A NOTE ON SINGULAR PERTURBATION PROBLEMS VIA
AUBRY-MATHER THEORY

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Abstract. Exploiting the metric approach to Hamilton-Jacobi equation recently
introduced by Fathi and Siconolfi [13], we prove a singular perturbation
result for a general class of Hamilton-Jacobi equations. Considered in the
framework of small random perturbations of dynamical systems, it extends a
result due to Kamin [19] to the case of a dynamical system having several
attracting points inside the domain.

1. Introduction. In a series of papers [19], [8], [10], Kamin and Eizenberg con-
considered the singular perturbation problem

\[
\begin{align*}
-\varepsilon \Delta v_\varepsilon + H(x, Dv_\varepsilon) - \varepsilon c(x) &= 0 & x \in D \\
v_\varepsilon(x) &= 0 & x \in \partial D
\end{align*}
\]

(1.1)

where \(H(x, p) = |p|^2/2 - b(x) \cdot p\) and \(c\) is a nonnegative function.

The interest in these type of problems is motivated by the relation with large
deviations theory for small random perturbations of dynamical systems. In fact,
\(v_\varepsilon := -\varepsilon \log u_\varepsilon\) where \(u_\varepsilon\) is a solution to

\[
\begin{align*}
-\varepsilon \Delta u_\varepsilon - b(x) \cdot Du_\varepsilon + c(x)u_\varepsilon &= 0 & x \in D \\
u_\varepsilon(x) &= 1 & x \in \partial D.
\end{align*}
\]

(1.2)

It is well known that \(u_\varepsilon(x)\) can be represented as \(E_x(e^{-\int_0^\tau c(X(s))ds})\), where \(X(\cdot)\) is the
solution of the stochastic differential equation

\[
dX(t) = b(X(t)) \, dt + \sqrt{\varepsilon} \, dW(t)
\]

issuing from \(x\) and \(\tau\) is the first exit time of this trajectory from \(D\). So the analysis of
the limit behavior of \(u_\varepsilon\) and of its logarithmic transform \(v_\varepsilon\) as \(\varepsilon \to 0\) is related
to asymptotic estimates of the first exit times of the trajectories of the stochastic
system (we refer to [8] for more details).

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theory.
For any $\varepsilon > 0$, the singular perturbation problem (1.1) admits a unique solution, while the limit problem

$$\begin{cases}
H(x, Dv) = 0 & x \in D \\
v(x) = 0 & x \in \partial D,
\end{cases}$$

(1.4)
in general, does not satisfy an uniqueness property. A natural question is therefore to understand how the sequence $v_\varepsilon$ behaves for $\varepsilon$ going to $0^+$.

Recalling the relationship between $v_\varepsilon$ and the perturbed stochastic system (1.3), it is quite natural that the limit behavior of $v_\varepsilon$ is strictly connected to the properties of the dynamical system

$$\dot{x}(t) = b(x(t)),
$$

(1.5)
and the corresponding Freidlin-Ventcel quasi-potential (see [15]). Assuming that (1.5) has a single stable attractor inside $D$ which is a class of equivalence for the quasi-potential, Kamin characterized the limit of $v_\varepsilon$ when $c$ is positive in an open set containing the attractor, while Eizenberg showed that any solution of (1.4) can be achieved in the limit for an appropriate choice of $c$. We recall also that Eizenberg in [9] considered the more general problem

$$\begin{cases}
-\varepsilon Lv_\varepsilon + H(x, Dv_\varepsilon) - \alpha(\varepsilon) c(x) = 0 & x \in D \\
v_\varepsilon(x) = 0 & x \in \partial D,
\end{cases}$$

with $L$ a uniformly elliptic operator, $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $H(x, p) = K(x, p) - b(x) \cdot p$, where $K$ is convex, coercive and superlinear in $p$ (more details will be given in Example 4.2). In this case the asymptotic behaviour of the solutions is related to the asymptotics of $-\varepsilon \log \alpha(\varepsilon)$, but such results are out of the framework of our paper.

In [6], with the aim of reproving by PDE methods a classical Freidlin-Ventcel’s large deviations result on the exit points from a domain of a random process, it was pointed out and exploited the correspondence between dynamical properties of (1.3) and metric properties of critical Hamilton-Jacobi equations, a class of Hamilton-Jacobi equations which arises in Hamiltonian dynamics as well as in weak KAM theory (see [12, 13]). In particular, in [6] it was shown that the Freidlin-Ventcel quasi-potential coincides with a distance function associated in natural way to the Hamilton-Jacobi equation (see for details Section 2) and that the set of the $\omega$-limits of (1.3) is contained in the Aubry set of the distance, an uniqueness set for the Hamilton-Jacobi equation.

Building upon the ideas in [6], we prove a singular perturbation result for a general class of Hamilton-Jacobi equations with continuous, convex and coercive Hamiltonian $H$ (see assumptions (2.3), (2.4)) which includes (1.4), but also other interesting models such as the Eikonal equation (in this case $H(x, p) = |p| - f(x)$). Our main result, Theorem 3.3 describes the behavior of the solutions $v_\varepsilon$ to (1.1) as $\varepsilon \to 0$:

if the Aubry set satisfies the following conditions

- it consists of a finite number of classes of equivalence for the distance function (see condition (3.6)),
- it is contained in the set where $c$ is positive (see condition (3.5)),

then the limit procedure selects, among all the possible solutions to (1.4), the maximal viscosity solution assuming the boundary datum.
We then prove that, in the case of problem \[1.1\], our result generalizes the one in \[19\] to the case that the dynamical system \[1.5\] has several attracting points inside \(D\).

Our approach is purely PDE and makes use, for the first order problem, of viscosity solution theory. The viscosity solution approach to singular perturbation problems was initiated in \[11\], and then pursued, for example, in \[4, 17, \] and \[21\] (see \[3, \text{Ch. VI}\] for a review). Nevertheless, since a key ingredient in this framework is the uniqueness of the viscosity solution to the limit equation, some of the interesting problems arising in Freidlin-Ventcel large deviations theory have not been considered in the viscosity solution approach.

Combining viscosity solution methods with the metric approach to Hamilton-Jacobi equation we are able in this paper and in \[6\] to consider singular perturbation problems where the limit Hamilton-Jacobi equations have many solutions. Our result also shows that, in the case that there is no explicit dynamic associated to the Hamilton-Jacobi equation (even if, as it is well known, a dynamical interpretation of Hamiltonian is always possible), the distance function associated to the Hamilton-Jacobi equation replaces the role of the quasi-potential in the analysis of the singular perturbation problem.

The paper is organized as follows. In Section 2 we recall some basic facts about the metric properties of Hamilton-Jacobi equations and we give a representation formula for the solution of \(1.4\). In Section 3 we prove the singular perturbation result. In Section 4 we describe some examples of Hamiltonians satisfying our assumptions and we discuss the application to the Kamin-Eizeenberg model.

## 2. The distance, the Aubry set and the Dirichlet problem.

In this section we recall some basic properties of the metric approach to the Dirichlet problem

\[
\begin{align*}
H(x, Du) &= 0 & x \in D, \\
v(x) &= g(x) & x \in \partial D.
\end{align*}
\tag{2.1}
\]

Throughout the paper a solution to \(2.1\) will be intended in the viscosity sense (see \[3\]), moreover the boundary condition will be interpreted in a strong, i.e. pointwise, sense.

We assume that

\[
\begin{align*}
D & \text{ is a bounded set with Lipschitz boundary,} \\
H(x, p) & \text{ is continuous, Lipschitz continuous in } x, \\
& \text{ convex and coercive in } p, \\
H(x, 0) & \leq 0.
\end{align*}
\tag{2.2, 2.3, 2.4}
\]

We define the critical value of the Hamiltonian \(H\) as

\[
C := \min\{\lambda : H(x, Du) = \lambda \text{ admits a viscosity subsolution in } D\}.
\]

Note that by stability properties of viscosity solutions, \(C\) is a minimum, hence a subsolution at the critical level always exists (see \[13\] for more details). Moreover, under the critical level, there is no subsolution and therefore no solution to the equation.

By \(2.4\),

\[
C \leq 0
\]

since any constant is a subsolution to \(H(x, Du) = 0\). If \(C < 0\), then a subsolution at the critical level, i.e. of the equation \(H(x, Du) = C\), is a strict subsolution to the equation \(H(x, Du) = 0\) in \(D\). It follows that there exists a unique viscosity solution
of the Dirichlet problem (2.1) (see [16], [20]). In this case the singular perturbation problem can be studied via standard stability results in viscosity solution theory and it can be proved that \( v_\varepsilon \) converges to the unique viscosity solution of (2.1). Therefore, from now on, we will assume that \( C = 0 \) since in this case the non-uniqueness phenomenon appears.

For \( x \in \overline{D} \) and \( q \in \mathbb{R}^N \) we set
\[
Z(x) := \{ p \in \mathbb{R}^N \mid H(x, p) \leq 0 \},
\]
\[
\sigma(x, q) := \sup\{ p \cdot q : p \in Z(x) \}.
\]
The function \( \sigma \), which is the support function of the convex set \( Z(x) \), is continuous and, by (2.4), nonnegative, moreover it is convex and positive homogenous in \( q \) for any \( x \in \overline{D} \). We introduce a nonsymmetric distance by setting
\[
S(x, y) = \inf \left\{ \int_0^1 \sigma(\phi(s)), \dot{\phi}(s) ds : \phi \in C_{x,y} \right\}
\]
for any \( x, y \in \overline{D} \), where \( C_{x,y} := \{ \phi \in W^{1,\infty}([0,1], D) \mid \phi(0) = x, \phi(1) = y \} \).

The next proposition collects some basic properties of the distance function \( S \) (see [9], Proposition 4.3, and [13], Propositions 4.1, 4.2).

**Proposition 2.1.**

i) \( S(x, y) \geq 0 \) for any \( x, y \in \overline{D} \), \( S(x, x) = 0 \) and
\[
S(x, y) \leq S(x, z) + S(z, y)
\]
for any \( x, y, z \in \overline{D} \).

Moreover, there exists a constant \( M \) such that
\[
S(x, y) \leq Md(x, y)
\]
for any \( x, y \in \overline{D} \) (2.7)

where \( d \) is the Euclidean geodesic distance in \( \overline{D} \).

ii) For any \( x \in D \), \( S(x, \cdot) \) is a viscosity subsolution in \( D \) and a viscosity supersolution in \( D \setminus \{ x \} \) of the equation (2.1).

iii) \( v \) is a viscosity subsolution of (2.1) if and only if \( -S(x, y) \leq v(x) - v(y) \leq S(y, x) \) for any \( x, y \in D \).

**Remark 2.2.** \( S \) is non negative because of the assumption (2.4). Moreover a condition such as (2.4) is also necessary to assure the equi-boundedness of the sequence \( v_\varepsilon \), solutions of singular perturbation problem (see Section 3).

We denote by \( \sim \) the equivalence relation
\[
x \sim y \text{ if and only if } S(x, y) = S(y, x) = 0.
\]

By the continuity of \( S \), a class of equivalence \( A \) is closed and, by (2.6), \( S(x, \cdot) \) and \( S(\cdot, x) \) are constant in \( A \), for any \( x \in D \). We denote by \( S(x, A) \) and, respectively, \( S(A, x) \) these constants.

In this framework a set where the distance \( S \) degenerates, called the Aubry set, plays a central role in the analysis of existence and uniqueness of the solutions to (2.1). It acts as an hidden boundary on which a datum must be fixed. In fact, as the following simple example shows, it is not sufficient to fix the value of the solution on \( \partial D \) to have uniqueness for (2.1).

**Example 2.1.** Let \( D \) be the interval \([-2, 2]\) and \( g(-2) = g(2) = 9 \). Set \( U(x) = (x^2 - 1)^2 \) and define \( b(x) = U'(x) \). Consider problem (2.1) with \( H(x, p) = |p|^2/2 - \)
b(x) \cdot p. Then U(x) and

\[ V(x) = \begin{cases} 
1 & x \in [-1, 1] \\
U(x) & x \in \mathbb{T} \setminus [-1, 1]
\end{cases} \]

are two C^1, hence viscosity, solutions of (2.1). Note that the boundary condition is taken in strong, i.e. pointwise, sense.

**Definition 2.3.** A point \( x \) belongs to the Aubry set \( \mathcal{A} \) if there exists a sequence \( \{ \phi_n \} \subset \mathcal{C}_{x,x} \) with \( \ell(\phi_n) \geq \delta > 0 \) (\( \ell \) is the Euclidean length of the curve) s.t.

\[ \inf_n \left\{ \int_0^1 \sigma(\phi_n(s), \dot{\phi}_n(s)) ds \right\} = 0. \]

The previous definition says that it is possible to find a sequence of cycles through \( x \) with uniformly lower bounded Euclidean length, but vanishing intrinsic length, i.e. the length of the curve with respect to the metric \( \sigma \).

On the other hand, out of the Aubry set, the intrinsic distance behaves as the Euclidean geodesic distance. We will use this property in the following so we state it precisely.

**Lemma 2.4.** Assume \( x \notin \mathcal{A} \). Then for every \( \varepsilon > 0 \) there exists a positive \( \delta_\varepsilon < \varepsilon \) such that for every \( y \in B(x, \delta_\varepsilon) \), the Euclidean ball of radius \( \delta_\varepsilon \) centered at \( x \), we have

\[ S(x, y) = \inf \left\{ \int_0^1 \sigma(\phi(s), \dot{\phi}(s)) ds : \phi \in \mathcal{C}_{x,y}, \ell(\phi) < \varepsilon \right\}, \]

and analogously

\[ S(y, x) = \inf \left\{ \int_0^1 \sigma(\phi(s), \dot{\phi}(s)) ds : \phi \in \mathcal{C}_{y,x}, \ell(\phi) < \varepsilon \right\}. \]

This result is proved in Lemma 5.5 of [13] for the periodic setting, but the same arguments hold in our case. In the following proposition we recall some basic properties of the Aubry set.

**Proposition 2.5.**

i) \( \mathcal{A} \) is a nonempty, closed subset of \( D \).

ii) There exists a C^1-subsolution to \( H(x, Dv) = 0 \) in \( D \) which is strict in any open subset \( U \) of \( D \) with closure disjoint from \( \mathcal{A} \), i.e.

\[ H(x, Dv) \leq -\delta_U \quad \text{in } U \]

in viscosity sense, for some \( \delta_U > 0 \).

The proof of i) can be found in [13], Corollary 5.7 and 5.9, while the proof of ii) is in [6], Proposition 5.11. For a general treatment of the Aubry set in the periodic setting we refer to [13]: most of the arguments can be repeated with minor changes also for critical Hamilton-Jacobi equations in bounded domains.

**Theorem 2.6.**

i) Let \( u \) and \( v \) be, respectively, a upper semicontinuous subsolution and a lower semicontinuous supersolution to (2.1) such that

\[ u \leq v \quad \text{on } \partial D \cup \mathcal{A}. \]

Then

\[ u \leq v \quad \text{in } \overline{D}. \]
ii) If \( g: \partial D \cup A \to \mathbb{R} \) is a continuous function which satisfies the compatibility condition
\[
-S(y,x) \leq g(x) - g(y) \leq S(y,x) \quad \text{for any } x, y \in \partial D \cup A
\]
then the unique viscosity solution of (2.1) such that
\[
u(x) = g(x) \quad \text{for any } x \in \partial D \cup A
\]
is given by
\[
u(x) = \min_{\partial D \cup A} \{ g(y) + S(y,x) \}. \tag{2.10}
\]

iii) The maximal viscosity solution of the Dirichlet problem (2.1) is
\[
G(x) = \min_{y \in \partial D} \{ g(y) + S(y,x) \}. \tag{2.11}
\]

Proof. The proof of i) relies on the properties of Aubry set recalled in Proposition 2.5 and on standard properties of viscosity solutions. We give only a sketch of it. Assume by contradiction
\[
\arg \max_{D} \{ u - v \} \subset D', \tag{2.12}
\]
where \( D' \) is an open set in \( D \) with closure disjoint from \( A \). Thanks to Proposition 2.5, there exists a subsolution \( u \) to (2.1), which is strict in \( D' \). These properties are inherited by the function \( w := \lambda u + (1 - \lambda)v \), for \( \lambda \in (0,1) \), thanks to the convex character of \( H \). We can furthermore take \( \lambda \) so small that (2.12) still holds with \( w \) in place of \( u \). Now, following the classical Ishii’s argument for comparison principle (11) and using the standard trick of doubling variables, it is possible to get a contradiction between (2.12) with \( w \) in place of \( u \) and supersolution and strict subsolution properties of, respectively, \( v \) and \( w \). Item ii) follows immediately from i), recalling Proposition 2.1. Finally, the maximality of (2.11) is obtained observing that, if \( u \) is a solution to (2.1) and \( x \in A \),
\[
u(x) = g(x) \leq g(y) + S(y,x) \quad \text{for any } y \in \partial D.
\]
Hence \( u \leq G \) on \( A \) and, by i), \( u \leq G \) in \( \overline{D} \).

3. The singular perturbation result. We consider the singular perturbation problem
\[
\begin{aligned}
-\varepsilon \Delta v_\varepsilon + H(x, Dv_\varepsilon) - \varepsilon c(x) &= 0 \quad x \in D \\
v_\varepsilon(x) &= g(x) \quad x \in \partial D
\end{aligned} \tag{3.1}
\]
where
\[
c: \overline{D} \to \mathbb{R} \text{ is continuous, nonnegative function}. \tag{3.2}
\]
Existence and uniqueness of the solution to (3.1) have been proved by several authors under some general structure conditions (see for example [1], [9], [20, Chapter VI]). By standard comparison principle between sub and supersolution, we obtain that there exists \( C \) independent of \( \varepsilon \) such that
\[
\|v_\varepsilon\|_\infty \leq C.
\]
Indeed for every \( \varepsilon \) the constant \( K := \inf_{\partial D} g \) is a viscosity subsolution to (3.1), because of assumption 2.4. Moreover, using the coercivity of \( H \) and the continuity of \( g \), we can find \( R \in \mathbb{R}^N \) and \( T \) such that \( R \cdot x + T \) is a supersolution to (3.1).

We assume that there exists \( C \) independent of \( \varepsilon \) such that
\[
\|Dv_\varepsilon\|_\infty \leq C. \tag{3.3}
\]
Remark 3.1. Condition (3.3) can be proved for particular coercive Hamiltonians adapting appropriately the Bernstein method. We refer to [11, Lemma 1.2.2.2] and [14] for the proof of this condition for Hamiltonians as $H(x, p) = |p|^2 + b(x) \cdot p - \lambda$, to Chapter VI for Hamiltonians as $H(x, p) = |p| - f(x)$.

We define $T$ := sup $\{ t \in [0, 1] : |\phi_n(t) - x| \geq \delta_\varepsilon/2 \}$. Passing to a subsequence, we get that $\phi_n(T_n) \to y \in \partial B(x, \delta_\varepsilon/2)$. It is easy to deduce that $S(x_0, y) = 0 = S(y, x)$, using the properties of $\phi_n$ and the construction of $y$.

Now we consider $\gamma \in C_{y, x}$ with $\ell(\gamma) < \varepsilon$. For such curve, we get that $\gamma(t) \in B(x, \varepsilon)$ for every $t \in [0, 1]$. This implies, as $f$ is a strict subsolution to (2.1)
\( D \setminus A \), that \( D f(\gamma(t)) \) is contained in the interior of \( Z(\gamma(t)) \) for every \( t \in [0,1] \). Hence
\[
d(D f(\gamma(t)), \partial Z(\gamma(t)) \geq \rho \quad t \in [0,1],
\]
for some \( \rho > 0 \), and, by the definition of \( \sigma \) and the convexity of the set \( Z(x) \), we get
\[
f(y) - f(x) = \int_0^1 D f(\gamma(t)) \dot{\gamma}(t) \, dt \leq \int_0^1 [D f(\gamma(t)) \dot{\gamma}(t) - \sigma(\gamma, \dot{\gamma}(t))] \, dt
\]
\[+ \int_0^1 \sigma(\gamma, \dot{\gamma}(t)) \, dt \leq -\rho \frac{\delta_r}{2} + \int_0^1 \sigma(\gamma, \dot{\gamma}(t)) \, dt.
\]
Taking the infimum in the previous inequality over all the curves \( \gamma \in C_{y,x} \) with \( l(\gamma) < \varepsilon \) and recalling Lemma 2.4, we conclude that \( f(x) - f(y) < S(y,x) = 0 \). On the other side the compatibility condition gives that \( f(y) - f(x) \leq S(x,y) = 0 \), hence we get (3.9).

Lemma 3.5. Let \( v \) be a solution of (2.1) and let \( g \) be extended onto the union of \( \partial D \) and \( A \) as in Remark 2.2. If \( v(x) = g(A_i) + S(A_i, x) \) for \( x \) in a neighborhood of \( A_i \), then \( v \) cannot be the uniform limit of a sequence \( v_\varepsilon \) of solution of (3.1).

Proof. Let \( v = \lim_{\varepsilon \to 0} v_\varepsilon \), up to a subsequence. Assume by contradiction for \( x \) in a neighborhood \( A_\delta \) of \( A_i \), we have \( v(x) = g(A_i) + S(A_i, x) \). We can choose \( A_\delta \) sufficiently small such that it intersects \( A \) only in \( A_i \). Let \( f \) to be as in (3.7) and define \( \psi(x) = v(x_0) + \alpha(f(x) - f(x_0)) \) where \( x_0 \in A_i \) and \( \alpha \in (0,1) \) is such that
\[
\alpha \|D^2 f\|_\infty \leq \min\{v(x) : x \in A_i\}
\] (note that the right hand side is positive by (3.5)). The functions \( \psi \) and \( v \) are constant on \( A_i \), with value \( \psi(x_0) = v(x_0) = g(A_i) \), and, for \( x \in A_\delta \), by Proposition 2.4 iii)
\[
\psi(x) \leq v(x_0) + \alpha S(x_0, x) \leq v(x_0) + S(x_0, x) = g(A_i) + S(A_i, x) = v(x).
\]

We show that actually \( \psi \) is a strictly less than \( v \) out of \( A_i \). We distinguish two cases.

If \( S(x_0, x) > 0 \), then, since \( \alpha \in (0,1) \), we have
\[
\psi(x) \leq v(x_0) + \alpha S(x_0, x) < v(x_0) + S(x_0, x) = v(x).
\]

If \( S(x_0, x) = 0 \), for \( x \in A_\delta \setminus A_i \), then by Lemma 3.4 we have that \( f(x) < f(x_0) \) and therefore we obtain \( \psi(x) < \psi(x_0) = v(x_0) \leq v(x) \).

So in both cases we get that \( v - \psi \) has as a local strict minimum at \( A_i \).

Let \( \rho_\eta \) be a standard mollifier, i.e. \( \rho_\eta(x) = \rho(x/\eta) = \eta^N \) where \( \rho : \mathbb{R}^N \to \mathbb{R} \) is a smooth, nonnegative function such that \( \text{supp}\{\rho\} \subset B(0,1) \) and \( \int_{\mathbb{R}^N} \rho(x) \, dx = 1 \). Set \( f_\varepsilon = f * \rho_{\varepsilon^2} \). Then, by the convexity of \( H \) and the regularity of \( f \), we have
\[
H(x, D f_\varepsilon) \leq \int_{\mathbb{R}^N} H(y, D f) \rho_{\varepsilon^2}(x - y) \, dy \leq H(x, D f) + C \varepsilon^2,
\]
\[
\|D^2 f_\varepsilon\|_\infty \leq \|D^2 f\|_\infty.
\]

Set \( \psi_\varepsilon(x) = v(x_0) + \alpha(f_\varepsilon(x) - f_\varepsilon(x_0)) \). Since \( v_\varepsilon - \psi_\varepsilon \) converges uniformly to \( v - \psi \) and \( \psi \) is strictly less than \( v \) out of \( A_i \), there exists a sequence \( x_\varepsilon \) of minimum points for \( v_\varepsilon - \psi_\varepsilon \) such that \( d(x_\varepsilon, A_i) \to 0 \) for \( \varepsilon \to 0 \). Hence, recalling (3.10)-(3.12), for \( \varepsilon \)}
sufficiently small, we get
\[
0 \leq -\varepsilon \Delta \psi(x_\varepsilon) + H(x_\varepsilon, D\psi(x_\varepsilon)) - \varepsilon c(x_\varepsilon) \\
= -\varepsilon \alpha \Delta f(x_\varepsilon) + H(x_\varepsilon, \alpha Df(x_\varepsilon)) - \varepsilon c(x_\varepsilon) \\
\leq \varepsilon \alpha \|D^2f\|_\infty + \alpha(H(x_\varepsilon, Df(x_\varepsilon)) + C\varepsilon^2) - \varepsilon c(x_\varepsilon) < 0
\] (3.13)
and therefore a contradiction.

**Proof of Theorem 3.3.** Let \( v = \lim_{\varepsilon \to 0} v_\varepsilon \) uniformly, up to a subsequence. By standard stability results in viscosity solution theory \( v \) is a solution of (2.14). So we can extend \( g \) onto the union of \( \partial D \) and \( \mathcal{A} \) as in Remark 3.2. Since \( v \) is a solution of (2.14), \( v \leq G \) in \( \mathcal{T} \). If we show that \( v = G \) on \( \mathcal{A} \), then the statement follows by Theorem 2.6 iii). We argue by contradiction and we assume without loss of generality that
\[
v(A_1) = g(A_1) < G(A_1).
\] (3.14)
Hence, in a neighborhood of \( A_1 \), \( g(A_1) + S(A_1, x) < G(x) \) and therefore, by formula (3.13),
\[
v(x) = \min_{i=1,..,n} \{ g(A_i) + S(A_i, x) \}. \tag{3.15}
\]
We claim that
\[
g(A_1) = \min_{j=2,..,n} \{ g(A_j) + S(A_j, A_1) \}. \tag{3.16}
\]
By (2.9) and (3.6), \( g(A_1) \leq g(A_j) + S(A_j, A_1) \) for \( j = 2,..,n \). If a strict inequality held in (3.16), then it would follow that
\[
v(x) = g(A_1) + S(A_1, x)
\]
in a neighborhood of \( A_1 \) and therefore a contradiction to Lemma 3.5. By relabeling the sets \( A_i \), we can assume that
\[
g(A_1) = g(A_2) + S(A_2, A_1). \tag{3.17}
\]
We observe that \( g(A_2) < G(A_2) \), otherwise
\[
g(A_1) = G(A_2) + S(A_2, A_1) \geq G(A_1)
\]
in contradiction with (3.14). Moreover
\[
g(A_1) + S(A_1, x) = g(A_2) + S(A_2, A_1) + S(A_1, x) \geq g(A_2) + S(A_2, x).
\]
It follows that in a neighborhood of \( A_2 \), we have
\[
v(x) = \min_{i=2,..,n} \{ g(A_i) + S(A_i, x) \}.
\]
Arguing as in (3.16) we have that
\[
g(A_2) = \min_{j=3,..,n} \{ g(A_j) + S(A_j, A_2) \}
\]
and therefore, relabeling the sets \( A_i \), we have
\[
g(A_2) = g(A_3) + S(A_3, A_2).
\]
Iterating the previous procedure \( n \)-times, we eventually get \( g(A_{n-1}) = g(A_n) + S(A_n, A_{n-1}) \) and \( g(A_n) < G(A_n) \), hence
\[
v(x) = \min \{ g(A_{n-1}) + S(A_{n-1}, x), g(A_n) + S(A_n, x) \} = g(A_n) + S(A_n, x)
\]
in a neighborhood of \( A_n \). But this gives a contradiction to Lemma 3.5. \( \square \)
4. Examples.

**Example 4.1.** The model problem for (3.1) is the one considered in [19], [8], i.e.

\[ H(x, p) = \frac{|p|^2}{2} - b(x) \cdot p, \]  

(4.1)

where the vector field \( b \) satisfies

\[ b(x) \cdot n_{ext}(x) < 0 \quad \text{for} \quad x \in \partial D \]  

(4.2)

(we assume that the boundary of \( D \) is \( C^1 \)) and with a boundary datum \( g \equiv 0 \). In those papers, the case of a single attractor in \( D \) for the dynamical system

\[ \dot{x}_t = b(x_t) \]  

(4.3)

is considered. Theorem 3.3 allows us to generalize the result to the case of a more complex structure of the set of the attractors of the dynamical system (4.3).

Following [10] and [15, Chapter VI], we assume that there exists a finite number of compacta \( A_1, ..., A_n \) such that the set \( \Omega_b \) of the \( \omega \)-limits of the dynamical system

\[ V(x, y) = \inf \left\{ \int_0^T \frac{1}{2} \dot{\phi}(s) - b(\phi(s)) \right\|^2 ds : \phi(0) = x, \phi(T) = y, \phi \in W^{1, \infty}([0, T], D), T > 0 \}. \]  

(4.4)

We assume that

\[ V(x, y) + V(y, x) = 0 \quad \text{if and only if} \quad x, y \in A_i \]  

(4.5)

for some \( i \in \{1, \ldots, n\} \).

In [10, Lemma 5.2], it is shown that, because of the assumption (4.2), the critical value for the Hamiltonian (4.1) is \( C = 0 \). We have in this case

\[ Z(x) = B(b(x), |b(x)|) \]  

\[ \sigma(x, q) = |b(x)||q| + b(x) \cdot q. \]

We denote by \( S \) the corresponding distance defined as in (2.5) and by \( \mathcal{A} \) the Aubry set of \( S \). We recall some results proved in [11] which establish the relation between (4.3) and the quasi-potential \( V \) on one side and the distance \( S \) and \( \mathcal{A} \) on the other side.

**Proposition 4.1.**

i) \( V(x, y) = S(y, x) \) for any \( x, y \in D \).

ii) \( \Omega_b \subset \mathcal{A} \) and \( \mathcal{A} \) is contained in the interior of \( D \).

iii) \( \mathcal{A} \) is forward invariant for (4.3).

(iv) If \( x_0 \in \mathcal{A} \), the integral curve \( x(\cdot) \) of \( b \), starting at \( x_0 \) and defined in \( [0, +\infty) \), satisfies \( V(x(t), x(s)) = 0 \), for any \( t, s \) in \( [0, +\infty) \).

The interesting point in (iv) is that \( V(x(t), x(s)) = 0 \) holds true for \( t > s \), while for \( t < s \), this identity immediately follows from the definition of quasi-potential.

Note that by item ii) \( \mathcal{A} \) is contained in the interior of \( D \), see assumption (3.6).

Concerning hypothesis (3.6), let us show that

\[ \mathcal{A} = \bigcup_{i=1}^n A_i. \]

By (4.5), it follows that either \( A_i \), for \( i = 1, \ldots, n \), is an equilibrium for (4.3) or \( A_i \) is a class of equivalence for \( S \). Hence, by Prop. (11) and the very definition of \( \mathcal{A} \), we have \( \bigcup_{i=1}^n A_i \subset \mathcal{A} \). To prove the reverse inclusion, if \( x_0 \in \mathcal{A} \) and \( x_0 \) is not an
equilibrium, then, by \(4.1\) (iii) and iv), it follows that there exists \(x_1\) in \(\Omega_b\) such that 
\(V(x_0, x_1) = V(x_1, x_0) = 0\). Hence \(x_0 \in A_i\), for some \(i\), and \(\bigcup_{i=1}^{n} A_i \supset A\).

Concerning assumption (3.7), the existence of a \(C^{1,1}\) strict-subsolution has been recently proved in [5] for Tonelli Hamiltonians, i.e. \(C^2\) Hamiltonians, strictly convex and superlinear with respect to \(p\). In this paper it is proved that, whenever a \(C^1\) strict subsolution exists (see Prop. 2.5), then it can be assumed to be \(C^{1,1}\). Moreover counter-examples shows that this regularity is optimal. The Hamiltonian (4.1) is a Tonelli Hamiltonian if \(b\) is \(C^2\). Note that we need this regularity only in a neighborhood of \(\bigcup_{i=1}^{n} A_i\).

Finally, as in [19], [8], [11], it is possible to prove that, for any \(\varepsilon\), there exists a unique solution \(v_\varepsilon\) to (3.1) satisfying (3.3).

By Theorem 3.3 we get that \(\lim_{\varepsilon \to 0} v_\varepsilon = G\) where 
\[
G(x) = \min_{y \in \partial D} \{V(x, y)\} \quad x \in \overline{D}.
\]

Example 4.2. A more general model of Hamiltonian is the one considered in [9], i.e.
\[
H(x, p) = -b(x) \cdot p + K(x, p)
\]
with \(b\) satisfying (4.2) and \(K\) satisfying (2.3), (2.4) and
\[
\lim_{|p| \to \infty} \frac{K(x, p)}{|p|} = +\infty \quad \text{for any} \ x \in \overline{D}.
\]
The existence of a sequence \(v_\varepsilon\) satisfying (3.3) is proved in [9]. Assumption (3.7) is satisfied, by [5], if \(K\) is \(C^2\) and strictly convex with respect to \(p\).

It is worthwhile to observe that the quasi-potential \(V\), defined as in (4.4), in general does not coincide with the distance \(S\), as in the previous example. In this case, the distance \(S\) replace the role of \(V\) in describing the asymptotic behavior of the singular perturbation problem (3.1) corresponding to the dynamical system (4.3) (see also [9]).

Remark 4.2. If we assume
\[
K(x, p) \leq C|p|^2 \quad \text{for any} \ x \in \overline{D},
\]
then
\[
V(x, y) \leq CS(y, x)
\]
where \(V\) is defined as in (4.4). In fact, by (4.4), \(|p|^2 - b(x) \cdot p/C \leq 0\), i.e. \(|p - b(x)/C| \leq |b(x)|/C\), implies \(H(x, p) \leq 0\). Hence \(B(b(x)/C, |b(x)|/C) \subset Z(x)\) and therefore for the corresponding support function
\[
\frac{|b(x)|}{C} \cdot q = \frac{b(x)}{C} \cdot q \leq \sigma(x, q).
\]

If we denote by \(S_b\) the distance introduced in Example 4.1 we then have \(V(x, y) = S_b(y, x) \leq CS(y, x)\). By \(S_b(x, y) \leq CS(x, y)\) it also follows that, for the corresponding Aubry sets, \(A \subset A_b\).

Example 4.3. Another class of examples can be obtained considering the Hamiltonian
\[
H(x, p) = F(p) - f(x)
\]
with \(F\) a continuous, convex function satisfying \(F(0) = 0\), and \(F(p) > 0\) for \(|p| \neq 0\) and \(f : \overline{D} \to \mathbb{R}\) a continuous, nonnegative function.
An Hamilton-Jacobi equation with Hamiltonian (4.9) is called an Eikonal equation and, since it arises in many applications, it is well studied in literature (see for example [7], [18] and also [2] for a related singular perturbation problem).

If $f$ vanishes in some points inside $D$, then the critical value for the Hamiltonian (4.9) is $C = 0$. Moreover the Aubry set for the distance $S$ defined as in (2.5) is given by

$$A = \{ x \in D : f(x) = 0 \}$$

and a strict subsolution out of $A$ is $\psi \equiv 0$. The Hamiltonian (4.9) is a Tonelli Hamiltonian if $F$ is smooth, strictly convex and superlinear (for example $F(p) = |p|^2$) and $f$ is smooth. Under the same hypothesis the sequence $v_\varepsilon$ satisfies (3.3) (see [20, Chapter VI]).

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