Maximum principles for elliptic and parabolic operators 2.

Exercise 1. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $Lu(x,t) = -\operatorname{tr} a(x,t)D_x^2 u + b(x,t) \cdot D_x u$, $x \in \Omega$, $t > 0$, with $a(x,t)$ a symmetric $n \times n$ positive semidefinite matrix. Let $c(x,t)$ be a bounded positive function, such that $c(x,t) \geq \gamma > 0$ for every x, t. Let $u \in \mathcal{C}^{2,1}(\Omega \times (0,+\infty)) \cap \mathcal{C}(\overline{\Omega} \times [0,+\infty))$ be the solution to

$$(C) \begin{cases} u_t + Lu + c(x,t)u = 0 & (x,t) \in \Omega \times (0,+\infty) \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,+\infty) \\ u(x,0) = u_0(x) & x \in \overline{\Omega}. \end{cases}$$

Show that there exists $C > 0$ such that $|u(x,t)| \leq Ce^{-\gamma t}$ for all $t \geq 0$ and all $x \in \overline{\Omega}$.

Exercise 2. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $f \in \mathcal{C}^2(\overline{\Omega})$, $g \in \mathcal{C}(\partial\Omega$. Let $v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ be a subsolution to

$$(D) \begin{cases} -\Delta v \leq f(x) & x \in \Omega \\ v(x) \leq g(x) & x \in \overline{\Omega}. \end{cases}$$

Let $u \in \mathcal{C}^{2,1}(\Omega \times (0,+\infty)) \cap \mathcal{C}(\overline{\Omega} \times [0,+\infty))$ be the solution to

$$(C) \begin{cases} u_t - \Delta u = f(x) & (x,t) \in \Omega \times (0,+\infty) \\ u(x,t) = g(x) & (x,t) \in \partial\Omega \times (0,+\infty) \\ u(x,0) = v(x) & x \in \overline{\Omega}. \end{cases}$$

Show that the funtion $t \mapsto u(x,t)$ is increasing for every $x \in \overline{\Omega}$.

Hint. Consider $w(x,t) = u(x,t) - v(x)$. Then $w_t - \Delta w \geq 0$, $w(x,0) = 0$, and if $x \in \partial\Omega$, $w(x,t) \geq g(x) - g(x) = 0$. So, by minimum principle, $w \geq 0$ everywhere, $u(x,t) \geq v(x)$ for every $t \geq 0$.
Fix $h > 0$ and consider $w^h(x,t) = u(x,t+h) - u(x,t)$. Then $w_t^h - \Delta w^h = 0$, $w^h(x,0) = u(x,h) - v(x) \geq 0$ (by previous argument), and if $x \in \partial\Omega$, $w^h(x,t) = g(x) - g(x)$. So, by minimum principle, $w^h \geq 0$ everywhere.

Exercise 3. Let $F : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$ function, with F' bounded (so F is Lipschitz). Consider the semilinear equation

$$u_t - \Delta u = F(u) \qquad (x,t) \in \mathbb{R}^n \times (0,+\infty).$$

i) Let $u, v \in \mathcal{C}^{2,1}(\mathbb{R}^n \times (0,+\infty)) \cap \mathcal{C}(\mathbb{R}^n \times [0,+\infty))$ 2 solutions to the problem such that for all $T > 0$ there exist $C_T, \alpha_T > 0$ such that

$$|u(x,t)| + |v(x,t)| \leq C_T e^{\alpha_T |x|^2} \quad \forall t \in [0,T].$$

Show that if $u(x,0) \leq v(x,0)$ then $u(x,t) \leq v(x,t)$ for all $t > 0$.

ii) Consider the Cauchy problem

$$(C)\begin{cases} u_t - \Delta u = u(1-u)(u-\frac{1}{2}) & (x,t) \in \mathbb{R}^n \times (0,+\infty) \\ u(x,0) = u_0(x) & x \in \mathbb{R}^n. \end{cases}$$

Assume that $0 \leq u_0(x) \leq 1$. Show that the unique solution (with exponential growth) satisfies

$$0 \leq u(x,t) \leq 1 \qquad \forall x \in \overline{\Omega}, t \geq 0.$$

Compute $\lim_{t\to+\infty} u(x,t)$ when $\|u_0\|_\infty < \frac{1}{2}$.