n - \nabla^2 G(x)) \nabla u, \nabla u \rangle = 0 \quad \forall x \in \mathbb{R}^n. \]  \hfill (22)

In particular, if \( u_0 \) is not constant, there are \( C \) and \( g \) of class \( C^2 \) such that

\[ G(x) = C(\langle x, \omega \rangle) + g(x'), \]  \hfill (23)

where \( x' := x - \langle x, \omega \rangle \omega \), and \( \lambda_G(x) = h(x) = C^\infty(\langle x, \omega \rangle) \) for all \( x \in \mathbb{R}^n \).

**Proof.** Let us fix \( R > 1 \) and let us define \( \varphi(x) := \Phi(|x|) \) where \( \Phi \in C^\infty(\mathbb{R}) \), \( |\Phi'(t)| \leq 3 \) for any \( t \in [R, R+1] \)

\[ \Phi(t) := \begin{cases} 1 & \text{if } t \leq R \\ 0 & \text{if } t \geq R+1. \end{cases} \]  \hfill (24)

Obviously \( \varphi \in C^\infty_c(\mathbb{R}^n) \) and \( |\nabla \varphi(x)| \leq |\Phi'(|x|)| \leq 3 \). Hence for every \( R > 1 \) \hfill (4)

yields

\[
\int_{\{\nabla u \neq 0\} \cap B_R} \left( |\nabla u|^2 \mathcal{K}^2 + |\nabla_T \nabla u|^2 \right) \, d\mu(x) \leq 9 \int_{B_{R+1}\setminus B_R} |\nabla u|^2 \, d\mu(x)
\]
where \( B_R := \{ y \in \mathbb{R}^n \mid |y| < R \} \).

If \( \nabla u \in L^2_\mu(\mathbb{R}^n) \); then
\[
\lim_{R \to \infty} \int_{B_{R+1} \setminus B_R} |\nabla u|^2 \, d\mu(x) = 0.
\]

Hence (3) and (3) yield
\[
k_j(x) = 0 \quad \text{and} \quad |\nabla_T |\nabla u||_1(x) = 0
\]
for every \( j = 1, \ldots, n-1 \) and every \( x \in \{ \nabla u \neq 0 \} \). From this and Lemma 2.11 in [13] we get the one-dimensional symmetry of \( u \).

If \( n = 2 \) and \( |\nabla u|^2 e^G \in L^\infty(\mathbb{R}^n) \), we take in (4) the following test function
\[
\varphi(x) = \max \left[ 0, \min \left( 1, \frac{\ln R^2 - \ln |x|}{\ln R} \right) \right],
\]
(25)

Reasoning as in [13, Cor. 2.6], we then obtain
\[
\int_{\{ \nabla u \neq 0 \} \cap \mathbb{B}_R} \left( |\nabla u|^2 K^2 + |\nabla T |\nabla u||_1^2 \right) \, d\mu(x) \leq \int_{\mathbb{B}_R \setminus \mathbb{B}_R} \frac{1}{|x|^2 (\ln R)^2} |\nabla u|^2 e^{G(x)} \, dx.
\]

When \( R \to +\infty \), since \( |\nabla u|^2 e^G(x) \) is bounded, the r.h.s. term of the previous inequality vanishes, and we conclude again that \( u \) is one-dimensional.

Assume now that \( u \) is not constant. If we take in (20) the same test functions as above, we get
\[
\int_{\mathbb{R}^n} \langle (h(x) I_n - \nabla^2 G(x)) \nabla u, \nabla u \rangle \, d\mu(x) = 0.
\]

Using the fact that \( u(x) = u_0(\langle \omega, x \rangle) \), we obtain that \( \langle (h(x) I_n - \nabla^2 G(x)) \omega, \omega \rangle = 0 \) for all \( x \) such that \( u_0'(\langle \omega, x \rangle) \neq 0 \). Since \( u \) is not constant and is a solution to the elliptic equation (11), the set of points such that \( u_0'(\langle \omega, x \rangle) = 0 \) has zero measure, so, by the regularity of \( G \) we conclude that
\[
\langle (h(x) I_n - \nabla^2 G(x)) \omega, \omega \rangle = 0 \quad \forall \ x \in \mathbb{R}^n,
\]
which gives (22) and (23).

\[ \square \]

Theorem 3.2 directly implies the following Liouville type result (cfr. [14]).

**Corollary 5.** Let \( h \in C^0(\mathbb{R}^n) \) with \( h \geq \lambda_G \), and \( u \) be a h-stable solution solution to (11) with finite energy. If \( \lambda_G(x) < h(x) \) for some \( x \in \mathbb{R}^n \), then \( u \) is constant. In particular, if \( u \) is a stable solution and \( \lambda_G(x) < 0 \) for some \( x \in \mathbb{R}^n \), then \( u \) is constant.

**Remark 5.** Recalling Remark 3, when the measure \( \mu \) is finite and \( G \) is concave, Theorem 3.2 implies that bounded solutions to (11) which are \( \lambda_G \)-stable are one-dimensional.

4. **Monotonicity implies \( \lambda_G \)-stability.** In this section we assume that, for every \( x \in \mathbb{R}^n \), \( e_n \) is the eigenvector associated to the maximal eigenvalue \( \lambda_G(x) \) of \( \nabla^2 G(x) \). This implies that there exist two functions \( g \) and \( C \) such that
\[
G(x) = g(y) + C(z) \quad \text{and} \quad \lambda_G(x) = C''(z).
\]

(26)

We prove that solutions to (11) which are monotone along the \( z \)-axis are stable.

**Theorem 4.1.** Assume that \( G \) satisfies (26) and \( u \) is a solution to (11) satisfying (3). Then \( u \) is \( \lambda_G \)-stable.
3.2. Letting $R$ If we let $\phi \in C_c^1(\mathbb{R}^n)$. Taking as test function $u_{x_1}^2$ in (27), we get
\[ 0 = \int_{\mathbb{R}^n} \left( \nabla u_z, \nabla \left( \frac{\phi^2}{u_z} \right) \right) - C''(z)\phi^2 - f'(u)\phi^2 \, d\mu(x) \]
\[ = \int_{\mathbb{R}^n} |\nabla \phi|^2 - \left| \frac{\phi}{u_z} \nabla u - \nabla \phi \right|^2 - C''(z)\phi^2 - f'(u)\phi^2 \, d\mu(x) \]
\[ \leq \int_{\mathbb{R}^n} |\nabla \phi|^2 - C''(z)\phi^2 - f'(u)\phi^2 \, d\mu(x), \]
which is the stability condition (2.1).

5. Proof of Theorem 1.1. Observe that in (2), $G(x) = g(y) + C(z)$, and by assumption $C''(z) \geq \nabla^2 g(y)$. So (26) holds, and by Theorem 4.1 every solution to (2) satisfying (3) is $\lambda_C$-stable.

If either a) or c) holds, the thesis follows from Theorem 3.2.

Let us assume that $u$ satisfies b). We define $\psi_R(y) := \Phi(|y|)$ where $\Phi$ is as in (24) and $\varphi_S(z)$ as follows. We fix $S > 1$ and let
\[ \varphi_S(z) := \begin{cases} 3 & \text{if } |z| \leq S \\ 4 - \frac{z^2}{S^2} & \text{if } S \leq |z| \leq 2S \\ 0 & \text{if } |z| \geq 2S. \end{cases} \]

We compute (4) with test function $\psi_R(y)\varphi_S(z)$ and obtain, recalling (6),
\[ \left( |\nabla u|^2 \right) \psi_R^2 \varphi_S^2 \, d\mu(x) \leq \int_{\mathbb{R}^n} |\nabla u|^2 \varphi_S^2 \psi_R(y) \, d\mu(x) \]
\[ \leq \frac{4}{S^2} \int_{\mathbb{R}^n} |\nabla u|^2 \varphi_S^2 \psi_R(y) \, d\mu(x) \leq \frac{36K}{S^2}. \]
If we let $R \to +\infty$ we obtain
\[ \int_{\{\nabla u \neq 0\} \cap \{ |z| \leq S \}} \left( |\nabla u|^2 K^2 + |\nabla T| |\nabla u|^2 \right) \, d\mu(x) \leq \frac{4K}{S^2}. \]
Letting $S \to +\infty$ we then obtain (3) and we conclude as in the proof of Theorem 3.2. \qed

REFERENCES


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