Introduzione alle equazioni alle derivate parziali,
Laurea Magistrale in Matematica, A.A. 2013/2014

Maximum principles for elliptic operators.

Let $\Omega \subset \mathbb{R}^n$ be a open set and L be the following linear elliptic operator

$$Lu(x) := -\operatorname{tr}\ a(x)D^2u(x) + b(x)\cdot Du(x) \qquad x \in \Omega.$$

We assume the following general conditions on the coefficients of L.

Assumption 1. $a : \Omega \to S_n$ is a bounded continuous function, where S_n is the space of symmetric $n \times n$ matrices).
$b : \Omega \to \mathbb{R}^n$ is a bounded continuous function.

Moreover we assume that L is a degenerate elliptic operator according to this definition.

Definition. The operator L is degenerate elliptic if for every $x \in \Omega$, $a(x)$ is a $n \times n$ symmetric positive semidefinite matrix (i.e. all the eigenvalues of $a(x)$ are real and nonnegative).

Moreover we consider the following function.

Assumption 2. $c : \Omega \to R$ a bounded nonnegative function (so $0 \le c(x) \le c_0$ for every $x \in \Omega$).

Remark. Note that we are not asking that c is a continuous function, only that c is nonegative and bounded.

The previous assumptions will hold throughout this note.

Weak Maximum principles for elliptic operators

In this section we will consider degenerate elliptic operators of the form $Lu + c(x)u$ where $x \in \Omega$ which satisfy, besides the standing assumptions, also the following.

Assumption 3. For all $x \in \Omega$ such that $c(x) = 0$ there exist $\mu_x > 0$ and $\delta_x > 0$ such that

$$a_{11}(y) > \mu_x \qquad \forall y \in B(x, \delta_x). \tag{1}$$

Remark. Note that since $a(x)$ is positive semidefinite for every x, then necessarily $a_{11}(x) \ge 0$.
Condition (1) implies that all $x \in \Omega$ such that $c(x) = 0$ there exists $\delta_x > 0$ such that the matrix $a(y)$ admits at least one positive eigenvalue for every $y \in B(x, \delta_x)$.

Theorem 1 (Weak maximum principle)**.** *Let Ω be a bounded open set and $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that $Lu + c(x)u \le 0$, where L and c are as above.*
Then

- *if $c \equiv 0$, then $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$,*
- *if $c \not\equiv 0$, then $\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u^+$, where $u^+(y) := \max(u(y), 0)$.*

Proof. We start considering the case $c \equiv 0$. Assume by contradiction that $\max_{\overline{\Omega}} u > \max_{\partial\Omega} u$.
Let $\varepsilon > 0$ and define

$$u_\varepsilon(x) = u(x) + \varepsilon e^{\gamma x_1} \tag{2}$$

where $\gamma > 0$ will be fixed later. Then $u_\varepsilon \to u$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$. Let $x_\varepsilon \in \overline{\Omega}$ such that $u_\varepsilon(x_\varepsilon) = \max_{\overline{\Omega}} u_\varepsilon$. Then, passing to a subsequence $x_\varepsilon \to x_0$, and by uniform convergence we get that $u(x_0) = \max_{\overline{\Omega}} u$. Since we assumed that $\max_{\overline{\Omega}} u > \max_{\partial\Omega} u$, necessarily $x_0 \in \Omega$ and then also $x_\varepsilon \in \Omega$ for ε sufficiently small.

Now, since u_ε is $\mathcal{C}^2$ in Ω and x_ε is a maximum point, $Du_\varepsilon(x_\varepsilon) = 0$ and $D^2u_\varepsilon(x_\varepsilon) \leq 0$ (the hessian in x_ε is negative semidefinite). This implies that

$$Lu_\varepsilon(x_\varepsilon) = -\operatorname{tr}\ a(x_\varepsilon)D^2u_\varepsilon(x_\varepsilon) + b(x_\varepsilon)\cdot Du(x_\varepsilon) \geq 0. \tag{3}$$

On the other hand, by linearity of the operator L and the assumption that u is a subsolution we get

$$Lu_\varepsilon(x_\varepsilon) = Lu(x_\varepsilon) - \gamma^2\varepsilon a_{11}(x_\varepsilon)e^{\gamma(x_\varepsilon)_1} + \gamma b_1(x_\varepsilon)e^{\gamma(x_\varepsilon)_1} \leq \gamma\varepsilon e^{\gamma(x_\varepsilon)_1}\left(-a_{11}(x_\varepsilon)\gamma + b_1(x_\varepsilon)\right), \tag{4}$$

Now, by assumption (1), there exists δ_{x_0} and μ_{x_0} such that $a_{11}(x) > \mu_{x_0}$ for every $x \in B(x_0, \delta_{x_0})$. Since $x_\varepsilon \to x_0$, choosing ε sufficiently small we have that $x_\varepsilon \in B(x_0, \delta_{x_0})$. For such ε, (4) reads

$$Lu_\varepsilon(x_\varepsilon) \leq \gamma\varepsilon e^{\gamma(x_\varepsilon)_1}\left(-\mu_{x_0}\gamma + \|b\|_\infty\right) < 0, \tag{5}$$

choosing $\gamma > 0$ sufficiently large. But then (5) contradicts (3).

We consider now the case $c \not\equiv 0$. Assume by contradiction that $M := \max_{\overline{\Omega}} u > \max_{\partial\Omega} u$ and $M > 0$.

Let $\mathcal{M} = \{x \in \Omega \text{ such that } u(x) = M\}$. If $x \in \mathcal{M}$, then $Du(x) = 0$ and $D^2u(x) \leq 0$. So, using this fact for the first inequality and the assumption that u is a subsolution for the second, we get

$$c(x)u(x) \leq Lu(x) + c(x)u(x) \leq 0.$$

This implies that $c(x) = 0$ for every $x \in \mathcal{M}$.

We define as above the function u_ε and the sequence $x_\varepsilon \to x_0 \in \mathcal{M}$. So repeating the arguments in (3),(5) above we get that

$$c(x_\varepsilon)u_\varepsilon(x_\varepsilon) \leq Lu_\varepsilon(x_\varepsilon) + c(x_\varepsilon)u(x_\varepsilon) + \varepsilon c(x_\varepsilon)e^{\gamma(x_\varepsilon)_1} \leq \varepsilon e^{\gamma(x_\varepsilon)_1}\left(-\mu_{x_0}\gamma^2 + \gamma\|b\|_\infty + \|c\|_\infty\right) < 0$$

choosing γ sufficiently large.

The fact that $c(x_\varepsilon)u_\varepsilon(x_\varepsilon) < 0$ implies that $u_\varepsilon(x_\varepsilon) < 0$ for every ε sufficiently small and then, passing to the limit as $\varepsilon \to 0$, $u(x_0) \leq 0$, in contradiction with the assumption $M > 0$. □

Remark. The weak minimum principle reads as follows.

Let $v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that $Lv + c(x)v \geq 0$, where L and c are as above. Then

- if $c \equiv 0$, then $\min_{\overline{\Omega}} v = \min_{\partial\Omega} v$,
- if $c \not\equiv 0$, then $\min_{\overline{\Omega}} v \geq \min_{\partial\Omega} v^-$, where $v^-(y) := \min(v(y), 0)$.

A first consequence of the theorem is the comparison principle.

Corollary 1 (Weak comparison principle)**.** *Let $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that $Lu + c(x)u \leq 0$, and $Lv + cv \geq 0$ in Ω, where L and c satisfies the same assumptions as above.*

If $u \leq v$ in $\partial\Omega$, then $u \leq v$ in $\overline{\Omega}$.

Proof. Let $w = u - v$, then $Lw + cw \leq 0$ in Ω and $w \leq 0$ on $\partial\Omega$. By the weak maximum principle $\max_{\overline{\Omega}} w \leq 0$, which gives the conclusion. □

The comparison principle implies as usual a uniqueness result.

Corollary 2 (Uniqueness for the Dirichlet problem)**.** *Let Ω be a bounded open set, then the Dirichlet problem*

$$(D)\begin{cases} Lu + c(x)u = f(x) & x \in \Omega \\ u(x) = g(x) & x \in \partial\Omega \end{cases}$$

admits at most one solution $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$.

Proof. If u_1, u_2 are two solutions, then $w = u_1 - u_2$ satisfies $Lw + cw = 0$ in Ω and $w = 0$ on $\partial\Omega$. By the weak maximum and minimum principle $\max_{\overline{\Omega}} |w| = 0$, which gives the conclusion. □

Finally we have the following continuous dependance estimates.

Proposition 1 (Continuous dependance estimates). *Let Ω be a bounded open set, $g \in \mathcal{C}(\partial\Omega)$ and $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ the solution to Dirichlet problem*

$$\begin{cases} Lu + c(x)u = 0 & x \in \Omega \\ u(x) = g(x) & x \in \partial\Omega \end{cases}.$$

Then

$$\|u\|_\infty \leq \|g\|_\infty.$$

Proof. Let $w(x) = u(x) - \|g\|_\infty$. Then $Lw + cw = Lu + cu - c(x)\|g\|_\infty = -c(x)\|g\|_\infty \leq 0$ in Ω and $w \leq 0$ on $\partial\Omega$. By the weak maximum principle $w \leq 0$, so $u(x) \leq \|g\|_\infty$ for every $x \in \overline{\Omega}$.

Let $v(x) = u(x) + \|g\|_\infty$. Then $Lv + cv = Lu + cu + c(x)\|g\|_\infty = c(x)\|g\|_\infty \geq 0$ in Ω and $v \geq 0$ on $\partial\Omega$. By the weak minimum principle $v \geq 0$, so $u(x) \geq -\|g\|_\infty$ for every $x \in \overline{\Omega}$. □

Proposition 2 (Continuous dependance estimates). *Let Ω be a bounded open set, $f \in \mathcal{C}(\Omega)$ and $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ the solution to Dirichlet problem*

$$\begin{cases} Lu = f(x) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases}.$$

Then there exists a constant $C = C(\Omega, L)$, depending of the coefficients a, b and of Ω, such that

$$\|u\|_\infty \leq C\|f\|_\infty.$$

Proof. Let $w(x) = u(x) - \|f\|_\infty e^{\gamma x_1}$, where $\gamma > 0$ has to be fixed. Then

$$Lw = Lu + \|f\|_\infty e^{\gamma x_1}(-a_{11}(x)\gamma^2 + b_1(x)\gamma) = f(x) + \|f\|_\infty e^{\gamma x_1}(-a_{11}(x)\gamma^2 + b_1(x)\gamma) \leq 0$$

choosing γ sufficiently large. Moreover $w \leq 0$ on $\partial\Omega$. By the weak maximum principle $w \leq 0$, so $u(x) \leq \|f\|_\infty \inf_{x \in \overline{\Omega}} e^{\gamma x_1}$ for every $x \in \overline{\Omega}$.

Repeating the same argument using weak minimum principle we get the other inequality. □

Strong Maximum principles for uniformly elliptic operators

In this section we will consider uniformly elliptic operators L, according to the following definition.

Definition. Let L be a degenerate elliptic operator. Then L is uniformly elliptic in Ω if there exists $\lambda > 0$ such that

$$\xi^t a(x)\xi \geq \lambda|\xi|^2 \qquad \forall x \in \Omega\ \forall \xi \in \mathbb{R}^n.$$

Remark. Note that L is uniformly elliptic if for every x the minimal eigenvalue of $a(x)$ is positive (bigger than λ).

Lemma 1 (Hopf lemma). *Assume that L is a uniformly elliptic operator in Ω and that $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that $Lu + c(x)u \leq 0$.*

If there exists $x_0 \in \partial\Omega$ such that

- *there exists $y_0 \in \Omega$ and $r_0 > 0$ such that $B(y_0, r_0) \subset \Omega$ and $\overline{B(y_0, r_0)} \cap (\mathbb{R}^n \setminus \Omega) = \{x_0\}$*
- - *if $c \equiv 0$, $u(x) < u(x_0)$ for every $x \in \Omega$*
  - *if $c \not\equiv 0$, $u(x) < u(x_0)$ for every $x \in \Omega$ and $u(x_0) \geq 0$*

then for every $\gamma \in \mathbb{R}^n$ such that $\gamma \cdot \frac{x_0 - y_0}{r_0} > 0$ we get

$$\liminf_{h \to 0^+} \frac{u(x_0) - u(x_0 - h\gamma)}{h} > 0.$$

Remark. Note that by maximality of x_0, it is trivial to prove that

$$\liminf_{h\to 0^+} \frac{u(x_0)-u(x_0-h\gamma)}{h} \geq 0.$$

Moreover $\frac{x_0-y_0}{r_0}$ is the exterior normal to Ω (and also to $B(y_0,r_0)$) in x_0. So, if u is differentiable at x_0, Hopf lemma gives that

$$\frac{\partial u}{\partial n}(x_0) > 0.$$

Proof. We start considering the case $c \equiv 0$. Let $\alpha > 0$ to be fixed and define

$$v(x) = e^{-\alpha|x-y_0|^2} - e^{-\alpha r_0^2}.$$

Note that $v(x) = 0$ for every $x \in \partial B(y_0,r_0)$ and $v(x) > 0$ for every $x \in B(y_0,r_0)$. Define the function

$$w(x) = u(x) - u(x_0) + \varepsilon v(x)$$

where $\varepsilon > 0$ has to be fixed. Compute

$$Dv(x) = -2\alpha(x-y_0)e^{-\alpha|x-y_0|^2}$$

and

$$D^2v(x) = 2\alpha e^{-\alpha|x-y_0|^2}(-I + 2\alpha(x-y_0)\otimes(x-y_0)).$$

Observe that

$$\mathrm{tr}[a(x)(x-y_0)\otimes(x-y_0)] = \sum_{i,j=1,\dots n} a_{ij}(x)(x-y_0)_i(x-y_0)_j = (x-y_0)^t a(x)(x-y_0) \geq \lambda|x-y_0|^2.$$

Then for every $x \in \Omega$ such that $\frac{r_0}{2} < |x-y_0| < r_0$,

$$Lw(x) = Lu(x) + \varepsilon e^{\alpha|x-y_0|^2}\alpha\left(\mathrm{tr}\ a(x) - 2\alpha\lambda|x-y_0|^2 + b(x)\cdot(x-y_0)\right) \leq$$
$$\leq \varepsilon e^{\alpha r_0^2}\alpha\left(n\|a\|_\infty - \frac{\alpha}{2}\lambda r_0^2 + \|b\|_\infty r_0\right) < 0$$

choosing $\alpha > 0$ sufficiently large.

Moreover for every $x \in \partial B(y_0,r_0)$, $w(x) = u(x) - u(x_0) \leq 0$ by assumption, and for every $x \in \partial B(y_0, \frac{r_0}{2})$,

$$w(x) = u(x) - u(x_0) + \varepsilon v(x) \leq \sup_{x\in\partial B(y_0,\frac{r_0}{2}}(u(x)-u(x_0)) + \varepsilon \max_{\partial B(y_0,\frac{r_0}{2}} v(x) < 0$$

if we choose ε sufficienlty small (since by assumption $u(X_0) > u(x)$ for every $x \in \Omega$).

In conclusion, with these choices of ε, α, $Lw \leq 0$ in $B(y_0,r_0)\setminus B(y_0,r_0/2)$ and $w \leq 0$ in $\partial(B(y_0,r_0)\setminus B(y_0,r_0/2))$. We conclude by weak maximum principle that $w \leq 0$ for every $x \in B(y_0,r_0)\setminus B(y_0,r_0/2)$.

Let γ as in the statement and fix h_0 such that $x_0 - h_0\gamma \in B(y_0,r_0)\setminus B(y_0,r_0/2)$. Then for every $0 < h \leq h_0$

$$\frac{u(x_0)-u(x_0-h\gamma)}{h} \geq \varepsilon\frac{v(x_0-h\gamma)-v(x_0)}{h}.$$

Passing to the limit as $h \to 0$ we get

$$\liminf_{h\to 0^+}\frac{u(x_0)-u(x_0-h\gamma)}{h}\varepsilon\lim_{h\to 0^+}\frac{v(x_0-h\gamma)-v(x_0)}{h} = -\varepsilon Dv(x_0)\cdot\gamma = \varepsilon\alpha(x_0-y_0)\cdot\gamma > 0.$$

If $c \not\equiv 0$, we follow the same argument as above, define the same function v and obtain

$$Lw + c(x)w \leq -c(x)u(x_0) + \varepsilon e^{\alpha r_0^2}\alpha\left(n\|a\|_\infty - \frac{\alpha}{2}\lambda r_0^2 + \|b\|_\infty r_0\right) + \varepsilon c(x)v(x) \leq$$
$$\leq \varepsilon e^{\alpha r_0^2}\alpha\left(n\|a\|_\infty - \frac{\alpha}{2}\lambda r_0^2 + \|b\|_\infty r_0\right) + \varepsilon\|c\|_\infty < 0$$

choosing α sufficiently large. Note that we used the fact that $c(x)u(x_0) \geq 0$ for every x. The conclusion follows as above. $\square$

We are ready now to prove the strong maximum principle.

Theorem 2 (Strong maximum principle)**.** *Let Ω be a open connected set and $u \in \mathcal{C}^2(\Omega)$ such that $Lu + c(x)u \leq 0$, where L and c are as above.*

Then

- *if $c \equiv 0$, and there exist $x \in \Omega$ such that $u(x) = \sup_\Omega u$, then u is constant*
- *if $c \not\equiv 0$, and there exist $x \in \Omega$ such that $u(x) = \sup_\Omega u$, and $u(x) \geq 0$ then u is constant .*

Proof. Let $M = \sup_\Omega u$ and let $\mathcal{M} = \{x \in \Omega \mid u(x) = M\}$. We show that if $\mathcal{M} \neq \emptyset$ then $\mathcal{M} = \Omega$ (so u is constant). Let $C = \Omega \setminus \mathcal{M} \subset \Omega$. C is open in Ω. We claim that $\partial C = \emptyset$ in Ω, so $C = \overline{C}$ in Ω. This implies that C is closed and open, so $C = \emptyset$ since Ω is connected and $\Omega \setminus C \neq \emptyset$.

By contradiction assume that $\partial C \cap \Omega \neq \emptyset$. This implies that there exist $y \in C$ such that $\text{dist}(y, \partial\Omega) >\text{dist}(y, \partial C) = r$. So $B(y, r) \subset C$ and $\partial B(y, r) \cap \partial C \neq \emptyset$, whereas $\partial B(y, r) \cap \partial\Omega = \emptyset$. Let $\partial B(y, r) \cap \partial C$. So, there exists $r' < r$ and $y' \in [x, y]$ (y' is in the segment connecting y, x) such that $B(y', r') \subset C$ and $\partial B(y', r') \cap \partial C = \{x\}$. So $u(z) < M$ for every $z \in B(y', r')$ and $u(x) = M$ and, since $x \in \Omega$, $Du(x) = 0$. But Hopf lemma applied to the set $B(y', r')$ we would have $Du(x) \cdot (x - y') > 0$, in contradiction with the fact that $Du(x) = 0$. □

A first consequence of the theorem is the comparison principle.

Corollary 3 (Strong comparison principle)**.** *Let $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that $Lu + c(x)u \leq 0$, and $Lv + cv \geq 0$ in Ω, where L and c satisfies the same assumptions as above and Ω is connected.*

If $u \leq v$ in Ω, then either $u \equiv v$ or $u < v$ in Ω.

Proof. Let $w = u - v$, then $Lw + cw \leq 0$ in Ω and $w \leq 0$ on Ω. If there exists $x \in \Omega$ such that $w(x) = 0$, then by the strong maximum principle, w is constant, so $w \equiv 0$. If such x does not exist, then $w < 0$ in Ω. □

We get also the following result on the uniqueness up to constant of solutions to the Neumann problem.

Corollary 4 (Uniqueness up to constant for the Neumann problem)**.** *Let Ω be a bounded open set, which satisfies the interior sphere condition at every point. Then the Neumann problem*

$$(N) \begin{cases} Lu + c(x)u = f(x) & x \in \Omega \\ \frac{\partial u}{\partial n}(x) = g(x) & x \in \partial\Omega \end{cases}$$

admits at most one solution $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ up to additive constants. This means that if $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ are solutions to (N) then there exists a constant $k \in \mathbb{R}$ such that $u(x) = v(x) + k$ for every $x \in \overline{\Omega}$.

Proof. If u_1, u_2 are two solutions, then $w = u_1 - u_2$ satisfies $Lw + cw = 0$ in Ω and $\frac{\partial w}{\partial n} = 0$ on $\partial\Omega$. Let $M = \max_{\overline{\Omega}} w$, we can assume without loss of generality that $M \geq 0$ (otherwise define $w = u_2 - u_1$). If w is not constant, by strong maximum principle for every $y \in \Omega$, $u(y) < M$. Moreover there exists at least one point $x \in \partial\Omega$ such that $u(x) = M$. Then by Hopf lemma, $\frac{\partial w}{\partial n}(x) > 0$, in contradiction with the fact that $\frac{\partial w}{\partial n}(x) = 0$. So $w \equiv M$. □

Liouville type results

Using strong and weak maximum principle for uniformly elliptic operators we prove a Liouville type theorem for subsolutions of uniformy elliptic operators.

So, in this section L will be a uniformly elliptic operator in $\mathbb{R}^n$. Moreover we assume that there exists a supersolution to L, exploding at infinity. In particular we assume the following.

Assumption 4. There exists $M > 0$ and $w \in \mathcal{C}^2(\mathbb{R}^n \setminus B(0, M))$ such that

- $Lw(x) \geq 0$ for every $|x| > M$,
- $\lim_{|x| \to +\infty} w(x) = +\infty$.

Proposition 3 (Liouville type result)**.** *Assume that L is uniformly elliptic and that 4 holds. Let $u \in \mathcal{C}^2(\mathbb{R}^n)$ such that $Lu \leq 0$ in $\mathbb{R}^n$ and $u(x) \leq C$ for every $x \in \mathbb{R}^n$.*

Then u is constant.

Analogously, every bounded from below supersolution to L in $\mathbb{R}^n$ is constant.

Remark. This result applies also to the laplacian operator in $\mathbb{R}^2$. Indeed the function $\log |x|$ satisfies assumption 4. So every bounded from above subharmonic function is constant.

The same result is not true in $\mathbb{R}^n$ for $n \geq 3$ (since in this case, assumption 4 is not satisfied). Indeed there are bounded subharmonic functions in $\mathbb{R}^n$, e.g. $u(x) = -(1 + |x|^2)^{-1}$ is subharmonic and bounde in $\mathbb{R}^n$ with $n \geq 4$ and $u(x) = -(1 + |x|^2)^{-\frac{1}{2}}$ is subharmonic and bounded in $\mathbb{R}^3$.

Proof. Let u be bounded from above and $\varepsilon > 0$. Define $v_\varepsilon = u - \varepsilon w$ for $|x| > 2M$. Then $v_\varepsilon \in \mathcal{C}^2\{x \in \mathbb{R}^n,\ |x| \geq 2M\}$ and $\lim_{|x| \to +\infty} v_\varepsilon(x) = -\infty$ and $Lv_\varepsilon = Lu - \varepsilon Lw \leq 0$ for every $|x| > 2M$. Define $C_\varepsilon = \max_{|x|=2M} v_\varepsilon(x)$. So, since $\lim_{|x| \to +\infty} v_\varepsilon(x) = -\infty$, there exists $K_\varepsilon > 2M$ such that $v_\varepsilon(x) < C_\varepsilon$ for every $|x| \geq K_\varepsilon$.

Moreover, by weak maximum principle in the set $\{x \in \mathbb{R}^n \mid 2M < |x| < K_\varepsilon\}$ we have that

$$\max_{\{x \in \mathbb{R}^n \ \mid \ 2M \leq |x| \leq K_\varepsilon\}} v_\varepsilon(x) = \max_{\{x \in \mathbb{R}^n \ \mid \ |x| = 2M \text{ or } |x| = K_\varepsilon\}} v_\varepsilon(x). \tag{6}$$

Since $v_\varepsilon(x) < C_\varepsilon$ for every $|x| \geq K_\varepsilon$, we obtain from (6) that for every $|y| \geq 2M$

$$v_\varepsilon(y) = u(y) - \varepsilon w(y) \leq \max_{\{x \in \mathbb{R}^n \ \mid \ |x| = 2M\}} v_\varepsilon(x) \leq \max_{\{x \in \mathbb{R}^n \ \mid \ |x| = 2M\}} u(x) - \varepsilon \min_{\{x \in \mathbb{R}^n \ \mid \ |x| = 2M\}} w(x). \tag{7}$$

Sending $\varepsilon \to 0$ in (7), we obtain

$$u(y) \leq \max_{\{x \in \mathbb{R}^n \ \mid \ |x| = 2M\}} u(x) \qquad \forall\ |y| > 2M.$$

Moreover by weak maximum principle appied in $B(0, 2M)$, we get that

$$u(y) \leq \max_{\{x \in \mathbb{R}^n \ \mid \ |x| = 2M\}} u(x) \qquad \forall\ |y| < 2M.$$

Putting together the last two inequalities we get

$$u(y) \leq \max_{\{x \in \mathbb{R}^n \ \mid \ |x| = 2M\}} u(x) \qquad \forall\ y \in \mathbb{R}^n.$$

This implies that u attains a maximum in some point in $\partial B(0, 2M)$, so, by strong masimum principle u is constant. $\square$