Large deviations for some fast stochastic volatility models by viscosity methods

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Nonlinear PDEs, in the occasion of Martino Bardi’s birthday
Joint work with Martino Bardi and Annalisa Cesaroni
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Outline

(i) Introduction
- Definition of the problem and motivations/applications

(ii) Main results and methods
- PDEs methods (mainly viscosity solutions methods) for homogenization (singular perturbation) problem of Hamilton-Jacobi-Bellman equations
Introduction on the stochastic models

The evolution of the price of a stock $S$ is described by

$$d\log S_t = \gamma dt + \sigma dW_t, \quad t=\text{time}, \ W_t = \text{Wiener proc.},$$

and the classical Black-Scholes formula for option pricing is derived assuming parameters are constants.

In reality the parameters of such models are not constants. In particular, the volatility $\sigma$, a measure for variation of price over time, is not constant but exhibits random behaviour.
Stochastic volatility

\( \sigma \) rather looks like an **ergodic mean reverting process**:
modeled as a positive function \( \sigma = \sigma(Y_t) \) of a stochastic process \( Y_t \) with
- **mean reversion** (the time it takes for agents to adjust their thresholds to current market conditions)-closely related to the concept of **ergodicity**

Main example: **Ornstein-Uhlenbeck process**

\[ dY_t = (m - Y_t)dt + \tau dW_t, \quad Y_0 = y_0 \in \mathbb{R}^m. \]

Refs.: Fleming-Soner, 2-end edition 2006, Fouque-Papanicolaou-Sircar 2000,...
Fast stochastic volatility

It is argued in the book

**Fouque, Papanicolaou, Sircar:** Derivatives in financial markets with stochastic volatility, 2000,

that $Y_t$ evolves on a faster time scale than the stock prices, modelling better the typical bursty behavior of volatility, see previous picture.

**Multiple time scale systems and singular perturbation:** for $\delta > 0$

$$dY_t = \frac{1}{\delta} b(Y_t) dt + \sqrt{\frac{2}{\delta}} \tau(Y_t) dW_t \quad Y_0 = y_0 \in \mathbb{R}^m.$$ 

Passing to the limit as $\delta \to 0$ is a classical singular perturbation problem.

Some Refs: **Bensoussan, Kushner, Khasminskii, Pardoux, Borkar, Gaitsgory, Alvarez, Bardi...**
Stochastic system with fast oscillating random parameter

Consider a stochastic system in $\mathbb{R}^n$ with random coefficients

$$dX_t = \phi(X_t, Y_t)dt + \sqrt{2}\sigma(X_t, Y_t)dW_t, \quad X_0 = x_0 \in \mathbb{R}^n.$$ 

$Y_t$ modelled as a markov process evolving on a faster time scale $\tau = \frac{t}{\delta}$:

$$dY_t = \frac{1}{\delta}b(Y_t)dt + \sqrt{\frac{2}{\delta}}\tau(Y_t)dW_t, \quad Y_0 = y_0 \in \mathbb{R}^m.$$ 

Notation: $X_t$ are the slow components of the system, and $Y_t$ are the fast components.
Main assumptions

Periodic case

Assumptions implying some compactness of the process $Y_t$:
Periodicity of the coefficients with respect to the $y$-variable

Non compact case

Assumptions implying the ergodicity of the process $Y_t$.
We treat Ornstein-Uhlenbeck type fast processes. Motivated by the applications in finance.

We will give more precise assumptions later in the talk.

Further assumptions (which will hold for the rest of the talk):

- $\tau(\cdot)$ is bounded and uniformly non degenerate;
- $\sigma(\cdot, x)$ is bounded for any $x \in \mathbb{R}^n$. 
Small time asymptotics for the system

Small time behaviour of the system: rescale time as

\[ t \to \varepsilon t. \]

Put

\[ \delta = \varepsilon^\alpha, \text{ with } \alpha > 1. \]

We consider the limit of the system for \( \varepsilon \to 0 \)

\[
\begin{aligned}
\begin{cases}
\quad dX_t^\varepsilon = \varepsilon \phi(X_t^\varepsilon, Y_t^\varepsilon)dt + \sqrt{2\varepsilon} \sigma(X_t^\varepsilon, Y_t^\varepsilon)dW_t, & X_0^\varepsilon = x_0 \in \mathbb{R}^n \\
\quad dY_t^\varepsilon = \frac{1}{\varepsilon^{\alpha-1}} b(Y_t^\varepsilon)dt + \sqrt{\frac{2}{\varepsilon^{\alpha-1}}} \tau(Y_t^\varepsilon)dW_t, & Y_0^\varepsilon = y_0 \in \mathbb{R}^m.
\end{cases}
\end{aligned}
\]
AIM: prove a Large Deviation Principle (LDP) for the process $X_{t}^{\varepsilon}$ (i.e. for probability measures generated by the laws of $X_{t}^{\varepsilon}$).

In other words, for every $t > 0$ and for any open set $B \subseteq \mathbb{R}^{n}$

**Large Deviation Principle**

$$P(X_{t}^{\varepsilon} \in B) = e^{-\inf_{x \in B} \frac{I(x;\mathbf{x}_{0},t)}{\varepsilon} + o\left(\frac{1}{\varepsilon}\right)}, \text{ as } \varepsilon \to 0.$$  

for some (good) rate function $I$, non-negative and continuous, which we will define in the next slides.
Asymptotic estimates for volatility of option prices near maturity

Avellaneda and collaborators (2002, 2003) used theory of large deviations to give asymptotic estimates for the Black-Scholes implied volatility of option prices near maturity (small time) in models with constant (local) volatility.

Remark

We derive analogous asymptotic estimates in the previously defined models (with stochastic volatility). In this case, further condition of periodicity/ergodicity on the fast process.

Remark

In this model:

- $\varepsilon$: short maturity of the option
- $\delta = \varepsilon^\alpha$: rate of mean reversion of the volatility.

- different scaling;
- different methods based on weak convergence.

*J. Feng, J.-P. Fouque, R. Kumar* (2012):

- systems of the form that we defined for $\alpha = 2, 4$,
  in the one-dimensional case $n = m = 1$, assuming that $Y_t$ is an Ornstein-Uhlenbeck process
  $(b(y) = m - y, \quad \tau(y) = \tau)$,
  the coefficients in the equation for $X_t$ do not depend on $X_t$;
- methods are based on the monograph by Feng and Kurtz, *Large deviations for stochastic processes* 2006.
Bryc’s inverse Varadhan lemma

Assume that for all $t > 0$

test1

1 $X_t^\varepsilon$ is exponentially tight.

2 Notation: for every $h$ bounded and continuous

\[ \nu^\varepsilon (t, x_0, y_0) = \varepsilon \log E \left[ e^{\frac{-1}{\varepsilon} h(X_t^\varepsilon)} \mid X_0 = x_0, Y_0 = y_0 \right]. \]

Let the limit

\[ \lim_{\varepsilon \to 0} \nu^\varepsilon (t, x_0, y_0) = \nu (t, x_0) \]

exists finite.

Then $X_t^\varepsilon$ satisfies a large deviation principle with good rate function

\[ I(x, t, x_0) = \sup_{h \in BC(\mathbb{R}^n)} \{ h(x) - \nu (t, x_0) \}. \]
Bryc’s inverse Varadhan lemma

Assume that for all $t > 0$

1. $X_t^\varepsilon$ is exponentially tight.
2. Notation: for every $h$ bounded and continuous

$$v^\varepsilon(t, x_0, y_0) = \varepsilon \log E \left[ e^{\varepsilon^{-1} h(X_t^\varepsilon)} \mid X_0 = x_0, Y_0 = y_0 \right].$$

Let the limit

$$\lim_{\varepsilon \to 0} v^\varepsilon(t, x_0, y_0) = v(t, x_0)$$

exists finite.

Then $X_t^\varepsilon$ satisfies a large deviation principle with good rate function

$$I(x, t, x_0) = \sup_{h \in BC(\mathbb{R}^n)} \{ h(x) - v(t, x_0) \}.$$ 

We focus on point 2 of the Lemma, which can be considered our main result.
Define

$$u^\varepsilon(t, x_0, y_0) := E \left[ g(X_t^\varepsilon) \mid X_0 = x_0, Y_0 = y_0 \right], \quad x_0 \in \mathbb{R}^n, \ y_0 \in \mathbb{R}^m, \ t \geq 0,$$

with $g$ bounded and continuous function.

The associated HJB equation to $u^\varepsilon$ is the following parabolic linear pde ($b, \tau$ computed in $y$, $\phi, \sigma$ in $(x, y)$):

$$u_t^\varepsilon = \varepsilon \left( \text{tr}(\sigma\sigma^T D_{xx}^2 u^\varepsilon) + \phi \cdot D_x u^\varepsilon \right) + 2\varepsilon^{1-\alpha} \text{tr}(\sigma\tau^T D_{xy}^2 u^\varepsilon)$$

$$+ \varepsilon^{1-\alpha} \left( b \cdot D_y u^\varepsilon + \text{tr}(\tau\tau^T D_{yy}^2 u^\varepsilon) \right)$$

complemented with

$$u^\varepsilon(0, x, y) = g(x).$$
Logarithmic transformation method

Take \( g(x) = e^{\frac{h(x)}{\varepsilon}} \) with \( h \) bounded and continuous and

\[
v^\varepsilon(t, x_0, y_0) := \varepsilon \log E \left[ e^{\varepsilon^{-1} h(X_t^\varepsilon)} \middle| X_0^\varepsilon = x_0, Y_0^\varepsilon = y_0 \right], \quad x_0 \in \mathbb{R}^n, y_0 \in \mathbb{R}^m, \quad t \geq 0.
\]
Logarithmic transformation method

Take \( g(x) = e^{\frac{h(x)}{\varepsilon}} \) with \( h \) bounded and continuous and

\[
v^\varepsilon(t, x_0, y_0) := \varepsilon \log E \left[ e^{\varepsilon^{-1} h(X_\varepsilon^t)} \mid X_0^\varepsilon = x_0, Y_0^\varepsilon = y_0 \right], x_0 \in \mathbb{R}^n, y_0 \in \mathbb{R}^m, t \geq 0.
\]

The associated HJB equation to \( v^\varepsilon \) is the following parabolic pde with quadratic nonlinearity in the gradient \((b, \tau \) computed in \( y \), \( \phi, \sigma \) in \( (x, y) \)).

\[
v_t^\varepsilon = \left| \sigma^T D_x v^\varepsilon \right|^2 + \varepsilon \left( \text{tr}(\sigma\sigma^T D_{xx}^2 v^\varepsilon) + \phi \cdot D_x v^\varepsilon \right) + 2\varepsilon^{-\frac{\alpha}{2}} (\tau\sigma^T D_x v^\varepsilon) \cdot D_y v^\varepsilon + 2\varepsilon^{1-\frac{\alpha}{2}} \text{tr}(\sigma\tau^T D_{xy}^2 v^\varepsilon) + \varepsilon^{1-\alpha} (b \cdot D_y v^\varepsilon + \text{tr}(\tau\tau^T D_{yy}^2 v^\varepsilon)) + \varepsilon^{-\alpha} \left| \tau^T D_y v^\varepsilon \right|^2
\]

complemented with

\[
v^\varepsilon(0, x, y) = h(x).
\]
Main result—Convergence by PDE methods

**Theorem**

Let $h$ be continuous and bounded. Under our standing assumptions (see sequent slides),

$$v^\varepsilon(x, y, t) = \varepsilon \log Ee^{\frac{h(X^\varepsilon_t)}{\varepsilon}} \to v(x, t)$$

locally uniformly in $y$ where $v$ is the unique viscosity solution to the effective equation

$$\begin{cases}
    v_t - \bar{H}(x, Dv) = 0 & \text{in } ]0, T[ \times \mathbb{R}^n, \\
    v(0, x) = h(x) & \text{in } \mathbb{R}^n.
\end{cases}$$

where $\bar{H}$ is the limit or effective Hamiltonian.
The effective Hamiltonian

Three regimes depending on $\alpha$; i.e. on how fast the volatility oscillates relative to the horizon length:

\[
\begin{align*}
\alpha > 2 & \quad \text{C.P.: uniformly elliptic sec. ord. linear equation} \\
\alpha = 2 & \quad \text{C.P.: uniformly elliptic sec. ord. quadratic in the gradient} \\
\alpha < 2 & \quad \text{C.P.: first order coercive in the gradient.}
\end{align*}
\]

Remark

*In all the regimes $\tilde{H}$ is continuous on $\mathbb{R}^n \times \mathbb{R}^n$ and convex in $p$ and*

\[
\inf_y \left| \sigma^T (\bar{x}, y) \bar{p} \right|^2 \leq \tilde{H}(\bar{x}, \bar{p}) \leq \sup_y \left| \sigma^T (\bar{x}, y) \bar{p} \right|^2.
\]
Periodic case

Under periodicity condition with respect to $y$ of the coefficients:

- all the ranges of $\alpha$;
- several representation formulas for $\bar{H}$;
- once proved the comparison for $\bar{H}$ (not trivial), the convergence is quite standard.
Non compact case

Without compactness assumptions: fast process “mainly” of Ornstein-Uhlenbeck type

\[ b(y) = m - y, \quad \tau(y) = \tau, \]

- \( \alpha \geq 2; \)
- further assumption on the volatility (only if \( \alpha = 2 \)): ex. \( \sigma \) of the type
  \[ \| \sigma(x, y) - \sigma(x, z) \|_\infty \leq \frac{1}{1 + |z| + |y|} |z - y| \]

Remark

*Main difficulties due to the unboundedness of the fast variable. From now on we focus on the non compact case.*
The ergodic problem, non compact case

For every $\bar{x}, \bar{p}$ fixed, find the unique constant $\tilde{H}(\bar{x}, \bar{p})$ such that there is a solution $w$ to

\[
\begin{cases}
-(b + 2\tau\sigma^T\bar{p}) \cdot Dw - \text{tr}(\tau\tau^TD^2w) - |\tau^TDw|^2 - |\sigma^t\bar{p}|^2 = -\tilde{H} \quad \text{in } \mathbb{R}^m \\
\exists C > 0 \text{ s.t. } w(y) \leq C(1 + |y|^2)
\end{cases}
\]

- **EXISTENCE OF $(\tilde{H}, w)$**: ergodic approximation

  \[
  \delta w_\delta - (b + 2\tau\sigma^t p) \cdot Dw_\delta - \text{tr}(\tau\tau^TD^2w_\delta) - |\tau^TDw_\delta|^2 - |\sigma^t p|^2 = 0,
  \]
  then $\delta w_\delta \to \tilde{H}, w_\delta - w_\delta(0) \to w$.

  Lipschitz bound uniform in $\delta$ for $w_\delta$.

- **UNIQUENESS OF $\tilde{H}$**: Growth assumption on $w$ (Ichihara 2011, Khaise-Sheu 2006).

  It holds if $\exists C > 0, R > 0$ such that $b(x) \cdot x \leq -C|x|^2 \quad \forall |x| \geq R$.

  The main assumption on $b$ implies that there exists $K > 0$ such that $K(1 + |y|^2)$ is a supersolution to the approximate problem outside a compact set. $K$ independent of $\delta$. 
Supercritical case $\alpha > 2$: some remarks

Cell problem: for $(\bar{x}, \bar{p})$ fixed

$$-(b + 2\tau \sigma^T \bar{p}) \cdot Dw - \text{tr}(\tau \tau^T D^2 w) - |\sigma^t \bar{p}|^2 = -\bar{H} \quad \text{in } \mathbb{R}^m.$$ 

We prove the existence of $(\bar{H}, w)$ solution and we characterize $\bar{H}$ by the following formula

$$\bar{H}(\bar{x}, \bar{p}) = \int_{\mathbb{R}^m} |\sigma(\bar{x}, y)^T \bar{p}|^2 d\mu(y),$$

where $\mu$ is the unique invariant probability measure of

$$dY_t = (m - Y_t)dt + \sqrt{2\tau} dW_t,$$

Ref: M. Bardi, A. Cesaroni, L. Manca, Convergence by viscosity methods in multiscale financial models with stochastic volatility, 2010
Global Lipschitz bound

**Theorem**

Let $\bar{x}, \bar{p}$ be fixed. There exists a positive constant $C$, depending on the data, such that

$$\sup_{y \in \mathbb{R}^m} |D_y w(y; \bar{x}, \bar{p})| \leq C,$$

where $w \in C^2(\mathbb{R}^m)$ be a solution of the cell problem.

**Key elements of the proof**

- Uniform ellipticity, inspired by Ishii-Lions method
- **IMPORTANT**: Ornstein-Uhlenbeck nature of the fast process, in order to get bounds independent of the $L^\infty$ norm of $w_\delta$:

  $$(b(y) - b(x)) \cdot (x - y) \geq |x - y|^2.$$
Convergence: key points of the method

AIM: \( v^\varepsilon(t, x, y) \to v(t, x) \) loc unif in \( y \).

MAIN DIFFICULTY: unboundedness of the fast variable.

Procedure of the relaxed-semi limits (by Barles-Perthame).

Perturbed test function with Lyapounov function \( \chi \):

\[
\psi^\varepsilon(t, x, y) = \psi(t, x) + \varepsilon(w(y) + \eta \chi(y)), \quad \varepsilon, \eta > 0
\]

- \( \chi(y) \to +\infty \) as \( |y| \to +\infty \)
- \( -(b + 2\tau \sigma^T \bar{p}) \cdot D\chi - |\tau^T D\chi|^2 - \text{tr}(\tau \tau^T D^2\chi) \to +\infty \) as \( |y| \to +\infty \)
\[
\psi^\varepsilon(t,x,y) = \psi(t,x) + \varepsilon(w(y) + \eta \chi(y)), \varepsilon, \eta > 0
\]

**KEY POINTS:**

- **Recall:** $v^\varepsilon$ are bounded and $\chi \to +\infty$ as $|y| \to +\infty$. Existence of maximum points of $v^\varepsilon - \psi^\varepsilon$.

- **Control uniformly in $\varepsilon$** the sequence $y_\varepsilon$ of maximum of $v^\varepsilon - \psi^\varepsilon$:
  
  Main idea: use the Lyapounov property

  \[
  -(b + 2\tau \sigma^T \bar{p}) \cdot D\chi - |\tau^T D\chi|^2 - \text{tr}(\tau \tau^T D^2\chi) \to +\infty \text{ as } |y| \to +\infty
  \]

  and get a contradiction with the viscosity inequality.

- **tricky point:** Nonlinearity of the equation: $|D(w + \eta \chi)|^2$: Global Lipschitz gradient estimate of $w$;

  Key element to conclude: comparison principle for the limit equation.
Ongoing works and some perspectives

- Generalization of the hypothesis on $Y_t$: Ornstein-Uhlenbeck only outside a ball.
  - Local uniform gradient estimates
- Gradient bounds for more general Hamiltonian with Ornstein-Uhlenbeck terms
- Homogenization and vanishing viscosity with similar scalings in fully nonlinear elliptic equations with random coefficients.
- Problems of portfolio optimization (Merton’s portfolio).
Thank you for the attention and
Happy birthday Martino!
Supercritical case $\alpha > 2$:

$$\bar{H}(x, p) = \int_{\mathbb{T}^m} |\sigma(x, y)^T p|^2 d\mu(y)$$

where $\mu$ is the invariant probability measure on the torus $\mathbb{T}^m$ of the stochastic process $dY_t = b(Y_t)dt + \sqrt{2}\tau(Y_t)dW_t$;

Critical case $\alpha = 2$:
- stochastic control formula, by an average with respect to the invariant measure of a suitable stochastic process,
- ergodic control formula,
- logarithmic-exponential type criterion per unit time on infinite time;

Subcritical case $\alpha < 2$:
- deterministic control formula,
- no correlation $\tau\sigma^T = 0$: $\bar{H}(x, p) = \max_{y \in \mathbb{R}^m} |\sigma^T(x, y)p|^2$
Representation formula for $\tilde{H}$, periodic case

SUPERCritical CASE $\alpha > 2$:

$$\tilde{H}(x, p) = \int_{\mathbb{T}^m} | \sigma(x, y)^T p |^2 d\mu(y)$$

where $\mu$ is the invariant probability measure on the torus $\mathbb{T}^m$ of the stochastic process $dY_t = b(Y_t)dt + \sqrt{2}\tau(Y_t)dW_t$;

CRITICAL CASE $\alpha = 2$:

- stochastic control formula, by an average with respect to the invariant measure of a suitable stochastic process,
- ergodic control formula,
- logarithmic-exponential type criterion per unit time on infinite time;

SUBCRITICAL CASE $\alpha < 2$:

- deterministic control formula,
- no correlation $\tau\sigma^T = 0$: $\tilde{H}(x, p) = \max_{y \in \mathbb{R}^m} | \sigma^T (x, y)p |^2$
Premilinary remark: an eigenvalue problem for $\alpha = 2$

For $x, p \in \mathbb{R}^n$ define the following perturbed generator $L^{x,p}$

$$L^{x,p} g(y) := Lg(y) + 2(\tau \sigma(x, y)^T p) \cdot D_y g(y),$$

where $L = b \cdot D_y + \text{tr}(\tau \tau^T D_{yy}^2)$. Then the ergodic problem is equivalent to

$$\bar{H} g(y) - (L^{\bar{x},\bar{p}} + V^{\bar{x},\bar{p}}) g(y) = 0,$$

where $g(y) = e^{w(y)}$ ($w$ being the corrector) and $V^{\bar{x},\bar{p}}(y) = |\sigma^T(\bar{x}, y)\bar{p}|^2$ is a multiplicative potential operator.

**Remark**

$\bar{H}$ is the first eigenvalue of the linear operator $L^{\bar{x},\bar{p}} + V^{\bar{x},\bar{p}}$, with eigenfunction $g = e^w$, where $w$ is a solution of the ergodic problem.
The effective Hamiltonian: comparison principle

In all the regimes we prove a comparison principle for $\tilde{H}$ by relating the regularity in $x$ of $\tilde{H}$ with that of the pseudo-coercive Hamiltonian $|\sigma^T(x, y)p|^2$:

$$\mu \tilde{H}(x, \frac{p}{\mu}) - \tilde{H}(z, q) \geq \frac{1}{\mu - 1} \sup_{y \in \mathbb{R}^m} |\sigma^T(x, y)p - \sigma^T(z, y)q|^2$$

0 < $\mu$ < 1.

Then, we follow the same argument for pseudo-coercive Hamiltonian as in Barles-Perthame (1990).
\( \alpha = 2: \) Representation formulas

We find representation formulas for \( \bar{H} \) which exploit the control interpretation of the cell equation (in \( \mathbb{R}^m \)):

\[
\bar{H} - \text{tr}(\tau \tau^T D_{yy}^2 w(y)) - (2\tau \sigma^T \bar{p} + b) \cdot D_y w(y) - |\tau^T Dw(y)|^2 - |\sigma^T \bar{p}|^2 = 0 \quad \text{in} \quad \mathbb{R}^m.
\]

Note

\[
-|\tau^T Dw(y)|^2 = \inf_{\beta \in \mathbb{R}^r} \{(2\tau \beta \cdot Dw(y) + |\beta|^2),
\]

then the cell equation reads

\[
\bar{H} + \inf_{\beta \in \mathbb{R}^r} \{-\text{tr}(\tau \tau^T D_{yy}^2 w(y)) + (2\tau \beta - 2\tau \sigma^T \bar{p} + b) \cdot D_y w(y) - (|\sigma^T \bar{p}|^2 + |\beta|^2)\}
\]

\[
= 0, \quad \text{in} \quad \mathbb{R}^m.
\]
\( \tilde{H} \): stochastic control formula

\( \tilde{H} = \tilde{H}(\bar{x}, \bar{p}) \) can be represented through stochastic control as

\[
\tilde{H}(\bar{x}, \bar{p}) = \lim_{\delta \to 0} \sup_{\beta(\cdot)} \delta E \left[ \int_0^\infty \left( |\sigma(\bar{x}, Z_t)^T \bar{p}|^2 - |\beta(t)|^2 \right) e^{-\delta t} dt \mid Z_0 = z \right]
\]

and

\[
\tilde{H}(\bar{x}, \bar{p}) = \lim_{t \to \infty} \sup_{\beta(\cdot)} \frac{1}{t} E \left[ \int_0^t \left( |\sigma^T(\bar{x}, Z_s)\bar{p}|^2 - |\beta(s)|^2 \right) ds \mid Z_0 = z \right],
\]

where \( \beta(\cdot) \) is an admissible control process taking values in \( \mathbb{R}^r \) for the stochastic control system

\[
dZ_t = (b(Z_t) + 2\tau(Z_t)\sigma^T(\bar{x}, Z_t)\bar{p} - 2\tau(Z_t)\beta(t)) \, dt + \sqrt{2\tau(Z_t)}dW_t; \quad (1)
\]
\[ \tilde{H}: \text{ergodic control formula} \]

Direct control interpretation of the cell equation:

\[ \tilde{H} + \inf_{\beta \in \mathbb{R}^r} \left\{ -\text{tr}(\tau T D_{yy} w) + (2\tau \beta - 2\tau \sigma^T \bar{p} + b) \cdot D_y w - (|\sigma^T \bar{p}|^2 + |\beta|^2) \right\} = 0. \]

Ergodic control problem of maximizing

\[
\lim_{T \to +\infty} \frac{1}{T} E \left[ \int_0^T \left( |\sigma(\bar{x}, z)^T \bar{p}|^2 - |\beta(s)|^2 \right) ds \mid Z(0) = z \right]
\]

\( (2) \)

where

\[ dZ_t = \left( b(Z_t) + 2\tau(Z_t)\sigma^T(\bar{x}, Z_t)\bar{p} - 2\tau(Z_t)\beta(t) \right) dt + \sqrt{2\tau(Z_t)} dW_t; \]

- \( Z \) is ergodic, then it has an invariant measure \( \mu \) on the torus \( \mathbb{T}^m \).
- The ergodic theorem says that the limit in (2) exists and it is the space average in \( d\mu \) of the running payoff.

Then we maximize

\[
\int_{\mathbb{T}^m} \left( |\sigma(\bar{x}, z)^T \bar{p}|^2 - |\beta(s)|^2 \right) d\mu(z)
\]
We maximize
\[ \int_{\mathbb{T}^m} (|\sigma(\bar{x}, z)^T \bar{p}|^2 - |\beta(s)|^2) \, d\mu(z) \]

Recall the cell equation
\[ \bar{H} + \inf_{\beta \in \mathbb{R}^r} \left\{ -\text{tr}(\tau \tau^T D_{yy} w) + (2\tau \beta - 2\tau \sigma^T \bar{p} + b) \cdot D_y w - |\sigma^T \bar{p}|^2 + |\beta|^2 \right\} = 0. \]

Since the cell equation has a smooth solution \( w \), a classical verification theorem says that the feedback control that achieves the minimum in the Hamiltonian

\[ 2\tau \beta \cdot D_y w + |\beta|^2 \]

is optimal.

The feedback control is \( \beta(z) = -\tau^T(z) Dw(z) \).
\( \bar{H} : \) ergodic control formula

\[
\bar{H}(\bar{x}, \bar{p}) = \int_{\mathbb{T}^m} \left( |\sigma(\bar{x}, z)^T \bar{p}|^2 - |\tau(z)^T Dw(z)|^2 \right) d\mu(z),
\]

where \( w = w(\cdot; \bar{x}, \bar{p}) \) is the smooth solution to the cell equation

\[
\bar{H}(\bar{x}, \bar{p}) - \text{tr}(\tau \tau^T D_{yy} w) - |\tau^T D_y w|^2 +
- (2\tau \sigma^T \bar{p} + b) \cdot D_y w - |\sigma^T \bar{p}|^2 = 0 \quad \text{in } \mathbb{R}^m
\]

\( \mu = \mu(\cdot; \bar{x}, \bar{p}) \) invariant probability measure on the torus \( \mathbb{T}^m \) of the process (1) (previous slide) with feedback \( \beta(z) = -\tau^T(z) Dw(z) \), that is the following process

\[
dZ_t = \left( b(Z_t) + 2\tau(Z_t)\sigma^T(\bar{x}, Z_t)\bar{p} + 2\tau(Z_t)\tau^T(Z_t) \right) \cdot Dw(Z_t) dt + \sqrt{2}\tau(Z_t) dW_t.
\]
Logarithmic-exponential type criterion per unit time on infinite time

$\bar{H}$ can be represented through the following formula

$$\bar{H}(\bar{x}, \bar{p}) = \lim_{t \to \infty} \frac{1}{t} \log E \left[ e^{\int_0^t |\sigma^T(\bar{x}, Y_s)\bar{p}|^2 \, ds} \mid Y_0 = y \right],$$

where $Y_t$ is the stochastic process defined by

$$dY_t = (b(Y_t) + 2\tau(Y_t)\sigma^T(\bar{x}, Y_t)\bar{p}) \, dt + \sqrt{2}\tau(Y_t) \, dW_t.$$

Sketch of the proof:

Take $v = v(t, x; \bar{x}, \bar{p})$ solution of the $t$-cell problem and define the function $f(t, y) = e^{v(t,y)}$. Then $f$ solves the following equation

$$\begin{cases}
\frac{\partial f}{\partial t} - f|\sigma^T \bar{p}|^2 - (2\tau \sigma^T \bar{p} + b) \cdot Df - \text{tr}(\tau\tau^T D^2 f) = 0 & \text{in } (0, \infty) \times \mathbb{R}^m \\
f(0, z) = 1 & \text{in } \mathbb{R}^m.
\end{cases}$$

and we conclude using the Feynam-Kac formula.
\( \tilde{H} \): Representation formulas

- \( \tilde{H} \) satisfies

\[
\tilde{H}(\bar{x}, \bar{p}) = \lim_{\delta \to 0} \sup_{\beta(\cdot)} \delta \int_0^{+\infty} (|\sigma(\bar{x}, y(t))^T \bar{p}|^2 - |\beta(t)|^2) e^{-\delta t} dt,
\]

where \( \beta(\cdot) \) varies over measurable functions taking values in \( \mathbb{R}^r \), \( y(\cdot) \) is the trajectory of the control system

\[
\begin{cases}
\dot{y}(t) = 2\tau(y(t))\sigma^T(\bar{x}, y(t))\bar{p} - 2\tau(y(t))\beta, & t > 0, \\
y(0) = y
\end{cases}
\]

and the limit is uniform with respect to the initial position \( y \) of the system;

- Moreover under the condition \( \tau \sigma^T = 0 \) of non-correlations among the components of the white noise acting on the slow and the fast variables in the system, we have

\[
\tilde{H}(x, p) = \max_{y \in \mathbb{R}^m} |\sigma^T(x, y)p|^2.
\]
The Large Deviation Principle

Assume $\sigma$ is uniformly non degenerate. Then the process $X^\varepsilon_t$ satisfies a large deviation principle with good rate function

$$I(x, x_0, t) = \inf \left[ \int_0^t \bar{L}(\xi(s), \dot{\xi}(s)) \, ds \mid \xi \in AC(0, t), \, \xi(0) = x_0, \, \xi(t) = x \right].$$

where $\bar{L}$ is the Legendre transform of $\bar{H}$, i.e.

$$\bar{L}(x, q) = \max_{p \in \mathbb{R}^n} \{p \cdot q - \bar{H}(x, p)\}.$$

Note that

- $\bar{L}(x, \cdot)$ is a convex nonnegative function;
- $\bar{L}(x, 0) = 0$ for all $x \in \mathbb{R}^n$.

(since $\bar{H}(x, \cdot)$ is convex nonnegative and $\bar{H}(x, 0) = 0$ for all $x \in \mathbb{R}$.)
The rate function

Then the rate function is defined as follows

\[ I(x; x_0, t) := \inf \left[ \int_0^t \bar{L}(\xi(s), \dot{\xi}(s)) \, ds \mid \xi \in AC(0, t), \, \xi(0) = x_0, \xi(t) = x \right]. \]

- \( I \geq 0 \) and \( I(x_0; x_0, t) = 0 \).
- \( I \) depends only on the volatility \( \sigma \) and on the fast process \( Y_t^\varepsilon \);
- \( I \) does not depend on the drift \( \phi \) of the log-price \( X_t^\varepsilon \) and on the initial value \( y_0 \) of the process \( Y_t \).
- \( I \) satisfies the following growth condition for some \( \nu, \, C > 0 \) and all \( x, x_0 \in \mathbb{R}^n \)

\[ \frac{1}{4C} \frac{|x - x_0|^2}{t} \leq I(x; x_0, t) \leq \frac{1}{4\nu} \frac{|x - x_0|^2}{t}; \]

- if \( \sigma \) does not depend on \( x \), i.e. \( \tilde{H} = \tilde{H}(\rho) \), the rate function is

\[ I(x; x_0, t) = t\bar{L} \left( \frac{x - x_0}{t} \right). \]
The rate function

- If $\alpha > 2$ and $n = 1$ and $\bar{H} = \bar{H}(p)$, then

$$I(x; x_0, t) = \frac{|x - x_0|^2}{4\bar{\sigma}^2 t}$$  \hspace{1cm} (3)

where

$$\bar{\sigma} = \sqrt{\int_{\mathbb{T}^m} \sigma(y)^2 d\mu(y)}$$

and $\mu$ is the invariant measure of the process $Y_t$

$$dY_t = b(Y_t)dt + \sqrt{2\tau}(Y_t)dW_t,$$

Remark

We observe that the rate function defined in (3) is the same as the rate function for the Black-Scholes model with constant volatility $\bar{\sigma}$. In other words, in the ultra fast regime, to the leading order, it is the same as averaging first and then taking the short maturity limit.
Applications-Out-of-the-money option pricing

Let $S^\varepsilon_t$ be the asset price, evolving according to the following stochastic differential system

\[
\begin{align*}
    dS^\varepsilon_t &= \varepsilon \xi(S^\varepsilon_t, Y^\varepsilon_t) S^\varepsilon_t dt + \sqrt{2\varepsilon \zeta(S^\varepsilon_t, Y^\varepsilon_t)} S^\varepsilon_t dW_t \\
    dY^\varepsilon_t &= \varepsilon^{1-\alpha} b(Y^\varepsilon_t) dt + \sqrt{2\varepsilon^{1-\alpha} \tau(Y^\varepsilon_t)} dW_t
\end{align*}
\]

\begin{align*}
    S^\varepsilon_0 &= S_0 \in \mathbb{R}_+ \\
    Y^\varepsilon_0 &= y_0 \in \mathbb{R}^m,
\end{align*}

(4)

where $\alpha > 1$, $\tau$, $b$ are $\mathbb{Z}^m$-periodic in $y$ with $\tau$ non-degenerate and $\xi : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}$, $\zeta : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{M}^{1,r}$ are Lipschitz continuous bounded functions, periodic in $y$.

Observe that $S^\varepsilon_t > 0$ almost surely if $S_0 > 0$.

We consider out-of-the-money call option with strike price $K$ and short maturity time $T = \varepsilon t$, by taking

\[S_0 < K \quad \text{or} \quad x_0 < \log K.\]

Similarly, by considering out-of-the-money put options, one can obtain the same for $S_0 > K$. 
As an application of the Large Deviation Principle, we prove

**Corollary**

*For fixed $t > 0$*

$$
\lim_{\varepsilon \to 0^+} \varepsilon \log E \left[ (S^\varepsilon_t - K)^+ \right] = - \inf_{y > \log K} l(y; x_0, t).
$$

When $\zeta(s, y) = \zeta(y)$, the option price estimate reads

$$
\lim_{\varepsilon \to 0^+} \varepsilon \log E \left[ (S^\varepsilon_t - K)^+ \right] = -l(\log K; x_0, t).
$$
Implied volatility

We recall that given an observed European call option price for a contract with strike price $K$ and expiration date $T$, the implied volatility $\sigma$ is defined to be the value of the volatility parameter that must go into the Black-Scholes formula to match the observed price.

We consider out-of-the-money European call option, with strike price $K$, and we denote by $\sigma_{\varepsilon}(t, \log K, x_0)$ the implied volatility.
Applications—An asymptotic formula for implied volatility

As a further application, we prove

**Corollary**

\[ \lim_{\varepsilon \to 0^+} \sigma_{\varepsilon}^2(t, \log K, x_0) = \frac{(\log K - x_0)^2}{2 \inf_{y > \log K} l(y; x_0, t) t}. \]

Note that the infimum in the right-hand side, is always positive by the assumption on \( S_0 \) and by the growth of the rate function.

**Remark**

*When \( \alpha > 2 \), the implied volatility is \( \bar{\sigma} \) that is*

\[ \bar{\sigma} = \sqrt{\int_{T^+} \sigma^2(y) d\mu(y)}. \]
A related homogenization result

Let $\varepsilon > 0$ and consider $u^\varepsilon$ solution to

$$
\begin{cases}
u_t^\varepsilon - b \left( \frac{x}{\varepsilon} \right) D u^\varepsilon - \varepsilon \text{tr}(\tau \tau^T \left( \frac{x}{\varepsilon} \right) D^2 u^\varepsilon) - |\tau \left( \frac{x}{\varepsilon} \right) D u^\varepsilon|^2 - l \left( x, \frac{x}{\varepsilon} \right) = 0 \\
u^\varepsilon(x, 0) = h(x).
\end{cases}
$$

If the main assumptions hold for $b, \tau$ (the underlying process to the fast variables is ergodic), then $u^\varepsilon$ converges locally uniformly, as $\varepsilon \to 0$, to the solution $u$ of

$$u_t - \bar{H}(x, Du) = 0.$$