Introduzione alle Equazioni alle Derivate Parziali, L.M. in Matematica, A.A. 2013-2014.

Maximum principles for elliptic and parabolic operators.

Exercise 1 (Liouville theorem for uniformly elliptic operators)**.** Let

$$Lu(x) = -\text{tr } a(x)D^2u(x) + b(x) \cdot Du(x),$$

where $a(x)$ for every $x \in \mathbb{R}^n$ is a symmetric, positive definite matrix such that there exist $\lambda_0, \Lambda_0 > 0$ for which

$$\lambda_0|\xi|^2 \le \xi^t a(x)\xi \le \Lambda_0|\xi|^2 \qquad \forall x \in \mathbb{R}^n, \ \forall \xi \in \mathbb{R}^n.$$

Assume moreover that there exists $w \in \mathcal{C}^2(\mathbb{R}^n)$ such that

- $\lim_{|x|\to+\infty} w(x) = +\infty$,
- there exists $M > 0$ such that $Lw(x) \ge 0$ for every $|x| > M$.

Let $u \in \mathcal{C}^2(\mathbb{R}^n)$ a bounded from above function such that $Lu \le 0$ in $\mathbb{R}^n$.
Show that u is constant.
Hint. : For $\varepsilon > 0$, define $u_\varepsilon(x) = u(x) - \varepsilon w(x)$. Then $Lu_\varepsilon \le 0$ in $|x| > M$ and $\lim_{|x|\to+\infty} u_\varepsilon = -\infty$. Then, applying weak max principle (in which set?) we get $u_\varepsilon(x) \le \max_{|y|=M} u_\varepsilon(y)$ for all $|x| \ge M$. This is true for every $\varepsilon > 0$, so also when $\varepsilon \to 0$. Moreover by weak max principle $u(x) \le \max_{|y|=M} u(y)$ for all $|x| \le M$. Conclude by strong max principle.

Exercise 2. Let $u \in \mathcal{C}^2(\mathbb{R}^n)$ such that

- there exists $C \in \mathbb{R}$ such that $u(x) \le C$ for every $x \in \mathbb{R}^n$,
- $-\Delta u(x) + x \cdot Du(x) \le 0$ for every $x \in \mathbb{R}^n$.

Show that u is constant.
Hint: look for a function w such that $-\Delta w + x \cdot Dw \ge 0$ for $|x|$ sufficiently large and apply Liouville theorem (the previous exercise). E.g consider $w(x) = |x|^2$.

Exercise 3. Let $Lu = -\operatorname{tr} a(x)D^2u(x) + b(x) \cdot Du(x)$ a uniformly elliptic operator and Ω a bounded connected open set of class C^2. Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ such that $Lu = 0$ in Ω.

(i) Let $\partial\Omega = S_1 \cup S_2$, with $S_1 \neq \emptyset$. If

$$u = 0 \quad x \in S_1, \qquad \frac{\partial u}{\partial n} = 0 \quad x \in S_2$$

then $u \equiv 0$.

(ii) Let $\gamma(x) \in \mathcal{C}(\partial\Omega, \mathbb{R}^n)$ such that $\gamma(x) \cdot n(x) > 0$ for every $x \in \partial\Omega$, and $\alpha(x) > 0$. If

$$\alpha(x)u(x) + Du(x) \cdot \gamma(x) = 0 \quad x \in \partial\Omega$$

then $u \equiv 0$.

Hint Apply strong and weak maximum principle.

Exercise 4. Let $u \in \mathcal{C}^{2,1}((0,1) \times (0,+\infty)) \cap \mathcal{C}[0,1] \times [0,+\infty)$ be the solution of

$$\begin{cases} u_t - u_{xx} = 0 & (0,1) \times (0,+\infty) \\ u(x,0) = x(1-x) & x \in (0,1) \\ u(0,t) = u(1,t) = 0 & t > 0. \end{cases}$$

- Find $\lambda \in \mathbb{R}$ such that
$$u(x,t) \leq x(1-x)e^{\lambda t}.$$
- Show that $\lim_{t \to +\infty} u(x,t) = 0$ uniformly for $x \in [0,1]$.

Hint: use weak maximum principle..

Exercise 5. Consider the Cauchy-Dirichlet problem in $B(0,R) \times (0,+\infty) \subseteq \mathbb{R}^4$

$$(C) \begin{cases} u_t - \Delta u = 1 & x \in B(0,R)\ t > 0 \\ u(x,0) = 0 & x \in B(0,R) \\ u(x,t) = 0 & x \in \partial B(0,R),\ t > 0. \end{cases}$$

(1) Find the stationary solution to (C), i.e. the solution to the Dirichlet problem
$$(D) \begin{cases} -\Delta v = 1 & x \in B(0,R) \\ v(x) = 0 & x \in \partial B(0,R). \end{cases}$$
(2) Show that the solution $u(x,t)$ of (C) converge uniformly in $B(0,R)$ to $v(x)$ solution of (D) for $t \to +\infty$.

Hint: find $\beta \in \mathbb{R}$ (using weak maximum principle for parabolic operators) such that
$$v(x)(1 - e^{-\beta t}) \leq u(x,t) \leq v(x) \qquad \forall t \geq 0, x \in \overline{B(0,1)}$$