Laplace and Poisson equation.

Exercise 1. i) Let $\beta \geq 0$. Find the solution $u_\beta \in \mathcal{C}^2(B(0,R)) \cap \mathcal{C}(\overline{B(0,R)})$ to the Dirichlet problem

$$\begin{cases} -\Delta u = |x|^\beta & |x| < R \\ u = 0 & |x| = R. \end{cases}$$

ii) Let $\Omega \in \mathbb{R}^n$ an open bounded set and $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ the solution to

$$\begin{cases} -\Delta u = 1 & x \in \Omega \\ u = 0 & x \in \partial\Omega. \end{cases}$$

Show that

$$u(x) \geq \frac{1}{2n}(\mathrm{dist}(x, \partial\Omega))^2.$$

Hint: i) Look for a radial solution $u(x) = f(|x|)$ with $f \in \mathcal{C}^2([0,R), f \in \mathcal{C}([0,R]))$.
ii) Observe that $u \geq 0$ in Ω. Let $x \in \Omega$ and $r =$dist$(x, \partial\Omega)$, consider the same problem but in the set $B(x,r) \subseteq \Omega$. Construct a solution for this new problem (using i with $\beta = 0$) and compare with u..

Exercise 2. Let $u : \mathbb{R}^n \to \mathbb{R}$ be harmonic and assume there exists $k \in \mathbf{N}$ and $M > 0$ such that

$$\frac{|u(x)|}{|x|^k + 1} \leq M \qquad \forall x \in \mathbb{R}^n.$$

Show that u is a polynomial of degree at most k.

Exercise 3. Let $\mathbb{R}^n_+$ be the halfspace $\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$ and $w \in \mathcal{C}(\overline{\mathbb{R}^n_+})$ harmonic in $\mathbb{R}^n_+$ such that $w = 0$ if $x_n = 0$ ($w = 0$ on $\partial\mathbb{R}^n_+$). Define the (antisymmetric) extension of w to $\mathbb{R}^n$ as follows:

$$\overline{w}(x) = \begin{cases} w(x_1, x_2, \dots, x_n) & \text{if } x_n \geq 0 \\ -w(x_1, x_2, \dots, -x_n) & \text{if } x_n \leq 0. \end{cases}$$

Show that $\overline{w}$ is harmonic in $\mathbb{R}^n$.

Exercise 4. We construct a solution to

$$(D) \begin{cases} -\Delta u = 0 & |x| < R \\ u = g(x) & |x| = R \end{cases}$$

(with $g \in \mathcal{C}$) without using Green functions.

i) Let g be a polynomial. So u solves (D) if and only if $v = u - g$ solves

$$(D') \begin{cases} -\Delta v = -\Delta g & |x| < R \\ v = 0 & |x| = R. \end{cases}$$

For every $k \in \mathbb{N}$ let

$$P_k = \{p : \mathbb{R}^n \to \mathbb{R} \mid p(x) \text{ is a polynomial of degree } \leq k\}$$

and the functional

$$T_k : P_k \to P_k \qquad T(p) = \Delta[(R^2 - |x|^2)p(x)].$$

Show that T is injective. Deduce that T is bijective.

Prove there exists a unique solution to (D').

ii) Show there exists a solution to (D) for every $g \in \mathcal{C}(\partial B)$.

Hint ii) Recall Stone-Weierstrass theorem.