Introduzione alle Equazioni alle Derivate Parziali, L.M. in Matematica, A.A. 2013-2014.

Heat equation.

Exercise 1. Let $c > 0$ and $u_0 \in \mathcal{C}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Provide the representation formula for the solution u to the Cauchy problem

$$(C)\begin{cases} u_t - \Delta u + cu = 0 & \mathbb{R}^n \times (0, +\infty) \\ u(x,0) = u_0(x) & x \in \mathbb{R}^n. \end{cases}$$

Compute $\lim_{t\to+\infty} u(x,t)$, if it exists, and write if the convergence is uniform.

Exercise 2 (Dissipation). Let $u_0 \in L^1(\mathbb{R}^n)$, and let $u(x,t) = u_0 * \Phi$.

Show that $u(\cdot,t) \in L^p(\mathbb{R}^n)$ for all $t > 0$ and for all $p \in [1, +\infty]$.

Prove that for every p that there exists a constant C_p depending on p, n such that

$$\|u(\cdot,t)\|_{L^p} \le \frac{C_p \|u_0\|_{L^1}}{t^{\frac{n}{2}\left(1-\frac{1}{p}\right)}}.$$

Hint: by Young inequality $\|u(\cdot,t)\|_{L^p} \le \|u_0\|_{L^1}\ \|\Phi(\cdot,t)\|_{L^p}$. So, it remains to compute $\|\Phi(\cdot,t)\|_{L^p}$.

Exercise 3. Let $u_0 \in \mathcal{C}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $\lim_{z\to+\infty} u_0(z) = a \in \mathbb{R}$ and $\lim_{z\to-\infty} u_0(z) = b \in \mathbb{R}$. Let u the solution to the Cauchy problem

$$(C)\begin{cases} u_t - u_{xx} = 0 & \mathbb{R} \times (0, +\infty) \\ u(x,0) = u_0(x) & x \in \mathbb{R}. \end{cases}$$

Compute $\lim_{t\to+\infty} u(x,t)$, if it exists, and write if the convergence is uniform.

Exercise 4. Let $u_0 \in L^2(\mathbb{R}^n)$, and let $u(x,t) = u_0 * \Phi \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,+\infty))$ the solution to

$$\begin{cases} u_t - \Delta u = 0 & x \in \mathbb{R}^n, t > 0 \\ \lim_{t\to 0^+} \|u(\cdot,t) - u_0(\cdot)\|_{L^2(\mathbb{R}^n)} = 0. \end{cases}.$$

Define for every $t > 0$ the energy

$$E(t) = \int_{\mathbb{R}^n} |u(x,t)|^2 dx.$$

Show that $E'(t) = -2\int_{\mathbb{R}^n} |Du|^2 dx < 0$ for all $t > 0$.

Deduce that $\|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \le \|u_0\|_{L^2(\mathbb{R}^n)}$.

Exercise 5. Let Ω be a bounded open set of class $\mathcal{C}^1$ and let $u \in \mathcal{C}^{2,1}(\Omega \times (0,+\infty)) \cap \mathcal{C}^{1,0}(\overline{\Omega} \times [0,+\infty))$ a solution to the Cauchy Neumann problem

$$(CN)\begin{cases} u_t - \Delta u = 0 & \Omega \times (0,+\infty) \\ \frac{\partial u}{\partial n}(x,t) = 0 & \partial\Omega \times (0,+\infty) \\ u(x,0) = u_0(x) & \Omega \end{cases}$$

with $u_0 \in \mathcal{C}(\overline{\Omega})$.

We define the thermic energy in Ω at time t as

$$E(t) = \int_\Omega u^2(x,t)dx, \qquad t \geq 0.$$

i) Show that $E'(t) \leq 0$ for $t \in (0,T)$.

ii) Using (i), prove that the Cauchy Neumann problem

$$\begin{cases} u_t - \Delta u = f(x,t) & \Omega \times (0,+\infty) \\ \frac{\partial u}{\partial n}(x,t) = g(x,t) & \partial\Omega \times (0,+\infty) \\ u(x,0) = u_0(x) & \Omega \end{cases}$$

admits at most one solution $u \in \mathcal{C}^{2,1}(\Omega \times (0,+\infty)) \cap \mathcal{C}^{1,0}(\overline{\Omega} \times [0,+\infty))$.

Exercise 6. Let $u_0 \in \mathcal{C}(\mathbb{R}^n)$ such that $u_0(x) \geq -K$ for all $x \in \mathbb{R}^n$. Consider the quasilinear problem

$$(Q) \begin{cases} u_t - \Delta u + |Du|^2 = 0 & x \in \mathbb{R}^n, t > 0 \\ u(x,0) = u_0(x) & x \in \mathbb{R}^n \end{cases}$$

where $|Du|^2 = \sum_i |u_{x_i}|^2$.

i) Let $u \in \mathcal{C}^{2,1}(\mathbb{R}^n \times (0,+\infty) \cap \mathcal{C}(\mathbb{R}^n \times [0,+\infty)$ be a solution of the problem. Define $v(x,t) = e^{-u(x,t)}$. Determine which is the Cauchy problem (C) solved by v.

ii) Compute the unique bounded solution of (C). Show that this solution is positive everywhere.

iii) Show that (Q) admits at most one bounded solution and provide a representation formula for this solution.