Introduzione alle equazioni alle derivate parziali,
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Maximum principle for parabolic operators: an application to semilinear equations

Let $\Omega \subset \mathbb{R}^n$ be a open set , $T > 0$ and $u_t + Lu$ a parabolic operator for which weak maximum principle holds (e.g we can think of $u_t - \Delta u$).

We want to study some qualitative properties of solutions to the semilinear Cauchy Dirichlet problem

$$(C) \begin{cases} u_t + Lu = F(u) & (x,t) \in \Omega \times (0,+\infty) \\ u(x,t) = g(x,t) & x \in \partial\Omega \ t \in (0,+\infty) \\ u(x,0) = u_0(x) & x \in \overline{\Omega}. \end{cases}$$

The operator is $u_t + L(u) - F(u)$, which is semilinear, so in general we cannot expect to apply all the theory developed for linear operators.

We will assume the following condition on the reaction term F.

Assumption 1. Let $F : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function, i.e. for every K compact of $\mathbb{R}$ there exists $L_K > 0$ such that

$$|F(a) - F(b)| \leq L_K |a - b| \qquad \forall a, b \in K.$$

Comparison priciple

Proposition 1 (Comparison principle)**.** *Let $u, v \in \mathcal{C}^{2,1}(\Omega_T) \cap \mathcal{C}(\overline{\Omega_T})$ such that $u_t + Lu \leq F(u)$ and $v_t + Lv \geq F(v)$ in Ω_T.*

Then if $u \leq v$ on $\partial^\star \Omega_T$, $u \leq v$ in $\overline{\Omega}_T$.

Proof. Define $C_T = \max_{\overline{\Omega}_T}(|u|+|v|)$ (this exists finite since u, v are continuous and $\overline{\Omega}_T$ is compact). Define now

$$c_T(x,t) = \begin{cases} \frac{F(u(x,t) - F(v(x,t))}{u(x,t) - v(x,t)} & (x,t) \in \Omega_T \ u(x,t) \neq v(x,t) \\ 0 & (x,t) \in \Omega_T \ u(x,t) = v(x,t). \end{cases} \tag{1}$$

Note that, since F is Lipschitz in the compact set $[-C_T, +C_T]$, then $c_T(x,t)$ is bounded (and its sup norm is given by the Lipschitz constant of F in $[-C_T, +C_T]$).

Define $w = u - v$. It satisfies $w_t + Lw \leq F(u) - F(v) = c_T(x,t)w$, so in particular w is a subsolution of the linear equation $w_t + Lw - c_T w \leq 0$ with $w \leq 0$ on $\partial^\star \Omega_T$. So, by weak comparison principle, we obtain $w \leq 0$ in $\overline{\Omega}_T$. □

Theorem 1 (Uniqueness)**.** *Let $f \in \mathcal{C}(\Omega \times (0,+\infty))$, $g \in \mathcal{C}(\partial\Omega \times (0,+\infty))$ and $u_0 \in \mathcal{C}(\overline{\Omega})$, such that $g(x,0) = u_0(x)$. There the problem (C) admits at most one solution $u \in \mathcal{C}^2(\Omega \times (0,+\infty) \cap \mathcal{C}(\overline{\Omega} \times [0,+\infty))$.*

Proof. Let u_1, u_2 two solutions. Fix $T > 0$ and consider the problem in Ω_T. Then $w = u_1 - u_2$ satisfies $w_t + Lw = c(x,t)w$, where c_T is defined as in (1). By comparison principle we get that $u_1 = u_2$ in $\overline{\Omega}_T$ and conclude by arbitrariness of $T > 0$. □

Stability properties of equilibria.

We consider the Cauchy Dirichlet problem in $\Omega \times (0,+\infty)$

$$(RD) \begin{cases} u_t + Lu = F(u) & (x,t) \in \Omega \times (0,+\infty) \\ u(x,0) = u_0(x) & x \in \Omega \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,+\infty). \end{cases}$$

Definition. *A value $\overline{u} \in \mathbb{R}$ such that $F(\overline{u}) = 0$ is called an equilibrium (or a stationary solution) of the system. Indeed the constant function $u(x,t) \equiv \overline{u}$ is a solution to the equation $u_t + Lu = F(u)$.*

We start with a simple observation.

Proposition 2. *Assume that $F \in \mathcal{C}^1(\mathbb{R})$ and*

$$F(s)s < 0 \qquad \forall s \neq 0.$$

This in particular impies that 0 is an equilibrium. Then, if u solves (RD) then

$$\lim_{t\to+\infty} u(x,t) = 0 \qquad \textit{uniformly in } \overline{\Omega}.$$

If moreover

$$F'(0) = -\alpha < 0,$$

then for every $0 < \beta < \alpha$ there exists $C_\beta > 0$ such that

$$|u(x,t) \leq C_\beta e^{-\beta t}$$

for every $x \in \overline{\Omega}$ and $t > 0$.

Proof. Consider the ordinary differential equation

$$U' = F(U).$$

Then if $U(t)$ solve this ODE, then it is also a solution (independent of x) to $u_t + Lu = F(u)$.

Consider the Cauchy problem

$$\begin{cases} U'(t) = F(U(t)) & t > 0 \\ U(0) = \|u_0\|_\infty. \end{cases} \tag{2}$$

This Cauchy problem admits a unique solution U, moreover $U'(t) < 0$ for every $t > 0$, $0 < U(t) \leq \|u_0\|_\infty$ and $\lim_{t\to+\infty} U(t) = 0$. Moreover $U(t)$ is also a solution to $U_t + LU = F(U)$ with $U \geq u_0$ for every $x \in \Omega$, and $U > 0$. So by comparison principle, if $u(x,t)$ is the solution to (RD),

$$u(x,t) \leq U(t) \qquad \forall x \in \overline{\Omega}, t \geq 0. \tag{3}$$

Analogously we consider the solution $V(t)$ to the Cauchy problem (2) with initial data $-\|u_0\|_\infty$. Reasoning as before, $V'(t) > 0$ for every $t > 0$, $-\|u_0\|_\infty \leq V(t) < 0$ and $\lim_{t\to+\infty} V(t) = 0$. Again by comparison we get

$$u(x,t) \geq V(t) \qquad \forall x \in \overline{\Omega}, t \geq 0. \tag{4}$$

(3) and (4) give the conclusion.

Assume now that $F'(0) = -\alpha < 0$. Fix $0 < \beta < \alpha = |F'(0)|$ and take $C = C(\beta)$ such that

$$F(s) \leq -\beta s \qquad \forall s \in [0, C].$$

Take U solution to (2) with initial data $\|u_0\|_\infty$ and fix $T = T(C) = T(\beta) \geq 0$ such that $U(T) \leq C$ (recall that U is monotonically decreasing to 0, starting from $\|u\|_\infty$. So, for every $t \geq T$, $U(t) \leq C$ and moreover $F(U(t)) \leq -\beta U(t)$.

So, by a comparison argument for solutions to Cauchy problems,

$$U(t) \leq Ce^{\beta T} e^{-\beta t} \qquad \forall t \geq T.$$

Eventually taking a bigger constant $C = C(\|u_0\|_\infty, \beta)$, we obtain that

$$U(t) \leq Ce^{-\beta t} \qquad \forall t \geq 0.$$

So, again by comparions principle, we can rewrite inequality (3) as follows: for every $\beta < |F'(0)|$ there exists a constant $C(\|u_0\|\infty, \beta)$ such that

$$u(x,t) \leq Ce^{-\beta t} \qquad \forall x \in \overline{\Omega}, t \geq 0.$$

As for the inequality (4) we argue analogously. ☐

Definition. *Let $\overline{u} \in \mathbb{R}$ be a equilibrium. $\overline{u}$ is stable (with respect to uniform convergence) if there exists $\delta > 0$ such that for every $u_0 \in \mathcal{C}(\overline{\Omega})$ with*

$$|u_0(x) - \overline{u}| \leq \delta \qquad \forall x \in \overline{\Omega}$$

then the solution u to (RD) satisfies

$$|u(x,t) - \overline{u}| \to 0 \qquad \text{as } t \to +\infty, \text{ uniformly in } \overline{\Omega}.$$

An equilibrium $\overline{u}$ is unstable (with respect to uniform convergence) if for every $\delta > 0$ there exists $t_\delta > 0$ such that for every $u_0 \in \mathcal{C}(\overline{\Omega})$ with

$$|u_0(x) - \overline{u}| \leq \delta \qquad \forall x \in \overline{\Omega}$$

then the solution u to (RD) satisfies

$$\sup_{x \in \overline{\Omega}} |u(x,t) - \overline{u}| \geq \delta \qquad \text{if } t \geq t_\delta.$$

Proposition 2 gives a sufficient condition for an equilibrium to be stable, as we see in the following proposition.

Proposition 3. *Let $\overline{u}$ be an equilibrium, such that $F'(\overline{u}) < 0$. Then there exists $\delta > 0$ such that for every $\beta < |F'(\overline{u})|$, there exists $C = C(\beta, \delta)$ for which if $|u_0 - \overline{u}| \leq \delta$, the solution to (RD) with initial datum u_0 satisfies*

$$|u(x,t) - \overline{u}| \leq Ce^{-\beta t} \qquad \forall t > 0.$$

In particular $\overline{u}$ is a stable equilibrium (actually it is exponentially asymptotically stable).

Proof. The proof is a simply adaptation of the arguments in the proof of Proposition 2, once one observes the following.

Let $\delta > 0$ such that $F(s) < 0$ for every $s \in (\overline{u}, \overline{u} + \delta)$ and $F(s) > 0$ for every $s \in (\overline{u} - \delta, \overline{u})$. Consider the solution U to (2) with initial data $\overline{u} + \delta$. Then, by uniqueness of solutions to the Cauchy problem, $U'(t) < 0$, $U(t) \in [\overline{u}, \overline{u} + \delta]$ for every $t \geq 0$ and $\lim_{t \to +\infty} U = \overline{u}$. Analogously, if V is a solution to (2) with initial datum $\overline{u} - \delta$, then $V'(t) > 0$, $V(t) \in [\overline{u} - \delta, \overline{u}]$ for every $t \geq 0$ and $\lim_{t \to +\infty} V = \overline{u}$. So, by comparison, $|u(x,t) - \overline{u}| \to 0$ uniformly in $\overline{\Omega}$ as $t \to +\infty$.

To get the exponential decay estimates, we proceed in a similar way. □

Remark. Repeating exactly the same arguments in the previous proof, it is possible to prove that if $\overline{u}$ is an equilibrium with $F'(\overline{u}) > 0$, then it is unstable (with respect to uniform convergence).