Introduzione alle Equazioni alle Derivate Parziali, L.M. in Matematica, A.A. 2013-2014.

Harmonic functions: mean value property, Harnack inequality, maximum principles.

Exercise 1. Let u be harmonic in $\mathbb{R}^n$. Prove the following.

i) If $u \in L^p(\mathbb{R}^n)$, $p \in [1, +\infty)$, then $u \equiv 0$.

ii) If $|Du| \in L^2(\mathbb{R}^n)$ then u is constant.

Hint: i) show that $|u|^p$ is subharmonic (using Jensen inequality).

Exercise 2. Show that every convex function $u : \mathbb{R}^n \to \mathbb{R}$ is subharmonic. Is it true that every subharmonic function is convex?

Show that if u is a convex superharmonic function then there exists $b \in \mathbb{R}^n, c \in \mathbb{R}$ such that $u(x) = b \cdot x + c$.

Hint: recall that if u is convex, then u is continuous. Moreover by convexity we get, for every $y \in \partial B(0, 1)$ and $r > 0$,

$$u(x) \leq \frac{1}{2}u(x + ry) + \frac{1}{2}u(x - ry). \tag{1}$$

Exercise 3. Let u_n be a sequence of nonnegative harmonic functions in Ω, open connected set in $\mathbb{R}^n$. Assume there exists $x_0 \in \Omega$ such that $\sum_n u_n(x_0)$ converges.

Show that $\sum_n u_n$ converges locally uniformly in Ω.

Exercise 4 (Continuous dependance)**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set, $f \in \mathcal{C}(\overline{\Omega})$, $g \in \mathcal{C}(\partial\Omega)$ and $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ the solution to

$$\begin{cases} -\Delta u = f(x) & x \in \Omega \\ u = g(x) & x \in \partial\Omega. \end{cases}$$

i) Show that there exists $C > 0$ such that

$$\sup_{\overline{\Omega}} |u| \leq C(\sup_{\overline{\Omega}} |f| + \sup_{\partial\Omega} |g|).$$

ii) Let $f_n \in \mathcal{C}(\Omega)$ and $g_n \in \mathcal{C}(\partial\Omega)$ two sequences which converge uniformly to respectively f and g and $u_n \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ the solutions to

$$\begin{cases} -\Delta u = f_n(x) & x \in \Omega \\ u = g_n(x) & x \in \partial\Omega \end{cases}.$$

Show that u_n converges uniformly to u in Ω.

Hint: Consider the functions $u(x) \pm \frac{|x|^2}{2n} \max_{\overline{\Omega}} |f| \mp \left[\sup_{\partial\Omega} |g| + \frac{(diam\Omega)^2}{2n} \max_{\overline{\Omega}} |f|\right]$.