Introduzione alle equazioni alle derivate parziali,
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An application in population genetics.

We refer to the paper by Aronson and Weinberger **Multidimensional nonlinear diffusion arising in population genetics**, Advances in Mathematics 30, pag 33-76, 1978.

Let consider a population of diploid individuals in a habitat Ω. Assume that the gene in specific chromosome pair appears in 2 allelic forms a, A. Then the population is divided in 3 types: one with genotype aa or AA (homozygotes) and one with genotype aA (heterozygotes). Moreover we assume that the death rate in the population depends on the genotype, and we define 3 positive constants $\tau_1, \tau_2, \tau_3 \in (0,1)$, which are the death rates respectively of type aa, aA and AA. We can assume without loss of generality that one type of homozigotes has death rate bigger than the other, so we assume

$$\tau_1 \geq \tau_3.$$

Let
$\rho^1(x,t)$= density of the population with genotype aa at place x at time t,
$\rho^2(x,t)$= density of the population with genotype aA,
$\rho^3(x,t)$= density of the population with genotype AA.

Assume that the growth rate of the population is constant, say $r > 0$, and that the population is spreading in the habitat Ω with diffusion constant 1. Let $\rho(x,t) = \rho^1(x,t) + \rho^2(x,t) + \rho^3(x,t)$ the total density. Under our assumptions, the densities satisfy the following system

$$\begin{cases} \rho^1_t - \Delta\rho^1 = -\tau_1\rho^1 + \frac{r}{\rho}\left(\rho^1 + \frac{1}{2}\rho^2\right)^2 \\ \rho^2_t - \Delta\rho^2 = -\tau_2\rho^2 + \frac{r}{\rho}\left(\rho^1 + \frac{1}{2}\rho^2\right)\left(\rho^3 + \frac{1}{2}\rho^2\right) \\ \rho^3_t - \Delta\rho^3 = -\tau_3\rho^3 + \frac{r}{\rho}\left(\rho^3 + \frac{1}{2}\rho^2\right)^2. \end{cases}$$

Define

$$u(x,t) = \frac{\rho^3 + \frac{1}{2}\rho^2}{\rho}$$

be the relative density of allele A at place x at time t. Under appropriate assumptions (in particular assuming that $|\tau_1 - \tau_2| + |\tau_2 - \tau_3| + |\tau_3 - \tau_1|$ is small), it is possible to justify the fact that u solves the following semilinear problem

$$\begin{cases} u_t - \Delta u = F(u) & x \in \Omega \times (0,+\infty) \\ u(x,0) = u_0(x) & x \in \Omega \\ u(x,t) = 0 & x \in \partial\Omega, t > 0 \end{cases}$$

where

$$F(u) = u(1-u)[(\tau_1 - \tau_2) - (\tau_1 + \tau_3 - 2\tau_2)u].$$

The initial distribution is $u_0(x) \in \mathcal{C}(\overline{\Omega})$, with

$$0 \leq u_0(x) \leq 1 \qquad \forall x \in \Omega$$

(since it is a density).

The euristic argument for this derivation can be found in the paper by Fischer (1937) and in previous papers by Aronson and Weinberger.

First of all observe that $0, 1$ are equilibria of the system, and moreover $0 \leq u_0 \leq 1$. So by comparison principle, we obtain that

$$0 \leq u(x,t) \leq 1 \qquad \forall (x,t) \in \overline{\Omega} \times [0,+\infty).$$

We consider 3 different cases.

Heterozygote intermediate Assume that $\tau_3 \le \tau_2 < \tau_1$.

Under this case, note that $F(s) > 0$ for all $s \in (0,1)$, $F'(0) > 0$ and $F'(1) < 0$. In this case the state 1 is stable, whereas 0 is unstable. More precisely, we get that for every initial distribution u_0 such that $u_0 \not\equiv 0$, then

$$\lim_{t \to +\infty} u(x,t) = 1$$

uniformly, with exponential rate depending on $F'(1)$.

In this case, we expect that in large time allele a will disappear.

Heterozygote superior Assume that $\tau_2 < \tau_3 \le \tau_1$.

In this case

$$c := \frac{\tau_1 - \tau_2}{(\tau_1 - \tau_2) + (\tau_3 - \tau_2)} \in (0,1)$$

is another equilibrium of the system in $[0,1]$ besides $0, 1$. Moreover $F(s) > 0$ for every $s \in (0,c)$, $F(s) < 0$ for every $s \in (c,1)$, $F'(0) > 0, F'(1) > 0, F'(c) < 0$. So c is the only stable equilibrium and that for every initial distribution u_0 such that $u_0 \not\equiv 0, u_0 \not\equiv 1$, then

$$\lim_{t \to +\infty} |u(x,t) - c| = 0$$

uniformly, with exponential rate depending on $F'(c)$.

In this case, we expect that in large time, the distribution of allele A is constant (given by c) and then also the distribution of allele a is constant (given by $1-c$).

Heterozygote inferior Assume that $\tau_3 \le \tau_1 < \tau_2$.

Also in this case there is another equilibrium of the system, which the same as above

$$c := \frac{\tau_2 - \tau_1}{(\tau_2 - \tau_1) + (\tau_2 - \tau_3)} \in (0,1).$$

Differently from the previous case, we get $F(s) < 0$ for every $s \in (0,c)$, $F(s) > 0$ for every $s \in (c,1)$, $F'(0) < 0, F'(1) < 0, F'(c) > 0$. In this case $0, 1$ are two stable equilibria and c is unstable. If the initial distribution u_0 is such that $u_0 \not\equiv c$, and $u_0(x) \le c$ for every x, then

$$\lim_{t \to +\infty} u(x,t) = 0$$

uniformly, with exponential rate depending on $F'(0)$.

If the initial distribution u_0 is such that $u_0 \not\equiv c$, and $u_0(x) \ge c$ for every x, then

$$\lim_{t \to +\infty} u(x,t) = 1$$

uniformly, with exponential rate depending on $F'(1)$.

This means that, if the initial distribution of allele A is under a given threshold (c), then we expect that in large time allele A will disappear. On the other hand, if allele s is above the same threshold (c), then we expect that in large time allele a will disappear.

The latest case, the case of heterozygote inferior, is the most interesting one. In this case, we have 2 stable equilibria and one unstable equilibrium. The next point will be to study which one between the two stable equilibria is more attractive.