Introduzione alle Equazioni alle Derivate Parziali, L.M. in Matematica, A.A. 2013-2014.

Poisson integral representation formula, Green functions and Perron method.

Exercise 1. Let $u \in \mathcal{C}^2(B(0,R)) \cap \mathcal{C}(\overline{B(0,R)})$ be a harmonic function in $B(0,R)$.

(1) **(Harnack inequality)** Assume $u \geq 0$. Prove that for every $x \in B(0,r) \subset B(0,R)$

$$\left(\frac{R}{R+r}\right)^{n-2} \frac{R-r}{R+r} u(0) \leq u(x) \leq \left(\frac{R}{R-r}\right)^{n-2} \frac{R+r}{R-r} u(0).$$

(2) **(Hopf lemma)** Assume $x_0 \in \partial B(0,R)$ be a maximum point of u. Let $n(x_0) = \frac{x_0}{R}$ be the external normal to B in x_0. Show that

$$\liminf_{h \to 0^+} \frac{u(x_0) - u(x_0 - hn(x_0)}{h} \geq \frac{u(x_0) - u(0)}{2^{n-1}R} > 0.$$

In particular, if $\frac{\partial u}{\partial n}$ exists in x_0,

$$\frac{\partial u}{\partial n}(x_0) > 0.$$

Hint (1): use integral Poisson formula and the spherical mean value property, and recall $|y| - |x| \leq |x-y| \leq |x| + |y|$.
(2): apply previous (right side) inequality to the positive function $v(x) = u(x_0) - u(x)$ and at point $x = x_0 - hn(x_0)$ (observe that $|x| = R - h$).

Exercise 2.

(1) Let $n \geq 3$. Construct a bounded subharmonic function $u \in \mathcal{C}(\mathbb{R}^n)$.

(2) Let $n = 2$. Show that every bounded subharmonic function $u \in \mathcal{C}(\mathbb{R}^2)$ is constant.

Hint . (1) consider $-|x|^{n-2}$ and recall that the maximum of subharmonic is subharmonic. (2) Consider $v(x) = u(x) - \log|x|$. Then $v \to -\infty$ as $|x| \to +\infty$ and $v = u$ on $|x| = 1$. Consider u in the ball $|x| < 1$ and study where the maxima of u are located. Consider v in the set $|x| > 1$ and study where the maxima of v have to be located.

Exercise 3. Consider the half space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n = x \cdot e_n > 0\}$.

(1) Let $u \in \mathcal{C}^2(\mathbb{R}^n_+) \cap \mathcal{C}(\overline{\mathbb{R}^n_+})$ be harmonic in $\mathbb{R}^n_+$, bounded and such that $u = 0$ on $\partial\mathbb{R}^n_+$. Show that $u \equiv 0$.

(2) Let $g \in \mathcal{C}(\partial\mathbb{R}^n_+) \cap L^\infty(\partial\mathbb{R}^n_+)$ and $u, v \in \mathcal{C}^2(\mathbb{R}^n_+) \cap \mathcal{C}(\overline{\mathbb{R}^n_+})$ be two bounded solutions to the Dirichlet problem

$$\begin{cases} -\Delta u = 0 & x \in \mathbb{R}^n_+ \\ u = g & x \in \partial\mathbb{R}^n_+. \end{cases}$$

Show that $u \equiv v$.

So in particular the previous Dirichlet problem has a unique bounded solution.

Hint (1) use the antisymmetric extension of u, see Foglio 4, ex 3. (2) Let $w = u - v$ and apply (1).

Exercise 4. Let g be a bounded function and Ω a bounded open set. Let $S^g = \{v \in \mathcal{C}(\overline{\Omega}) \mid v \text{ is superharmonic in } \Omega \text{ and } v \geq g \text{ on } \partial\Omega\}$. Show that

$$H^g(x) = \inf_{v \in S^g} v(x),$$

is well defined and harmonic in Ω.

Exercise 5. Let $\Omega = \mathbb{R}^n \setminus \overline{B(0,r)} = \{x \in \mathbb{R}^n \mid |x| > r\}$ and $g \in \mathcal{C}(\partial B(0,r))$.

(1) Show that there exists at most one solution $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ of the following Dirichlet problem

$$\begin{cases} -\Delta u = 0 & |x| > r \\ u(x) = g(x) & |x| = r \\ \lim_{|x| \to +\infty} u(x) = \gamma \end{cases}$$

where $\gamma \in \mathbb{R}$ is fixed.

(2) Assume $n \geq 3$. Let $R > r$ and consider the solution u_R to the following Dirichlet problem

$$\begin{cases} -\Delta u = 0 & r < |x| < R \\ u(x) = g(x) & |x| = r \\ u(x) = 0 & |x| = R. \end{cases}$$

Extend u_R to a continuous function in the whole Ω. Prove that u_R is an equibounded sequence and that converge locally uniformly, as $R \to +\infty$, to the solution of

$$\begin{cases} -\Delta u = 0 & |x| > r \\ u(x) = g(x) & |x| = r \\ \lim_{|x| \to +\infty} u(x) = 0 \end{cases}$$

Hint. (1): Let u, v two solutions, then $w = u - v$ is 0 on $|x| = r$ and is going to 0 as $|x| \to +\infty$, so by Weierstrass, either it is 0 or it has a non trivial maximum and minimum...