

Asymptotic Stability of Continuous Time Systems with Saturation Nonlinearities

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Abstract

A new criterion is established for global asymptotic stability of second order systems modeled by equations of the type $\dot{x} = \sigma(Ax)$, where σ is the saturation function. The derivation is based on the Bendixon's theorem on limit cycles and a closer study of the trajectories of the systems. Applications to stabilization of more general cascade nonlinear systems are also discussed.

Keywords: Stability, saturation maps, Lyapunov functions, limit cycles, stabilization.

1 Introduction

We deal with the problem of characterizing global asymptotic stability of the origin for systems of the type:

$$\dot{x} = \bar{\sigma}(Ax), \tag{1}$$

with $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, and if $v \in \mathbb{R}^n$, $\vec{\sigma}(v) := (\sigma(v_1), \dots, \sigma(v_n))^T$, where σ is the saturation map, i.e.:

$$\sigma(x) = \begin{cases} -1 & \text{if } x \leq -1 \\ x & \text{if } -1 < x < 1 \\ 1 & \text{if } x \geq 1. \end{cases}$$

The model (1) is not invariant under a general change of coordinates of the type $z = Tx$. However, the dynamics are left unchanged if T is of the form $T = PS$, where P is a permutation matrix and $S = \text{Diag}(\epsilon_1, \dots, \epsilon_n)$, with $\epsilon_j = \pm 1$, $j = 1, \dots, n$.

The study of systems such as (1) is motivated by the fact that they model classes of analogic circuits, neural networks, and control systems and in general systems which present symmetrically saturating states after normalization.

The discrete time counterpart of (1) is

$$x^+ = \vec{\sigma}(Ax), \tag{2}$$

where x , A , and $\vec{\sigma}$ are as above. Recently, systems (2) received a growing amount of attention (see [12] and the references therein), due to their interest in digital filtering and saturation arithmetic.

The main result of this paper (Theorem 1, Section 3) establishes a new sufficient condition on the entries of the matrix A to conclude global asymptotic stability of systems (1) for the second order case. The paper is organized as follows. In Section 2 we give the basic definitions and some preliminary results. In particular, after a change of coordinates, system (1) will be placed in a form similar to the one of a Hopfield neural network for which there exists a large literature (see [6, 13, 2, 3, 22, 17, 11, 19] and the references therein, see also [10] for a different but related model). Section 3 gives sufficient conditions for asymptotic stability of the second order systems using an application of the Bendixon's theorem on limit cycles. In Section 4, we discuss an application of our results to the synthesis of a stabilizing controller for a system with saturated inputs.

2 Definitions and Preliminaries

First, we recall some basic definitions from stability theory for dynamical systems (see [20, chap. 5] for more detailed definitions and results). Let Σ be a dynamical system, with an unique equilibrium point x_0 . If A is a nonsingular matrix, then this is the case of model (1) where $x_0 = 0$. The equilibrium state x_0

is said to be *stable* if, for each neighborhood S of x_0 , there exists a neighborhood $W \subseteq S$, such that each trajectory starting in W remains in S , for all $t \geq 0$. Moreover, x_0 is said to be *locally asymptotically stable* if the above holds and, for any initial state in W , the corresponding trajectory converges to $x_0 = 0$, as $t \rightarrow \infty$. If x_0 is stable and any trajectory converges to it as $t \rightarrow \infty$, regardless of the initial state, then x_0 is said to be *globally asymptotically stable*.

In order to ascertain the stability properties of an equilibrium point, the direct method of Lyapunov is the most common tool. There are various versions of this method depending upon what kind of stability one wants to study. We state here, informally, only the criterion for global asymptotic stability: Assume there exists a C^1 proper function $V(\cdot)$ satisfying $V(x_0) = 0$, $V(x) > 0$ for $x \neq x_0$, and define $\dot{V}(\cdot)$ the derivative of this function, with respect to the time, calculated along the trajectories of the system. If $\dot{V}(\cdot) < 0$ along any non constant trajectory, then x_0 is globally asymptotically stable. In this case, the function $V(\cdot)$ is referred to as a *Lyapunov function*.

An obvious necessary condition for the local asymptotic stability of $x_0 = 0$ in (1) is the asymptotic stability of the linear system $\dot{x} = Ax$. Therefore we assume:

$$A \text{ is a Hurwitz matrix,} \tag{C1}$$

i.e. all the eigenvalues of A have a negative real part, in particular, A is nonsingular.

In order to deal with the asymptotic stability of the system (1) we will apply the following simple device. Define the change of coordinates

$$z = Ax.$$

We get

$$\dot{z} = A\dot{x} = A\vec{\sigma}(Ax) = A\vec{\sigma}(z),$$

thus the global asymptotic stability of system (1) is equivalent to the one of the system

$$\dot{x} = A\vec{\sigma}(x). \tag{3}$$

Systems (3) are simpler to deal with, so, from now on, we assume that a model of type (3) is given and we will prove our results for this model. Notice also that system (3) looks like a Hopfield neural network [6].

Condition (C1) is by no means sufficient to conclude global asymptotic stability of $x_0 = 0$ for systems of type (3). In fact, for example, (C1) does not avoid trajectories which remain in the regions of R^n where $|x_j| \geq 1$, for some $j \in \{1, \dots, n\}$. The condition stated in the next proposition excludes this possibility.

Proposition 2.1 Given a system of type (3), if $x_0 = 0$ is globally asymptotically stable, then

$$\min_j \epsilon_j \sum_{k=1}^n \epsilon_k a_{jk} < 0, \quad (\text{C2})$$

for each combination of $\epsilon_1, \dots, \epsilon_n = \pm 1$.

Proof. We prove this statement by contradiction. Assume that, for a given combination $\bar{\epsilon}_1, \dots, \bar{\epsilon}_n = \pm 1$, (C2) is not verified, that is we have

$$\bar{\epsilon}_j \sum_{k=1}^n \bar{\epsilon}_k a_{jk} \geq 0, \quad \forall j = 1, \dots, n. \quad (4)$$

Let $\bar{x} \in \mathbb{R}^n$ be such that $\bar{x}_j \geq 1$ for $\bar{\epsilon}_j = +1$ and $\bar{x}_j \leq -1$ for $\bar{\epsilon}_j = -1$. Denote by $\bar{x}(t)$ the trajectory of (3) starting at \bar{x} . Then, $\forall t \geq 0$, $\bar{x}_j(t) \geq \bar{x}_j \geq 1$ if $\bar{\epsilon}_j = 1$, and $\bar{x}_j(t) \leq \bar{x}_j \leq -1$, if $\bar{\epsilon}_j = -1$, since, from (4), we have:

$$\bar{\epsilon}_j \dot{\bar{x}}_j(t) = \bar{\epsilon}_j \sum_{k=1}^n \bar{\epsilon}_k a_{jk} \geq 0, \quad j = 1, \dots, n. \quad (5)$$

Therefore $|x_j(t)| \geq 1, \forall j = 1, \dots, n$, and $\forall t \geq 0$. ■

Remark 2.2 Condition (C2) requires that, for each combination $\epsilon_1, \dots, \epsilon_n = \pm 1$, there exists a $j \in \{1, \dots, n\}$ such that the quantity $\sum_{k=1}^n \epsilon_k a_{jk}$ is not zero, and it has not the same sign of ϵ_j . It consists of checking 2^n conditions for any possible combination of signs in $\epsilon_1, \dots, \epsilon_n$. However only 2^{n-1} conditions are significative since it is

$$\epsilon_j \sum_{k=1}^n \epsilon_k a_{jk} = -\epsilon_j \sum_{k=1}^n -\epsilon_k a_{jk}.$$

Next result states an useful sufficient condition for global asymptotic stability of the system (3) or equivalently (1). It has been first stated, for more general models than this specific one, in [16], and a complete proof has been given recently in [4, 9].

Proposition 2.3 Let Σ be a model of type (3) (or (1)). Assume that, there exists a positive definite diagonal matrix $D = \text{Diag}(k_1, \dots, k_n)$, for which the Lyapunov equation

$$A^T D + D A = Q, \quad (6)$$

holds, with Q negative definite. Then $x_0 = 0$ is a globally asymptotically stable equilibrium point for Σ .

The idea of the proof is to show that the following function:

$$V(x_1, \dots, x_n) = \sum_{i=1}^n k_i \int_0^{x_i} \sigma(t) dt.$$

is a global Lyapunov function for the model.

To understand when Proposition 2.3 can be applied, one needs to know when, for a given Hurwitz matrix A , there exists a diagonal positive definite D such that equation (6) holds. Reference [1] deals with this problem, and presents some necessary and sufficient conditions. An interesting case in which (6) is verified, is when A is a Hurwitz symmetric matrix, in this case it holds with $D = I$. A necessary condition is that every principal submatrix of A satisfies (6). This easily follows from the fact that principal submatrices of positive definite matrices are positive definite. In particular the diagonal elements of A must be all negative. For two dimensional matrices, this condition is also sufficient, in fact, the following fact holds:

Lemma 2.4 Let $A \in \mathbb{R}^{2 \times 2}$. There exists $D = \text{Diag}(k_1, k_2)$, with $k_i > 0$ for $i = 1, 2$, such that $A^T D + DA < 0$ if and only if i) A is a Hurwitz matrix, ii) $a_{11} < 0$ and $a_{22} < 0$.

Proof. The proof simply follows by applying the Sylvester criterion for symmetric negative definite matrix (see e.g. [7, pg.404]) to $A^T D + DA$. ■

3 Stability of the Second Order Model

In this section, we deal with second order systems of type (3). By combining Proposition 2.3 and Lemma 2.4, one has:

Corollary 3.1 Let Σ be a second order model of type (3) (or (1)). If the matrix A is Hurwitz, and both its diagonal elements are negative, then $x_0 = 0$ is a globally asymptotically stable equilibrium point for Σ .

In order to derive a sufficient condition, for global asymptotic stability of second order systems, which works when the above condition is not verified we state below two definitions and a property. This is done in the general context of a state space of dimension n .

Definition 3.2 A matrix $A \in \mathbb{R}^{n \times n}$ is i -row diagonally dominant if

$$a_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}| < 0.$$

Definition 3.3 A matrix $A \in \mathbb{R}^{n \times n}$ is row diagonally dominant if it is i -row diagonally dominant, for any $i = 1, \dots, n$.

Analogous definitions can be given for column dominance (see [7, pg.349]) but we shall not use them in the following. Row diagonal dominance has a special significance in terms of the trajectories of system (3), as described in the following lemma.

Lemma 3.4 Let Σ be a model of type (3). Assume that, for one $i \in \{1, \dots, n\}$, A is i -row diagonally dominant. Then $W_i := \{x : |x_i| < 1\}$ is a global attractor and a positively invariant set.

Proof. Assume $x_i(t) \geq 1$, then:

$$\dot{x}_i(t) = a_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} \sigma(x_j(t)) \leq a_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}| < 0.$$

Therefore $x_i(t) \leq (a_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}|)t + x_i(0)$, so there exists \bar{t} such that $x_i(\bar{t}) < 1$. Similar arguments show that the same holds for $x_i(t) \leq -1$. Thus W_i is a global attractor. Moreover, the velocity vector is transverse to the boundary of W_i , and it is oriented inward to W_i , so W_i is also a positively invariant set. ■

Notice that if A is row diagonally dominant, then, by Gershgorin theorem [7, pg. 344], it is Hurwitz. Moreover, since each W_i , $i = 1, \dots, n$ is a global attractor and a positively invariant set, the system behaves, from a certain instant on, as a linear system. So, in this case, we can conclude global asymptotic stability of $x_0 = 0$ for system (3). Since the diagonal dominance implies, in particular, that $a_{ii} < 0$, for $i = 1, \dots, n$, if $n = 2$, one may also apply Corollary 3.1 to get the same conclusion.

The rest of this section is devoted to show that, for second order systems, i -row diagonal dominance for one $i \in \{1, 2\}$ and the Hurwitz property of A are sufficient to conclude global asymptotic stability for (3).

Lemma 3.5 Let Σ be a model of type (3) with $n = 2$. If A is Hurwitz, and, for $i = 1$ or 2 it is also i -row diagonally dominant, then all the trajectories of Σ are bounded.

Proof. Assume $i = 1$ (same arguments hold for $i = 2$), and let $x(t)$ be a trajectory of Σ . We already know by Lemma 3.4 that there exists $t_1 \geq 0$ such that $|x_1(t)| < 1$, for any $t \geq t_1$. Assume $x_2(t_1) > 1$. Then, by continuity, $x_2(t) > 1$ for t in a open neighborhood of t_1 . As long as $x_2(t) > 1$, the equation for Σ are:

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12} \\ \dot{x}_2 &= a_{21}x_1 + a_{22}. \end{aligned}$$

An explicit computation gives

$$\begin{aligned} x_1(t) &= e^{a_{11}(t-t_1)}(x_1(t_1) + \frac{a_{12}}{a_{11}}) - \frac{a_{12}}{a_{11}}, \\ x_2(t) &= x_2(t_1) + \frac{a_{21}}{a_{11}}(x_1(t_1) + \frac{a_{12}}{a_{11}})(e^{a_{11}(t-t_1)} - 1) + \frac{\text{Det } A}{a_{11}}(t - t_1), \end{aligned} \tag{7}$$

where the 1-row dominance implies $a_{11} < 0$. Since A is Hurwitz, $\text{Det } A > 0$, thus there exists $t_2 > t_1$ such that $0 \leq x_2(t_2) \leq 1$. Same arguments give that, if $x_2(t_1) < -1$, then, as long as $x_2(t) < -1$, the trajectory is given by:

$$\begin{aligned} x_1(t) &= e^{a_{11}(t-t_1)}(x_1(t_1) - \frac{a_{12}}{a_{11}}) + \frac{a_{12}}{a_{11}}, \\ x_2(t) &= x_2(t_1) + \frac{a_{21}}{a_{11}}(x_1(t_1) - \frac{a_{12}}{a_{11}})(e^{a_{11}(t-t_1)} - 1) - \frac{\text{Det } A}{a_{11}}(t - t_1). \end{aligned} \tag{8}$$

Thus, also in this case, there exists $t_2 > t_1$ such that $-1 \leq x_2(t_2) \leq 0$.

So, in any case, the trajectory $x(t)$ reaches the region $W = \{(x_1, x_2) \mid |x_1| \leq 1, |x_2| \leq 1\}$. Now, even if $x(t)$ leaves W for some $t > t_2$, using the explicit form of the trajectory given by equations (7) or (8), we have:

$$|x_2(t)| < 1 + \left| \frac{a_{21}}{a_{11}} \right| \left(1 + \left| \frac{a_{12}}{a_{11}} \right| \right), \quad \forall t \geq t_2;$$

thus $x(t)$ is bounded, as desired. ■

Next lemma, under the same assumptions of Lemma 3.5, excludes the possibility of having nonconstant periodic orbits. We will use the classical Bendixon's theorem which gives sufficient conditions for the nonexistence of closed trajectories in a simply connected region of R^2 (see [20, pg.31]). However, in the following proof, we need a slight modification of the original proof of Bendixon, due to the fact that $\sigma(\cdot)$ is not continuously differentiable.

Lemma 3.6 Let Σ be a model of type (3) with $n = 2$. If A is Hurwitz, and, for $i = 1$ or 2 it is also i -row diagonally dominant, then Σ has no closed trajectories in the strip W_i .

Proof. Let $i = 1$ (the case $i = 2$ can be treated in the same way). Assume there is a closed trajectory Γ in W_1 . To make notation simpler, assume that Γ intersects the line $x_2 = 1$ at two points, and it does not intersect the line $x_2 = -1$. Denote by $P_1(x_1^1, 1)$ and $P_2(x_1^2, 1)$, with $x_1^1 < x_1^2$, the intersection points. The treatment of the general case goes in full analogy.

At any point of Γ the velocity vector of (3), that we denote by \vec{v} , is orthogonal to the normal vector to Γ , which we denote by \vec{n} . Thus we have

$$\int_{\Gamma} \vec{v} \cdot \vec{n} dl = 0. \tag{9}$$

Let $\Gamma_{x_1^1-x_1^2}$ be the arc of Γ which goes from P_1 to P_2 , and $\Gamma_{x_1^2-x_1^1}$ the one from P_2 to P_1 . Denote by $\Gamma_1 = \Gamma_{x_1^1-x_1^2} \cup \overline{P_2P_1}$, and by $\Gamma_2 = \Gamma_{x_1^2-x_1^1} \cup \overline{P_1P_2}$ (where $\overline{P_iP_j}$ denotes the segment from P_i to P_j). For $i = 1, 2$, let S_i be the region enclosed by Γ_i , and \vec{n}_i be its outward normal. Since

$$\int_{\overline{P_2P_1}} \vec{v} \cdot \vec{n}_1 dl = - \int_{\overline{P_1P_2}} \vec{v} \cdot \vec{n}_2 dl,$$

we may rewrite equation (9) as:

$$\int_{\Gamma_{x_1^1-x_1^2}} \vec{v} \cdot \vec{n}_1 dl + \int_{\overline{P_2P_1}} \vec{v} \cdot \vec{n}_1 dl + \int_{\Gamma_{x_1^2-x_1^1}} \vec{v} \cdot \vec{n}_2 dl + \int_{\overline{P_1P_2}} \vec{v} \cdot \vec{n}_2 dl = 0,$$

which is equivalent to

$$\int_{\Gamma_1} \vec{v} \cdot \vec{n}_1 dl + \int_{\Gamma_2} \vec{v} \cdot \vec{n}_2 dl = 0 \quad (10)$$

Notice that, the closed curves Γ_1 and Γ_2 are contained in two regions of the plane R^2 where the velocity vector \vec{v} is continuously differentiable. Thus applying the Divergence Theorem (see e.g. [18, pg.754]), we obtain,

$$\int_{\Gamma_1} \vec{v} \cdot \vec{n}_1 dl + \int_{\Gamma_2} \vec{v} \cdot \vec{n}_2 dl = \int_{S_1} \nabla \vec{v} dx + \int_{S_2} \nabla \vec{v} dx. \quad (11)$$

In S_1, S_2 , $\nabla \vec{v}$ is alternatively a_{11} or $a_{11} + a_{22}$, which, in both cases, is strictly less than zero. This implies that the right hand side of (11) is strictly less than zero, which contradicts equation (10). \blacksquare

Next theorem provides, for second order models, a new sufficient condition to conclude the global asymptotic stability of the origin.

Theorem 1 *Let Σ be a model of type (3) with $n = 2$. Assume that A is Hurwitz, and, for one i in $\{1, 2\}$, it is i -row diagonally dominant. Then $x_0 = 0$ is a globally asymptotically stable equilibrium point for Σ .*

Proof. Let $x(t)$ be any trajectory of Σ . By Lemma 3.4, $x(t)$ is, from a certain instant on, in the strip W_i and it is bounded by Lemma 3.5. Thus if Ω denotes the positive limit set of $x(t)$, by classical results on limit sets for 2-dimensional systems (see, for example [5], Theorem 1.3), we know that one of the following holds:

- Ω is a closed trajectory;
- Ω is an equilibrium point;
- Ω contains some equilibrium points and some trajectories whose positive and negative limit sets are among these equilibrium points.

Lemma 3.6 excludes the first possibility. The third situation is also to be excluded, since it would imply the existence of a trajectory whose negative limit set is zero. This would be in contradiction with the fact that A is Hurwitz. Thus $\Omega = \{0\}$, as desired. \blacksquare

As mentioned above, Theorem 1 covers some cases in which the assumptions of Corollary 3.1 fail. Nevertheless, it is not difficult to find examples of Hurwitz matrices with both diagonal elements negative which are not i -row diagonally dominant for any $i = 1, 2$.

4 An Application

We discuss an application of Theorem 1 to a specific design problem. It will appear clear, however, that the arguments used here work for more general, higher order, systems.

Consider feedback stabilization of the following nonlinear system

$$\begin{aligned}\dot{z} &= -\phi(z) + zx_1^2 \\ \dot{x}_1 &= \sigma(x_2) \\ \dot{x}_2 &= \sigma(u),\end{aligned}\tag{12}$$

where the scalar system $\dot{z} = -\phi(z)$ is globally asymptotically stable and u denotes a scalar input. If we consider the system as a cascade, we cannot apply classical results on stability of cascade nonlinear systems ([14],[15],[21]) because of the lack of the Lipschitz condition in the term zx_1^2 , and because the system is not C^1 . Moreover, no linear feedback can be chosen so that the subsystem x_1, x_2 satisfies the condition of Proposition 2.3, since the corresponding matrix A does not satisfy the necessary condition ii) of Lemma 2.4. However, it is easily seen that the linear feedback

$$u = -k_1x_1 - k_2x_2,\tag{13}$$

$k_1 > 0, k_2 > k_1$, gives global asymptotic stability of system (12).

The claim follows from combining Theorem 1 with recently developed results on the stability of nonlinear cascades [8]. In particular, notice that the matrix of the subsystem

$$\dot{x}_1 = \sigma(x_2),$$

$$\dot{x}_2 = \sigma(-k_1x_1 - k_2x_2),$$

is 2- row dominant. Therefore, from Theorem 1, there exists a $T \geq 0$ such that, for $t \geq T$, this subsystem behaves as a linear system and $|x(t)| \leq B$, with a certain bound B , for $t \leq T$.

For $t \leq T$, $z(t)$ is also bounded, namely it does not have finite escape time in the interval $[0, T]$. In order to see this, multiply the first equation of (12) by z . We get

$$\begin{aligned} z\dot{z} &= -z\phi(z) + z^2x_1^2, \\ \frac{1}{2} \frac{dz^2}{dt} &\leq z^2x_1^2 \leq z^2B^2, \end{aligned} \tag{14}$$

and integrating (14), we get

$$z^2(t) = z^2(0) + \int_0^t 2z^2(s)B^2 ds, \tag{15}$$

for $t \leq T$. Application of the Bellman-Gronwall Lemma (see e.g. [20][pg. 292]) to (15) gives

$$z^2(t) \leq z^2(0)e^{\int_0^t 2B^2 ds} \leq z^2(0)e^{2B^2T}.$$

The above shows that, in an asymptotic analysis, we can consider the cascade (12) with the control (13), as the cascade of a linear asymptotically stable system with an asymptotically stable nonlinear system. The global asymptotic stability of the overall system follows directly from application of the main result of [8].

We conclude noticing that the control (13) is still stabilizing if we multiply it for any positive constant γ since this does not destroy the dominance and Hurwitz property of the resulting matrix. This controller has infinite gain margin.

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