Introduzione alle equazioni alle derivate parziali,
Laurea Magistrale in Matematica, A.A. 2013/2014

Some comments and remarks on the heat equation

This note is just an integration. The main reference is Evans, Partial Differential Equations, chapter 2, section 3.

We define the heat kernel as the function

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & x \in \mathbb{R}^n, \ t > 0 \\ 0 & x \in \mathbb{R}^n, \ t \leq 0. \end{cases}$$

To determine the fundamental solution to the heat equation, we used Fourier transform method (see Evans, chapter 4, section 3, example 2- pag.192).

We enumerate the main properties of the heat kernel:

1. Φ is singular at $(0,0)$,
2. Φ is radial in the x variable and $\lambda^n \Phi(\lambda x, \lambda^2 t) = \Phi(x,t)$.
3. $\Phi(\cdot,t) \in L^p(\mathbb{R}^n)$ for every $p \in [1,+\infty]$. For every $t > 0$, for every $x \in \mathbb{R}^n$,
$$\|\Phi(\cdot,t)\|_1 = \int_{\mathbb{R}^n} \Phi(x-y,t)dy = 1.$$
Moreover
$$\|\Phi(\cdot,t)\|_p = \frac{1}{p^{\frac{n}{2p}} (4t)^{\frac{n}{2}(1-\frac{1}{p})}}.$$
4. $\Phi \in \mathcal{C}^\infty(\mathbb{R}^{n+1} \setminus (0,0)$, and all the derivatives of Φ (of all orders) are uniformly bounded in every set $\mathbb{R}^n \times [\delta,+\infty)$ for every $\delta > 0$.
5. $\Phi_t - \Delta\Phi = 0$ for all $x \in \mathbb{R}^n$, $t \neq 0$ (it is a direct computation).
6. Φ solves (in distributional sense) the Cauchy problem
$$\begin{cases} \Phi_t - \Delta\Phi = 0 & x \in \mathbb{R}^n \ t > 0 \\ \Phi = \delta_0 & x \in \mathbb{R}^n \ t = 0. \end{cases}$$

Heat equation with initial data in $L^1(\mathbb{R}^n)$.

Let $u_0 \in L^1(\mathbb{R}^n)$. Define the function

$$u(x,t) = \int_{\mathbb{R}^n} u_0(y) \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} dy = \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} u_0(x + \sqrt{4t}z) e^{-|z|^2} dz. \tag{1}$$

Theorem 1. *1. The function $u(x,t) \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,+\infty))$ (this is called* **istantaneous regularization***) and solves*

$$\begin{cases} u_t - \Delta u = 0 & x \in \mathbb{R}^n \ t > 0 \\ \lim_{t\to 0} \|u(\cdot,t) - u_0(\cdot)\|_1 = 0. \end{cases}$$

2. $u(\cdot,t) \in L^p(\mathbb{R}^n)$ for every $p \in [1,+\infty]$, *3.*

$$\int_{\mathbb{R}^n} u(x,t) = \int_{\mathbb{R}^n} u_0(x), \qquad \forall t > 0$$

this property is called **mass conservation** *(in particular if $u_0 \geq 0$ the L^1 norm is conserved*[1]*)*

4.

$$\lim_{t\to+\infty} \|u(\cdot,t)\|_p = 0 \qquad \forall p \in (1,+\infty].$$

This property is called **dissipation**.

Proof. 1. $u \in \mathcal{C}^\infty$ because of the properties of heat kernel and convolution. Moreover, again by the properties of heat kernel, it is a solution to the heat equation for $t > 0$. We prove that is assumes in L^1 sense the initial datum. Since $\int \Phi(x-y,t)dy = 1$, we get

$$\int_{\mathbb{R}^n} |u(x,t) - u_0(x)|dx \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u_0(y) - u_0(x)| \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} dydx \leq$$

$$\leq \int_{\mathbb{R}^n}\int_{B(x,r)} |u_0(y)-u_0(x)| \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} dydx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus B(x,r)} |u_0(y)-u_0(x)| \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} dydx \quad (2)$$

where $r > 0$. We estimate the first integral in (2) as follows:

$$\int_{\mathbb{R}^n}\int_{B(x,r)} |u_0(x+\sqrt{4t}z) - u_0(x)| e^{-\frac{|x-y|^2}{4t}} dydx \quad \leq$$

$$\int_{\mathbb{R}^n}\int_{B(x,r)} \sup_{h\in\mathbb{R}^n, |h|\leq r} |u_0(x+h) - u_0(x)| \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} dydx \quad \leq$$

$$\leq \int_{\mathbb{R}^n} \sup_{h\in\mathbb{R}^n,|h|\leq r} |u_0(x+h) - u_0(x)|dx = \sup_{h\in\mathbb{R}^n,|h|\leq r} \|u_0(\cdot+h) - u_0(\cdot)\|_1 \to 0 \qquad \text{as } r \to 0. \quad (3)$$

where the last limit is a general fact in L^p spaces: if $f \in L^p$, then

$$\lim_{|h|\to 0} \int_{\mathbb{R}^n} |f(x+h) - f(x)|^p dx = 0$$

(see Brezis, Analisi Funzionale, cor. IV.28).

So for every $\varepsilon > 0$, there exists r_ε such that for all $r \in (0, r_\varepsilon)$,

$$\int_{\mathbb{R}^n}\int_{B(x,r)} |u_0(x+\sqrt{4t}z) - u_0(x)| e^{-\frac{|x-y|^2}{4t}} dydx \leq \varepsilon \qquad \forall r \leq r_\varepsilon. \quad (4)$$

As for the second integral in (2), we get, with the change of variable $z = \frac{y-x}{\sqrt{4t}}$, and using Fubini-Tonelli,

$$\begin{aligned} & \textstyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus B(x,r)} |u_0(y)-u_0(x)| \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} dydx \\ \leq \quad & \textstyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus B(0,\frac{r}{\sqrt{4t}})} |u_0(x+\sqrt{4t}z) - u_0(x)| e^{-|z|^2} dzdx \\ = \quad & \textstyle\int_{\mathbb{R}^n\setminus B(0,\frac{r}{\sqrt{4t}})} e^{-|z|^2} \int_{\mathbb{R}^n} |u_0(x+\sqrt{4t}z) - u_0(x)| dxdz \quad \leq 2\|u_0\|_1 \displaystyle\int_{\mathbb{R}^n\setminus B(0,\frac{r}{\sqrt{4t}})} e^{-|z|^2} dz \to 0 \quad (5) \end{aligned}$$

as $t \to 0$.

So for every ε there exists t_ε such that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus B(x,r_\varepsilon)} |u_0(y)-u_0(x)| \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} dydx \leq \varepsilon \qquad \forall t \leq t_\varepsilon. \quad (6)$$

[1] in general the L^1 norm is not always conserved! differently as I said during the lesson..

So putting togheter (4) and (6), we obtain that for every ε there exists t_ε such that

$$\int_{\mathbb{R}^n} |u(x,t) - u_0(x)|dx \leq 2\varepsilon \qquad \forall t \leq t_\varepsilon$$

which gives the desired conclusion.

2. This follows from properties of convolutions, since $\Phi(\cdot, t) \in L^p(\mathbb{R}^n)$ for all p (see Brezis, Analisi Funzionale, thm IV.15).

3. By convolution properties, we get that for all $t > 0$, $u_0(y)\Phi(x-y,t) \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$. Then by Fubini Tonelli theorem

$$\int_{\mathbb{R}^n} u(x,t)dx = \int_{\mathbb{R}^n} u_0(y) \int_{\mathbb{R}^n} \Phi(x-y,t)dxdy = \int_{\mathbb{R}^n} u_0(y)dy.$$

The same fact can be proved bu showing that $\frac{d}{dt}\int_{\mathbb{R}^n} u(x,t)dx = 0$.

4. Form Young inequality and properties of the heat kernel, we get that

$$\|u(\cdot,t)\|_p \leq \|u_0\|_1 \|\Phi(\cdot,t)\|_p \leq \|u_0\|_1 \frac{1}{p^{\frac{n}{2p}}(4\pi t)^{\frac{n}{2}(1-\frac{1}{p})}} \to 0$$

as $t \to +\infty$ if $p > 1$. Moreover

$$\|u(\cdot,t)\|_\infty \leq \|u_0\|_1 \|\Phi(\cdot,t)\|_\infty \leq \|u_0\|_1 \frac{1}{(4\pi t)^{\frac{n}{2}}} \to 0.$$

□

Remark. If $u_0 \notin L^1$ we cannot expect dissipation. If $u_0 \in \mathcal{C}(\mathbb{R})$ and $\lim_{x\to+\infty} u_0(x) = U^+$ and $\lim_{x\to-\infty} u_0(x) = U^-$, then

$$\lim_{t\to+\infty} u(x,t) = \frac{U^+ + U^-}{2}$$

locally uniformly in x.

Heat equation with initial data in $L^p(\mathbb{R}^n)$.

Let $u_0 \in L^p(\mathbb{R}^n)$. Let u as defined in (1).

Theorem 2. *1. The function $u(x,t) \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,+\infty))$ (this is called* **istantaneous regularization***) and solves*

$$\begin{cases} u_t - \Delta u = 0 & x \in \mathbb{R}^n \ t > 0 \\ \lim_{t\to 0} \|u(\cdot,t) - u_0(\cdot)\|_p = 0. \end{cases}$$

2. $u(\cdot,t) \in L^p(\mathbb{R}^n)$, and $\|u(\cdot,t)\|_p \leq \|u_0(\cdot)\|_p$.

Proof. 1.The arguments is the same as in Theorem 1. The only little changement is in the proof that u is assumes in L^p sense the initial datum. By Jensen inequality, since $\int \Phi(x-y,t)dx = 1$,

$$\int_{\mathbb{R}^n} |u(x,t) - u_0(x)|^p dx \leq$$

$$\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} |u_0(y) - u_0(x)| \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} dy \right|^p dx \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u_0(y) - u_0(x)|^p \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} dydx \leq$$

$$\leq \int_{\mathbb{R}^n} \int_{B(x,r)} |u_0(y) - u_0(x)|^p \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} dydx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B(x,r)} |u_0(y) - u_0(x)|^p \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} dydx \quad (7)$$

where $r > 0$. We estimate the first integral in (7) as follows:

$$\int_{\mathbb{R}^n}\int_{B(x,r)} |u_0(x+\sqrt{4t}z) - u_0(x)|^p e^{-\frac{|x-y|^2}{4t}} dydx \leq$$

$$\int_{\mathbb{R}^n}\int_{B(x,r)} \sup_{h\in\mathbb{R}^n, |h|\leq r} |u_0(x+h) - u_0(x)|^p \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} dydx \quad \leq$$

$$\leq \int_{\mathbb{R}^n} \sup_{h\in\mathbb{R}^n, |h|\leq r} |u_0(x+h) - u_0(x)|^p dx = \sup_{h\in\mathbb{R}^n, |h|\leq r} \|u_0(\cdot + h) - u_0(\cdot)\|_p^p \to 0 \qquad \text{as } r \to 0. \tag{8}$$

As for the second integral in (2), we get, with the change of variable $z = \frac{y-x}{\sqrt{4t}}$, and using Fubini-Tonelli,

$$\begin{aligned}
&\textstyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus B(x,r)} |u_0(y) - u_0(x)|^p \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} dydx \\
\leq \; &\textstyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus B(0,\frac{r}{\sqrt{4t}})} |u_0(x+\sqrt{4t}z) - u_0(x)|^p e^{-|z|^2} dzdx \\
= \; &\textstyle\int_{\mathbb{R}^n\setminus B(0,\frac{r}{\sqrt{4t}})} e^{-|z|^2} \int_{\mathbb{R}^n} |u_0(x+\sqrt{4t}z) - u_0(x)| dxdz \quad \leq 2\|u_0\|_p \displaystyle\int_{\mathbb{R}^n\setminus B(0,\frac{r}{\sqrt{4t}})} e^{-|z|^2} dz \to 0
\end{aligned}$$

as $t \to 0$. We conclude as in the proof of Theorem 1.

2. It follows from the properties of convolutions (Young inequality) and of heat kernel. $\square$

Heat equation with bounded and continuous initial data.

Let $u_0 \in L^\infty(\mathbb{R}^n) \cap \mathcal{C}(\mathbb{R}^n)$. Let u as defined in (1).

First of all we observe that the heat equation forces infinite speed of propagation for disturbances.

Remark. Let $u_0 \geq 0$ and with compact support. Then $u(x,t) > 0$ for all $x \in \mathbb{R}^n$. In particular $u(\cdot, t)$ has NOT compact support, but for all $t > 0$ is strictly positive everywhere. This property is called **infinite speed of propagation**.

This means that if the initial temperature is nonnegative and positive somewhere, then the temperature at any later time (no matter how small) is everywhere positive.

Theorem 3. *Let $u_0 \in L^\infty(\mathbb{R}^n) \cap \mathcal{C}(\mathbb{R}^n)$ and let u defined in (1).*

1. $u \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,+\infty)) \cap \mathcal{C}(\mathbb{R}^n \times [0,+\infty))$ *and solves the Cauchy problem*

$$\begin{cases} u_t - \Delta u = 0 & x \in \mathbb{R}^n \; t > 0 \\ u(x,0) = u_0(x). \end{cases}$$

2. *There exists $C = C(n)$ such that*

$$\sup_{x\in\mathbb{R}^n} |D_x u(x,t)| \leq \frac{C}{\sqrt{t}} \|u_0\|_\infty$$

for all $t > 0$.

3. *Assume that $u_0 \in \mathcal{C}^{0,\alpha}(\mathbb{R}^n)$ for some $\alpha \in (0,1]$* [2]. *Then for every $R > 0$ and $T > 0$ there exists a constant $C = C(n, R, T, \alpha, \|u_0\|_\infty)$ such that*

$$\sup_{x\in B(0,2R)} |u_t(x,t)| + |\Delta u(x,t)| \leq \frac{C}{t^{1-\frac{\alpha}{2}}} \qquad \forall t \in (0,T).$$

[2] This means that for every compact $K \subset \mathbb{R}^n$ there exists a constant C_k such that $|u_0(x) - u_0(y)| \leq C_k|x-y|^\alpha$ for all $x, y \in K$

Proof. For the proof of 1 see Evans, chapter 2, section 3, theorem 1 (pag 47-48).

2.

$$|D_x u(x,t)| \leq \|u_0\|_\infty \int_{\mathbb{R}^n} \frac{|x-y|}{2t} \Phi(x-y,t)dy \leq$$

with the change of variable $z = \frac{x-y}{\sqrt{4t}}$,

$$\leq \frac{\|u_0\|_\infty}{\pi^{n/2}\sqrt{t}} \int_{\mathbb{R}^n} |z| e^{-|z|^2} dz.$$

3. Note that since $u_t = \Delta u$, it is sufficient to prove the estimate for $|u_{x_i x_j}(x,t)|$, $i,j \in \{1,\dots,n\}$. Observe that since $\int_{\mathbb{R}}^n \Phi(x-y,t)dy = 1$, then $\int_{\mathbb{R}}^n \Phi_{x_i x_j}(x-y,t)dy = 0$and so for every $x \in \mathbb{R}^n$,$\int_{\mathbb{R}}^n u_0(x)\Phi_{x_i x_j}(x-y,t)dy = 0$. Take $x \in B(0,R)$ and compute

$$|u_{x_i x_j}(x,t)| \leq \int_{\mathbb{R}^n} |u_0(y) - u_0(x)||\Phi_{x_i x_j}(x-y,t)|dy \leq \int_{\mathbb{R}^n} |u_0(y)-u_0(x)| \left(\frac{1}{2t} + \frac{|x-y|^2}{4t^2}\right) \Phi(x-y,t)|dy \leq$$

$$\int_{B(0,R)} |u_0(y)-u_0(x)| \left(\frac{1}{2t} + \frac{|x-y|^2}{4t^2}\right) \Phi(x-y,t)|dy + \int_{\mathbb{R}^n \setminus B(0,R)} |u_0(y)-u_0(x)| \left(\frac{1}{2t} + \frac{|x-y|^2}{4t^2}\right) \Phi(x-y,t)|dy.$$

We estimate the first integral as follows:

$$\begin{aligned} &\int_{B(0,R)} |u_0(y) - u_0(x)| \left(\frac{1}{2t} + \frac{|x-y|^2}{4t^2}\right) \Phi(x-y,t)|dy \\ &\leq C_R \int_{B(0,R)} |x-y|^\alpha \left(\frac{1}{2t} + \frac{|x-y|^2}{4t^2}\right) \Phi(x-y,t)|dy \\ &\leq \frac{C_R}{\pi^{n/2}} \int_{\mathbb{R}^n} (4t)^{\alpha/2} |z|^\alpha \frac{1}{2t}(1+|z|^2) e^{-|z|^2} dz \leq \frac{C(n,R)}{t^{1-\alpha/2}}. \end{aligned} \tag{9}$$

As for the second integral we get

$$\int_{\mathbb{R}^n \setminus B(0,R)} |u_0(y) - u_0(x)| \left(\frac{1}{2t} + \frac{|x-y|^2}{4t^2}\right) \Phi(x-y,t)|dy$$

$$\leq \frac{2\|u_0\|_\infty}{\pi^{n/2} t} \int_{\mathbb{R}^n \setminus B(0,\frac{R}{\sqrt{4t}})} (1+|z|^2) e^{-|z|^2} dz \leq \frac{2\|u_0\|_\infty e^{-\frac{R^2}{8t}}}{\pi^{n/2} t} \int_{\mathbb{R}^n \setminus B(0,\frac{R}{\sqrt{4t}})} (1+|z|^2) e^{-|z|^2/2} dz \leq \frac{C}{t^{1-\alpha/2}}$$

since, if $z \in \mathbb{R}^n \setminus B(0, \frac{R}{\sqrt{4t}})$, $e^{-|z|^2/2} \leq e^{-R^2/8t} \leq C_T t^{\alpha/2}$ for $t \in (0,T)$. □

Proposition 1. *Let $f \in \mathcal{C}(\mathbb{R}^n \times [0,T])$ bounded, and assume that for all t, $f(\cdot,t) \in \mathcal{C}^{0,\alpha}(\mathbb{R}^n)$. Then there exists a solution u to the inhomogenous Cauchy problem*

$$\begin{cases} u_t - \Delta u = f(x,t) & x \in \mathbb{R}^n \ t \in (0,T) \\ u(x,0) = u_0(x). \end{cases}$$

Proof. We start considering the case $u_0 \equiv 0$. Define for all $s \in (0,T)$ $v^s(x,t)$ to be the solution to the Cauchy problem

$$\begin{cases} v^s_t - \Delta v^s = 0 & x \in \mathbb{R}^n \ t \in (s,T) \\ v^s(x,s) = f(x,s). \end{cases}$$

Then

$$v^s(x,t) = \int_{\mathbb{R}^n} f(y,s)\Phi(x-y,t-s)dy.$$

Define

$$u(x,t) = \int_0^t v^s(x,t)ds.$$

Then, since f is Holder continuous, we get by the previous theorem that $|v_t^s|, |\Delta v^s| \leq Ct^{\alpha/2-1} \in L^1(0,T)$ for all $x \in B(0,R)$. This implies that we can differentiate u as follows

$$u_t = v^t(x,t) + \int_0^t v_t^s \qquad \Delta u = \int_0^t \Delta v^s ds.$$

From this and the definition of v^s we conclude that $u_t - \Delta u = f(x,t)$.

So the solution of the Cauch problem is given (by superposition principle) by

$$u(x,t) = \int_{\mathbb{R}^n} u_0(y)\Phi(x-y,t)dy + \int_0^t \int_{\mathbb{R}^n} f(y,s)\Phi(x-y,t-s)dyds.$$

□

Comparison principle and uniqueness in $\mathbb{R}^n \times (0,T)$.

We start recalling Tychonov counterexample to uniqueness.

Consider the following Cauchy problem

$$\begin{cases} u_t - u_{xx} = 0 & x \in \mathbb{R}\ t > 0 \\ u(x,0) = 0. \end{cases}$$

This problem has obviously the solution $u \equiv 0$. Let $h(t) \in \mathcal{C}^\infty(\mathbb{R})$ defined as $h(t) = e^{-t^2}$ for $t > 0$ and $h(t) = 0$ for $t \leq 0$. Then

$$u(x,t) = \sum_{k=0}^{\infty} \frac{h^{(k)}(t)}{(2k)!} x^{2k}$$

is another solution to the Cauchy problem (it is a simple computation, once one observes that the serie is locally uniform convergent and that we can derive it term by term).

Theorem 4 (Comparison principle)**.** *Let c, f be bounded functions. Let $u, v \in \mathcal{C}^{2,1}(\mathbb{R}^n \times (0,T)) \cap \mathcal{C}(\mathbb{R}^n \times [0,T])$ such that*

- $u_t - \Delta u + c(x,t)u \leq f(x,t)$,
- $v_t - \Delta v + c(x,t)v \geq f(x,t)$
- *there exists $C > 0, \alpha > 0$ such that $u(x,t) \leq Ce^{\alpha|x|^2}$ for all x, $t \in [0,T]$,*
- *there exists $C > 0, \alpha > 0$ such that $v(x,t) \geq Ce^{\alpha|x|^2}$ for all x, $t \in [0,T]$,*
- $u(x,0) \leq v(x,0)$.

Then $u(x,t) \leq v(x,t)$ for all $x \in \mathbb{R}^n, t \in [0,T]$.

Proof. (See also Evans, theorem 6, pag 57).

Define $w(x,t) = e^{(\inf c)t}(u(x,t) - v(x,t)$. Then $w_t - \Delta w + \tilde{c}(x,t)w \leq 0$ where $\tilde{c} = c(x,t) - \inf c \geq 0$, $w(x,t) \leq Ce^{\alpha|x|^2}$ and $w(x,0) \leq 0$. To conclude it is sufficient to show that $w(x,t) \leq 0$ for all x, t.

Take $S < T$ such that, for some $\varepsilon > 0$, $4\alpha(S+\varepsilon) < 1$ and define $\beta = \frac{1}{4\alpha(S+\varepsilon)} - \alpha > 0$. We prove that $w(x,t) \leq 0$ for all $x \in \mathbb{R}^n, t \in [0,S]$. To conclude we repeatly apply this result on time intervals $[0,S], [S,2S]$.. up to $[0,T]$.

Let $\delta > 0$ and define

$$w_\delta(x,t) = w(x,t) - \delta \frac{1}{(S+\varepsilon-t)^{n/2}} e^{\frac{|x|^2}{4(S+\varepsilon-t)}}.$$

It is easy to check that $(w_\delta)_t - \Delta w_\delta + \tilde{c}(x,t) w_\delta \leq 0$ for $x \in \mathbb{R}^n$, $t \in (0,S)$, and moreover $w_\delta(x,0) \leq 0$.

Using the growth condition and the definition of S, β we get that

$$w_\delta(x,t) \leq e^{\alpha|x|^2} \left(C - \delta \frac{1}{(S+\varepsilon)^{n/2}} e^{\beta|x|^2} \right).$$

So for $\delta > 0$ fixed there exists R_δ such that $w_\delta(x,t) \leq 0$ for all $t \in [0,S]$ and $|x| \geq R_\delta$.

We consider w_δ in the cylinder $B(0,R_\delta) \times [0,S]$. It solves $(w_\delta)_t - \Delta w_\delta + \tilde{c}(x,t) w_\delta \leq 0$ and moreover $w_\delta(x,t) \leq 0$ in $\partial_\star(B(0,R_\delta) \times [0,S])$. So by parabolic comparison principle we get

$$w_\delta(x,t) \leq 0 \qquad \forall x \in \mathbb{R}^n, t \in [0,S].$$

We conclude letting $\delta \to 0$.

□

Corollary 1. There exists at most one solution to the Cauchy problem

$$\begin{cases} u_t - \Delta u = f(x,t) \\ u(x,0) = u_0(x) \end{cases}$$

such that there exist $C, \alpha > 0$ with $|u(x,t) \leq Ce^{\alpha|x|^2}$.

Moreover $\sup_{x \in \mathbb{R}^n, t \in [0,T]} |u(x,t)| \leq \|u_0\|_\infty + \|f\|_\infty T$.