Introduzione alle Equazioni alle Derivate Parziali, L.M. in Matematica, A.A. 2013-2014.

Wave equation.

Exercise 1 (D'Alembert formula)**.** Consider the Cauchy problem in $\mathbb{R}$

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & (x,t) \in \mathbb{R} \times (0,+\infty) \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \\ u_t(x,0) = v_0(x) & x \in \mathbb{R}, \end{cases}$$

con $u_0 \in \mathcal{C}^2(\mathbb{R}), v_0 \in \mathcal{C}^1(\mathbb{R})$.

(i) Let

$$v(x,t) = u_t(x,t) - c u_x(x,t).$$

Ahow that v solves the homogeneous transport equation

$$v_t(x,t) + c v_x(x,t) = 0 \qquad (x,t) \in \mathbb{R} \times (0,+\infty)$$

with initial data $v(x,0) = v_0(x) - c u_0'(x)$.

Provide a representation formula for v.

(ii) Show that u solves the non homogeneous transport eqaution

$$u_t(x,t) - c u_x(x,t) = v(x,t) \qquad (x,t)\mathbb{R} \times (0,+\infty)$$

with initial data $u(x,0) = u_0(x)$, where v has been obtained in (i). Provide the representation formula for u.

Exercise 2. Consider the partial differential equation

$$(E) \qquad u_{tt} + u_{tx} - 2u_{xx} = 0 \qquad (x,t) \in \mathbb{R} \times (0,T).$$

Is the operator in (E) elliptic, parabolic or hyperbolic?
Determine the general form of the solutions to equation (E).

Exercise 3 (Equipartition of energy)**.** Let $u \in \mathcal{C}^2$ be the solution to the Cauchy problem

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & (x,t) \in \mathbb{R} \times (0,+\infty) \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \\ u_t(x,0) = v_0(x) & x \in \mathbb{R}. \end{cases}$$

Assume that u_0, v_0 are smooth functions with compact support contained in $(-k,k)$, $k > 0$. Define the kinetic energy as

$$k(t) = \frac{1}{2} \int_{\mathbb{R}} u_t^2(x,t)dx$$

and the potential energy as

$$p(t) = \frac{c^2}{2} \int_{\mathbb{R}} u_x^2(x,t)dx.$$

(i) Show that $u(x,t) = 0$ for $(x,t) \in \{|x| > k + ct\}$.
(ii) Show that $k(t) + p(t)$ is constant for all t.
(iii) Show that there exists $T > 0$ such that $k(t) = p(t)$ for $t \geq T$.

Hint for iii): compute explicitly $u_t^2 - c^2 u_x^2$ (using d'Alember formula) and show $u_t^2 - c^2 u_x^2 = 0$ for $x \in \mathbb{R}$ and $t > k/c$.

Exercise 4. Let $u \in C^2(\mathbb{R}^3 \times [0, +\infty)$ the solution to

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 & (x,t) \in \mathbb{R}^3 \times (0, +\infty) \\ u(x,0) = u_0(x) & x \in \mathbb{R}^3, \\ u_t(x,0) = v_0(x) & x \in \mathbb{R}^3, \end{cases}$$

with u_0, v_0 smooth functions with compact support contained in $B(0,r)$.

Show that for all fixed t the map

$$x \longrightarrow u(x,t)$$

has compact support contained in the annulus

$$(t-r)^+ \leq |x| \leq r + t$$

where $(t-r)^+ = \max(0, t-r)$.

Hint: use Kirchhoff formula.

Exercise 5. Let $n = 3$ and $\phi : \mathbb{R} \to \mathbb{R}$ a givem smooth function.

Prove that there exist smooth functions $\alpha : (0, +\infty) \to \mathbb{R}$, $\alpha \not\equiv 0$, and $\beta : (0, +\infty) \to [0, +\infty)$, with $\beta(0) = 0$, independent of ϕ, such that

$$u(x,t) = \alpha(|x|)\phi(t - \beta(|x|))$$

is a spherical wave (a solution of the spherical wave equation) in $\mathbb{R}^3 \setminus \{0\} \times \mathbb{R}$.

Show that such α, β do not exist for $n = 2$ and $n > 3$.