Introduzione alle equazioni alle derivate parziali, Laurea Magistrale in Matematica

Perron method for the Dirichlet problem.

We approach the question of existence of solution $u \in \mathcal{C}^{\infty}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ of the Dirichlet problem in an open bounded set $\Omega \subseteq \mathbb{R}^n$

$$\begin{cases} -\Delta u = 0 & x \in \Omega\\ u = g & x \in \partial \Omega \end{cases}$$
(1)

where $g \in C(\partial \Omega)$. Observe that by the weak maximum principle, if a solution of this problem exists, it is unique.

Subharmonic functions

We prove the following characterization of subharmonic functions.

Proposition 1. Let $u \in C(\Omega)$. Then the following are equivalent:

(i) for all $x \in \Omega$ and $B(x,r) \subset \subset \Omega$

$$u(x) \leq \int_{B(x,r)} u(y) dy$$

(i.e. u is subharmonic),

(ii) for all $B \subset \subset \Omega$ and for all $h : \overline{B} \to \mathbb{R}$ which satisfies

$$\begin{cases} -\Delta h = 0 & x \in B \\ h \ge u & x \in \partial B, \end{cases}$$

one has $h(x) \ge u(x)$ for all $x \in \overline{B}$

(iii) for all $x \in \Omega$ and $B(x,r) \subset \subset \Omega$

$$u(x) \leq \int_{\partial B(x,r)} u(y) dS(y),$$

(iv) for all $x \in \Omega$ and for all $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local maximum in x, then $-\Delta \phi(x) \leq 0$.

Proof. $(i) \to (ii)$. u - h is a subharmonic function in B such that $u - h \leq 0$ in ∂B . We conclude by weak Maximum Principle.

 $(ii) \rightarrow (iii)$. Let $x \in \Omega$ and $B(x, r) \subset \Omega$ and take h the solution to

$$\begin{cases} -\Delta h = 0 & y \in B(x, r) \\ h = u & y \in \partial B(x, r), \end{cases}$$

Then by the Poisson integral formula (or by the property of spherical mean for harmonic functions) $h(x) = \int_{\partial B(x,r)} h(y) dS(y) = \int_{\partial B(x,r)} h(y) dS(y)$. Moreover by (*ii*), $h(x) \ge u(x)$, so we conclude. (*iii*) \rightarrow (*i*). Let $x \in \Omega$ and $B(x,r) \subset \subset \Omega$. Then, by the formula of integral over spheres and by (*iii*),

$$\int_{B(x,r)} u(y)dy = \int_0^r \int_{\partial B(x,s)} u(y)dS(y)ds \ge \int_0^r u(x)n\omega_n s^{n-1}ds = u(x)\omega_n r^n$$

which gives the conclusion.

 $(iii) \to (iv)$. Let $x \in \Omega$, $\phi \in C^2(\Omega)$ and $B(x,r) \subseteq \Omega$ such that $u(y) - \phi(y) \leq u(x) - \phi(x)$ for all $y \in B(x,r)$. Since the inequality holds for every $y \in B(x,r)$ we get, by (iii), that for all $s \in (0,r)$

$$u(x) - \phi(x) \ge \int_{\partial B(x,s)} u(y) dS(y) - \int_{\partial B(x,s)} \phi(y) dS(y) \ge u(x) - \int_{\partial B(x,s)} \phi(y) dS(y)$$

We define for $s \in (0, r)$

$$\psi(s) = \int_{\partial B(x,s)} \phi(y) dS(y) = \int_{\partial B(0,1)} \phi(x+sz) dS(z)$$

We compute, using the divergence theorem,

$$\psi'(s) = \int_{\partial B(0,1)} D\phi(x+sz) \cdot z dS(z) = \frac{1}{n\omega_n s^{n-1}} \int_{B(x,s)} \Delta\phi(y) dy = \frac{s}{n} \oint_{B(x,s)} \Delta\phi(y) dy.$$

By Lebesgue theorem $\lim_{s \to 0^+} \psi(s) = \phi(x)$ so

$$\phi(x) = \psi(0) \le \psi(s) \qquad \forall s \in (0, r).$$
(2)

Assume now by contradiction that (iv) is not verified, then there exists $\delta > 0$ such that $\Delta\phi(x) < -2\delta < 0$. By continuity there exists s > 0 such that $\Delta\phi(y) < -\delta$ for all $y \in B(x,s)$. This gives that $\psi'(t) < -\delta t/n$ for all $t \leq s$, so $\psi(t) > \psi(s)$ for all $t \in (0,s)$ and then, integrating in $t \in (0,s)$, $\psi(s) - \psi(0) < -\frac{\delta s^2}{2n} < 0$ which contradicts condition (2).

 $(iv) \rightarrow (iii)$. Assume by contradiction that (iii) is not verified. So there exists $x \in \Omega$ and there exists r > 0 such that

$$u(x) > \int_{\partial B(x,r)} u(y) dS(y).$$

Fix c > 0 sufficiently small such that

$$u(x) - \int_{\partial B(x,r)} u(y) dS(y) > cr^2.$$
(3)

Let $U \in \mathcal{C}^2(B(x,r)) \cap \mathcal{C}(\overline{B(x,r)})$ to be the unique solution to the Dirichlet problem

$$\begin{cases} -\Delta U = 0 & B(x,r) \\ U(y) = u(y) & y \in \partial B(x,r) \end{cases}$$

Then, U(y) = u(y) on $\partial B(x, r)$ and (by Poisson integral formula) $U(x) = \int_{\partial B(x, r)} u(y) dS(y)$.

Define $\phi(y) = U(y) + c(r^2 - |y - x|^2)$. Then $u(y) - \phi(y) = 0$ if $y \in \partial B(x, r)$, $\phi \in C^2(B(x, r))$ and $u(x) - \phi(x) = u(x) - U(x) - cr^2 > 0$ by the choice of c in (3). Then $\max_{\overline{B(x,r)}} u(y) - \phi(y) > 0$ and there exists a point $z \in B(x, r)$ (the important thing is that z is in the interior of B(x, r)!) such that $u(z) - \phi(z) = \max_{\overline{B(x,r)}} u(y) - \phi(y)$

By (iv) this implies that $-\Delta\phi(z) \leq 0$, but $\Delta\phi(z) = \Delta U(z) - c\Delta(|z-x|^2) = 0 - 2cn < 0$, and so we reached a contradiciton.

Definition (Harmonic lifting). Let u be a subharmonic function in Ω and $B \subset \subset \Omega$. Then the harmonic lifting of u in B is the function U which coincides with u in $\Omega \setminus B$ and in B solves the Dirichlet problem

$$\begin{cases} -\Delta U = 0 & x \in B\\ U = u & x \in \partial B. \end{cases}$$

Remark. By weak maximum principle, it is immediate to show that $u \leq U$ in Ω .

Remark. Let U be the harmonic lifting of u in B, then U is a subharmonic function.

It is sufficient to show that U satisfies property (ii) in Proposition 1. Let $B' \subset \subset \Omega$ and h be a function

$$\begin{cases} -\Delta h = 0 \quad x \in B' \\ h \ge U \qquad x \in \partial B'. \end{cases}$$

If $B' \cap B = \emptyset$, then it is true that $h \ge u = U$ since u is subharmonic. For the same reason, if $B' \cap B \ne \emptyset$, in $B' \setminus (B' \cap B)$, $h \ge u = U$. In $B' \cap B$, h, U are both harmonic, and moreover on $\partial(B' \cap B)$, $h \ge U$. So we conclude by weak maximum principle.

Perron solution: existence result

Let Ω be a bounded open set and $g \in L^{\infty}(\partial \Omega)$. Define

 $S_q = \{ v \in \mathcal{C}(\overline{\Omega}) \mid v \text{ subharmonic in } \Omega, v(x) \leq g(x), x \in \partial \Omega \}.$

Remark. The set S_g is not empty and bounded from above.

In fact the constant function $\inf_{\partial\Omega} g$ is in S_g . Moreover by weak maximum principle for all $v \in S_g$ we get $v \leq \sup_{\partial\Omega} g$.

Theorem 1. Let Ω be an open and bounded set and $g \in L^{\infty}(\partial \Omega)$. Then the function

$$H_g(x) = \sup_{v \in S_g} v(x)$$

is harmonic in Ω .

Proof. Fix $x \in \Omega$. We want to show that H_g is harmonic in x.

Let v_n be a sequence in S_g such that $v_n(x) \to H_g(x)$.

Without loss of generality we can assume that v_n is equibounded. Indeed, if it is not the case we consider the sequence $\tilde{v}_n = \max(v_n, \inf_{\partial\Omega} g)$. Note that $\tilde{v}_n \in S_g$, \tilde{v}_n is equibounded (since $\inf_{\partial\Omega} g \leq \tilde{v}_n \leq \sup_{\partial\Omega} g$) and $\tilde{v}_n(x) \to H_g(x)$ (since $v_n(x) \leq \tilde{v}_n(x) \leq H_g(x)$). So v_n is an equibounded sequence in S_g with $v_n(x) \to H_g(x)$. We fix r > 0 such that

So v_n is an equibounded sequence in S_g with $v_n(x) \to H_g(x)$. We fix r > 0 such that $B(x,r) \subset \subset \Omega$ and consider for every n the harmonic lifting V_n of v_n in B(x,r). Then $V_n \in S_g$, V_n is equibounded (by weak maximum principle and the fact that v_n is equibounded) and $V_n(x) \to H_g(x)$. By Ascoli Arzelà theorem for harmonic functions, eventually passing to a subsequence (that we still denote with V_n) we get that $V_n \to V$ uniformly in $B(x, \rho)$ for every $\rho < r$. Moreover, V is harmonic in B(x,r), $V(y) \leq H_g(y)$ for every $y \in B(x,r)$ and finally $V(x) = H_g(x)$.

We claim now that there exists $\rho < r$ such that $V(y) = H_g(y)$ for every $y \in B(x, \rho)$. If it is true, we are done, since then H_g is harmonic in x.

We assume that the claim is not true, so for every ρ we find $z \in B(x, \rho)$ such that $V(z) < H_g(z)$. We prove that this leads to a contradiction.

Take a sequence $w_n \in S_g$ such that $w_n(z) \to H_g(z)$. As above, we can assume whoge that w_n is equibounded. Moreover, we can also assume that $w_n \ge V_n$ for every n. Indeed, if it is not the case we consider the sequence $\tilde{w}_n = \max(w_n, V_n)$. Note that $\tilde{w}_n \in S_g$, \tilde{v}_n is equibounded (since w_n and V_n are equibounded) and $\tilde{w}_n(z) \to H_g(z)$ (since $w_n(z) \le \tilde{w}_n(z) \le H_g(z)$).

For every n we consider the harmonic lifting W_n of w_n in $B(x,\rho)$. Then $W_n \in S_g$, W_n is equibounded, $V_n(y) \leq W_n(y)$ (in particular $V_n(x) \leq W_n(x) \leq H_g(x)$). By Ascoli Arzelà theorem for harmonic function, eventually passing to a subsequence (that we still denote with W_n) we get that $W_n \to W$ uniformly in $B(x,\rho')$ for every $\rho' < \rho$. Moreover, W is harmonic in $B(x,\rho)$, $V(y) \leq W(y)$ for every $y \in B(x,\rho)$, $W(x) = H_g(x) = V(x)$ and $W(z) = H_g(z) > V(z)$.

So, V, W are two harmonic functions in $B(x, \rho)$ such that $V - W \leq 0$, and V(x) - W(x) = 0. This implies by strong maximum principle that $V \equiv W$ in $B(x, \rho)$, in contradiction with the fact that W(z) > V(z).

Perron method: study of the boundary behaviour

In theorem 1, we proved that for every bounded function g, there exists a harmonic function $H_g \in \mathcal{C}^{\infty}(\Omega)$ which solves

$$\begin{cases} -\Delta H_g = 0 & x \in \Omega \\ H_g \ge g & x \in \partial \Omega \end{cases}$$

Now, we assume that $g \in \mathcal{C}(\partial \Omega)$ and we wonder under which conditions H_g is the solution of the Dirichlet problem (1), in particular under which conditions we can prove that, for all $x_0 \in \partial \Omega$,

$$\lim_{x \to x_0, \ x \in \Omega} H_g(x) = g(x_0). \tag{4}$$

Indeed if we prove this identity, we get that $H_g \in \mathcal{C}^{\infty}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ and coincides with g on the boundary of Ω .

Remark. Observe that in general we cannot expect that (4) holds true for every Ω bounded.

Consider the following example. Let $\Omega = \{x \in \mathbb{R}^2 \mid 0 < |x| < 1\}$ and $g \in \mathcal{C}(\partial\Omega)$ defined as follows: g(x) = 0 for |x| = 1, g(0) = 1. Then in this case $H_g \equiv 0$ (and in particular it is not a solution of the Dirichlet problem with boundary data g since $H_g(0) = 0 \neq 1$).

In fact, $0 \in S_g$, so $H_g(x) \ge 0$ for every $x \in \Omega$.

Let $v \in S_g$. So by weak maximum principle v(x) < 1 for every $x \in \Omega$. Fix $\delta > 0$ and $\varepsilon = \varepsilon(\delta) \in (0, 1)$ such that $-\delta \log(\varepsilon) > 1$. Consider now the function $w_{\delta}(x) = -\delta \log |x|$. This is harmonic in $\varepsilon < |x| < 1$, moreover $w_{\delta}(x) = 0$ if |x| = 1 and $w_{\delta}(x) = -\delta \log \varepsilon > 1$ if $|x| = \varepsilon$. This implies by weak maximum principle that $v(x) \leq w_{\delta}(x)$ for every $\varepsilon \leq |x| \leq 1$. Moreover $w_{\delta}(x) > 1 \geq v(x)$ also for every $|x| \leq \varepsilon$. Then $w_{\delta}(x) \geq v(x)$ for every $0 < |x| \leq 1$ and every $v \in S_g$, which implies that $H_g(x) \leq -\delta \log |x|$ for every $0 < |x| \leq 1$, and every $\delta > 0$, which gives the conclusion letting $\delta \to 0$.

The continuity assumption (4) on the boundary is connected with the geometric properties of the boundary through the concept of barrier.

Definition (Regular points). Let $x_0 \in \partial \Omega$. Then x_0 is a regular point (with respect to the Laplacian), if there exists a (local) barrier at x_0 .

Definition (Local barrier). Let $x_0 \in \partial\Omega$. Then w is a local barrier at x_0 if there exists a neighbourhood U of x_0 such that $w \in C(\overline{\Omega \cap U})$ and

- (i) w is superharmonic in $\Omega \cap U$,
- (ii) $w(x_0) = 0$ and w(x) > 0 for every $x \in \overline{\Omega \cap U} \setminus \{x_0\}$.

Remark. A barrier w in x_0 , is a local barrier, with $U = \mathbb{R}^n$. Given a local barrier in x_0 , it is always possible to construct a barrier in x_0 as follows. Let r > 0 such that $B(x_0, r) \subset U$ and $m = \inf_{(U \setminus B(x_0, r)) \cap \Omega} w > 0$. Define

$$w(x) = \begin{cases} \min(m, w(x)) & x \in \overline{B(x_0, r) \cap \Omega} \\ m & \text{elsewhere.} \end{cases}$$

Then w is a barrier in x_0 (check it).

Theorem 2. [Wiener theorem] Let Ω be an open bounded set, $g \in L^{\infty}(\partial \Omega)$. Let $x_0 \in \partial \Omega$.

If x_0 is regular (with respect to the Laplacian) and g is continuous in x_0 , then (4) holds in x_0 .

Proof. Let w be a barrier in x_0 . Fix $\varepsilon > 0$, then there exists $\delta > 0$ such that for all $y \in \partial \Omega$ with $|y - x_0| \leq \delta$, $g(y) - g(x_0)| \leq \varepsilon$. Let $M = ||g||_{\infty}$ and fix k > 0 such that kw(y) > 2M for all $y \in \partial \Omega$ with $|y - x_0| > \delta$.

Consider the function $u(x) = \phi(x_0) - \varepsilon - kw(x)$. Then this function is subharmonic in Ω (since w is superharmonic). Moreover, if $y \in \partial \Omega$ with $|y - x_0| \leq \delta$, $u(y) \leq \phi(x_0) - \varepsilon \leq \phi(x_0)$ and if

 $y \in \partial \Omega$ with $|y - x_0| > \delta$, $u(y) \le \phi(x_0) - \varepsilon - 2M \le \phi(y)$. So $u \in S_g$ and then $H_g(x) \ge u(x)$ for every $x \in \overline{\Omega}$.

Consider the function $v(x) = \phi(x_0) + \varepsilon + kw(x)$. Then v is superharmonic in Ω . Moreover, if $y \in \partial \Omega$ with $|y - x_0| \leq \delta$, $v(y) \geq \phi(x_0) + \varepsilon \geq \phi(x_0)$ and if $y \in \partial \Omega$ with $|y - x_0| > \delta$, $v(y) \geq \phi(x_0) + \varepsilon + 2M \geq \phi(y)$. So by weak maximum principle $v(x) \geq u(x)$ for every $u \in S_g$ and then $H_g(x) \leq v(x)$ for every $x \in \overline{\Omega}$.

Therefore we proved that

$$\phi(x_0) - \varepsilon - kw(x) \le H_g(x) \le \phi(x_0) + \varepsilon + kw(x) \qquad \forall x \in \Omega$$

We let $x \to x_0$ and we get the conclusion.

From theorem 1 and theorem 2 we get the following result:

Corollary 1. For every $g \in C(\partial\Omega)$, the Dirichlet problem (1) admits a unique solution $u \in C^{\infty}(\Omega) \cap C(\overline{\Omega})$ iff all the boundary points of Ω are regular.

Proof. If g is continuous and all the points of the boundary are regular, then H_g is a solution of (1), and it is unique by weak maximu principle.

If (1) admits a solution for every continuous boundary data, take $x_0 \in \partial \Omega$ and the solution u to (1) with $g(x) = |x - x_0|$. Then the solution u to (1) is a barrier in x_0 .

Regular boundary points

It remains open the question: for which domains Ω all the boundary points are regular? Sufficient conditions for this property to hold can be stated in terms of local geometric (for n > 2) or topological (for n = 2) properties of the boundary.

We mention some of these conditions.

Definition. Let Ω be a open set of \mathbb{R}^n . Ω has the exterior ball property if at every point $x \in \partial \Omega$, there exists $y \in \mathbb{R}^n \setminus \Omega$ and r > 0 such that $B(y,r) \subset \mathbb{R}^n \setminus \overline{\Omega}$, $\overline{B(y,r)} \cap \overline{\Omega} = \{x\}$.

Remark. Observe that if Ω is convex, then the exterior ball condition is satisfied, due to the Hahn Banach separation theorem. Indeed at every point of $\partial\Omega$ it is possible to construct an hyperplane passing through that point and such that Ω is entirely contained in one of the two half spaces in which the space is divided by the hyperplane.

Observe that Ω of class C^1 is not sufficient to assure that the exterior ball condition is satisfied. E.g consider $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > x_1^2 \log |x_1|\}$. Then Ω is of class C^1 but in (0, 0) the exterior ball condition is not satisfied. Indeed to prove this, let $f(x) = x^2 \log |x|$ and $g(x) = \sqrt{r^2 - x^2} - r$, for r > 0 to be fixed. Note that f(0) = g(0) = 0. If the exterior ball condition were satisfied in (0, 0), then there would exist r > 0 such that f(x) > g(x) for every $x \in (-r, r), x \neq 0$. But this is not the case, since f'(x) < g'(x) for $x \to 0^+$ and f'(x) > g'(x) for $x \to 0^-$.

If Ω is of class C^2 , then it satisfies the exterior ball condition (also the interior ball condition). Let $x \in \partial \Omega$. Up to a suitable choice of coordinates, we can assume that x = 0 and that the exterior normal at Ω in 0 is $e_n = (0, \ldots, 0, 1)$. Let r > 0 and $f \in C^2(\mathbb{R}^{n-1}, \mathbb{R})$, such that $\Omega \cap B(0, r) = \{x_n < f(x_1, \ldots, x_{n-1})\}$. Note that f(0) = 0, and that Df(0) = 0. By Taylor theorem, $|f(x)| \leq M(x_1^2 + \cdots + x_{n-1}^2)$ for some M and x in a neighbourhood of 0. Let $y = \delta e_n$, $\delta > 0$ (note that $y \in \mathbb{R}^n \setminus \Omega$). We show that we can choose $\delta > 0$ sufficiently small such that $B(y, \delta) \subset \mathbb{R}^n \setminus \overline{\Omega}, B(y, r) \cap \overline{\Omega} = \{0\}$. If $x \in \overline{B(y, \delta)}, x \neq 0$, then $|x - y|^2 = |x|^2 - 2\delta x_n + \delta^2 \leq \delta^2$. So $|x|^2 - 2\delta x_n \leq 0$. Then $f(x_1, \ldots, x_{n-1}) \leq M|x|^2 \leq 2M\delta x_n < x_n$ if $\delta < \frac{1}{2M}$, which implies that $x \in \mathbb{R}^n \setminus \Omega$.

Proposition 2. Let Ω be a bounded open set and $x_0 \in \partial \Omega$ such that in x_0 it is satisfied the exterior ball condition: there exists $y_0 \in \mathbb{R}^n \setminus \Omega$ and $r_0 > 0$ such that $\overline{B}(y_0, r_0) \cap \overline{\Omega} = \{x_0\}$. Then x_0 is regular (with respect to the Laplacian).

Proof. A barrier in x_0 is the function

$$w(x) = \begin{cases} \frac{1}{r_0^{n-2}} - \frac{1}{|x-y_0|^{n-2}} & n > 2\\ \log \frac{|x-y_0|}{r_0} & n = 2. \end{cases}$$

Proposition 3. Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $0 \in \partial \Omega$ and consider the polar coordinates r, θ with origin 0. Suppose there exists a neighbourhood U of 0 such that in $\Omega \cap U$ a single valued branch of θ is defined. Then 0 is regular.

Proof. A barrier in 0 is the function

$$w(\rho, \theta) = -\frac{\log r}{(\log r)^2 + \theta^2}$$

Note that w is the real part of the complex function $-\frac{1}{\log(z)}$.

Remark. Similarly, a point on the boundary of $\Omega \subset \mathbb{R}^2$ is regular if it is accessible from the complement of Ω by a simple arc. The same is not true in general for $n \geq 3$ (for more details, see the example due to Lebesgue ([DiBenedetto, PDEs, chapter 2, section 7.2].

Proposition 4. Let Ω be a bounded open set and $x_0 \in \partial \Omega$ such that in x_0 it is satisfied the exterior cone condition: there exists a cone C with int $C \neq \emptyset$ and a neighbourhood U of x_0 such that such that $(x_0 + C) \cap U \subset \mathbb{R}^n \setminus \Omega$. Then x_0 is regular (with respect to the Laplacian).

We recall the definition of cone.

Definition (Cone). $C \subseteq \mathbb{R}^n$ is a (convex) cone if for every $x, y \in C$ then $x + y \in C$ and $\lambda x \in C$ for every $\lambda > 0$.

Observe $C \subseteq \mathbb{R}^2$ is a convex circular cone iff there exists $\alpha \in (0, 2\pi)$ (called the opening angle) such that, up to a suitable change of coordinates,

$$C = \{ x \in \mathbb{R}^2 \mid \arg x \in (-\alpha, \alpha) \}$$

where $\arg x$ is the argument of x $(\arctan(x_2/x_1))$. A similar representation formula holds for spherical cones in \mathbb{R}^n (using spherical coordinates).

Remark. It is possible to prove that if Ω is of class \mathcal{C}^1 , then at every boundary point of Ω it is satisfied the exterior cone condition (and also the interior cone condition).

Actually, in order to the (exterior and interior) cone condition to be satisfied it is sufficient that the boundary of Ω is Lipschitz. This means that at every $x \in \partial \Omega$ there exists a neighbourhood U of x, a open bounded set $D \subseteq \mathbb{R}^{n-1}$ and a Lipschitz function $\phi: D \to \mathbb{R}$ such that (up to a suitable orthogonal transformation of coordinates)

$$\partial \Omega \cap U = \{ (x, \phi(x)) \mid x \in D \}.$$

Proof. We prove the statement only in dimension 2 (the proof in dimension $n \geq 3$ is completely analogous, only a bit more involved).

Fix $x_0 \in \partial \Omega$ and first of all observe that we can reduce to the case of spherical cone (with opening angle α). Moreover, up to a translation, we can assume $x_0 = 0$. Consider the function

$$w(\rho,\theta) = \rho^{\lambda} \cos(\lambda\theta)$$

with $\lambda > 0$ to be fixed. Put w(0) = 0.

The Laplacian in polar coordinates reads as follows (exercise)

$$u_{\rho\rho} + \frac{1}{\rho}u_{\rho} + \frac{1}{\rho^2}u_{\theta\theta}.$$

Then $-\Delta w(x) = 0$ for every $x \neq 0$. So in particular w is harmonic in Ω , and w(0) = 0.

Finally if we choose $\lambda = \frac{\pi}{2\alpha}$, then the function $-\rho^{\lambda}\cos(\lambda\theta)$ is a barrier at x_0 , since it is harmonic in Ω , it is zero in 0 and it is positive on points with argument $\theta \in (\alpha, 2\pi - \alpha)$.