ADVANCED MATHEMATICS FOR STATISTICS- A.A. 2020-2021

FUNCTIONAL ANALYSIS- TEACHER A. CESARONI

EXAM- NOVEMBER 25, 2020- 11-12.30 (90 MINUTES)

Exercise 1.

(1) Give the definition of absolutely continuous measure and of singular measure (with respect to the Lebesgue measure in $\mathbb{R}$).
(2) Write the characterization of σ-finite absolutely continuous measure (in terms of functions f...)
(3) Consider the increasing continuous. function

$$F(x) = \begin{cases} 1 & x \geq 0 \\ e^x & x < 0 \end{cases}$$

and let μ_F the Borel measure associated to this function.

(a) Show that this measure is finite and compute $\mu_F(\mathbb{R})$.
(b) Compute $\mu_F(0,1)$, $\mu_F(-3,-2)$ and $\mu_F(-1,2)$.
(c) Find, if it exists, a measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $\mu_F(a,b) = \int_a^b f(x)dx$.
(d) Is μ_F absolutely continuous with respect to Lebesgue? is it singular?

Exercise 2.

(1) State the Holder inequality.
(2) Prove that if $f \in L^2(0,2)$ then $f \in L^1(0,2)$ and moreover

$$\|f\|_1 \leq \sqrt{2}\|f\|_2.$$

(3) Prove that if $f \in L^1(\mathbb{R})$ then

$$\mathcal{L}\left(x \text{ s.t. } |f(x)| \geq t\right) \leq \frac{\|f\|_1}{t}.$$

Sketch of solutions

Solution 1.

(1) μ is absolutely continuous with respect to Lebesgue $\mathcal{L}$ if for every Borel set A such that $\mathcal{L}(A) = 0$ there holds that also $\mu(A) = 0$.
μ is singular with respect to Lebesgue $\mathcal{L}$ if there exist A, B Borel sets such that $\mathbb{R} = A \cup B$, $A \cap B = \emptyset$, and $\mathcal{L}(A) = 0 = \mu(B)$.

(2) A σ-finite measure μ is absolutely continuous with respect to Lebesgue $\mathcal{L}$ if and only if there exists a measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \geq 0$, $\int_{-M}^{M} f(x)dx < +\infty$ for every $M > 0$ and $\mu(a, b) = \int_a^b f(x)dx$.

(3) (a) $\mu_F(\mathbb{R}) = \sup F - \inf F = 1$.
(b) Note that F is continuous, therefore $\mu_F(a, b) = \mu_F(a, b]$. Therefore $\mu_F(0, 1) = F(1) - F(0) = 1 - 1 = 0$, $\mu_F(-3, -2) = F(-2) - F(-3) = e^{-2} - e^{-3}$ and $\mu_F(-1, 2) = F(2) - F(-1) = 1 - e^{-2} = F(0) - F(-2) = \mu_F(-2, 0)$.
(c) Note that $\mu_F(a, b) = 0$ if $b \geq a \geq 0$. Therefore $f(x) = 0$ if $x > 0$. On the other hand, if $a < b \leq 0$, $\mu_F(a, b) = e^b - e^a = \int_a^b e^x dx$. Therefore

$$f(x) = \begin{cases} e^x & x < 0 \\ 0 & x > 0 \end{cases}.$$

Note that $f \geq 0$ and $f \in L^1(\mathbb{R})$.
(d) By the previous point and characterization of absolutely continuous measure, μ_F is absolutely continuous with respect to Lebesgue.

Solution 2.

(1) Let $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$, where $q = \frac{p}{p-1}$ (and $q = +\infty$ if $p = 1$). Then $fg \in L^1(\mathbb{R})$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$.

(2) Note that $\chi_{(0,2)} \in L^2(\mathbb{R})$ and $\|\chi_{(0,2)}\|_2 = \sqrt{2}$. By assumption $f\chi_{(0,2)} \in L^2(\mathbb{R})$, therefore by Holder inequality $f\chi_{(0,2)} \in L^1(\mathbb{R})$, that is $f \in L^1(0, 2)$ and moreover

$$\|f\chi_{(0,2)}\|_1 \leq \|f\chi_{(0,2)}\|_2 \|\chi_{(0,2)}\|_2 = \|f\chi_{(0,2)}\|_2 \sqrt{2}.$$

(3) By definition and monotonicity of the integral

$$\|f\|_1 = \int_{\mathbb{R}} |f(x)|dx \geq \int_{(x \text{ s.t. } |f(x)| \geq t)} |f(x)|dx \geq \int_{(x \text{ s.t. } |f(x)| \geq t)} t dx = t\mathcal{L}\left(x \text{ s.t. } |f(x)| \geq t\right).$$