Further conditions on the Stability of Continuous Time Systems with Saturation

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Abstract

The global asymptotic stability of the origin for systems described by $\dot{x} = \bar{\sigma}(Ax)$, where $\bar{\sigma}$ is the saturation function, is studied. The case $x(t) \in I\!R^2$ is treated in detail and conditions which are necessary and sufficient are obtained. Examples are presented to illustrate the ideas developed in the paper.

Keywords: Stability, saturation maps, limit cycles.

1 Introduction and Preliminaries

In this paper, systems of the following form are studied.

$$\dot{x} = \bar{\sigma}(Ax), \tag{1}$$

where $x \in I\!R^n$, $A \in I\!R^{n\times n}$, and for $v \in I\!R^n$, $\bar{\sigma}(v) = (\sigma(v_1), \ldots, \sigma(v_n))^T$, where σ is the saturation map:

$$\sigma(x) = \begin{cases} -1, & x \leq -1 \\ x, & -1 < x < 1 \\ 1, & x \geq 1 \end{cases}$$

We try to find a characterization of the global asymptotic stability of the origin for such systems in terms of the entries of the matrix A.

Systems such as (1) model classes of analog circuits, neural networks and control systems with symmetrically saturating states, after normalization. The qualitative analysis of neural networks and, in particular, the study of their stability and asymptotic stability properties have received a great deal of attention, see for example [9, 3, 4, 8, 11] and references therein.

Some criteria on the entries of the matrix A for global asymptotic stability of systems (1) are summarized in the first section. Necessary and sufficient conditions for the general case are studied initially, summarizing some known results (see [1], [5], [7], [9]). Then in Section 3 a detailed consideration of the second-order case, $n = 2$, is presented, and a complete solution is obtained. The conditions given here require that two particular trajectories converge to zero. We conjecture, however, that this condition is indeed redundant.

Examples of the various types of instability that may be observed for a Hurwitz matrix $A \in I\!R^{2\times 2}$ are also presented, and these are used as a basis in proving our main result (Theorem 9). A main ingredient of the proof of Theorem 9 is based on the idea of using a Poincaré cut to analyze the behavior of the system. In principle, this method may be used to deal with higher

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dimensional cases, however, when $n \geq 3$, it does not appear that a simple characterization of stability is possible.

Given a system of differential equations $\dot{x} = f(x)$, we say that $x_0 = 0$ is *locally asymptotically stable* if for each neighborhood V of the origin there exists a neighborhood $W \subseteq V$ such that each solution trajectory starting in W, remains in V for all $t \geq 0$ and converges to zero as $t \to \infty$. Moreover $x_0 = 0$ is said to be *globally asymptotically stable* if it is locally asymptotically stable and if all solution trajectories converge to zero as $t \to \infty$. The trajectory corresponding to the initial conditions $x(0) = x_0$ will be denoted by $\phi(t; x_0)$. Notice that for models as in (1), there exists a unique solution for every initial condition, and furthermore all solutions are defined for all $t \geq 0$.

2 Conditions for Global Asymptotic Stability

Some results for global asymptotic stability are reviewed. Necessary and sufficient conditions are treated separately.

2.1 Necessary Conditions

An obvious necessary and sufficient condition for the local asymptotic stability of $x_0 = 0$ in (1) is the asymptotic stability of the linear system $\dot{x} = Ax$. This yields the following straightforward necessary condition for global stability.

Proposition 1 *Given a system of type (2), if $x_0 = 0$ is globally asymptotically stable, then*

$$A \text{ is a Hurwitz matrix,} \tag{C1}$$

i.e. *all the eigenvalues of A have a negative real part.* □

Under (C1), the asymptotic stability of (1) is equivalent to that of the system :

$$\dot{x} = A\bar{\sigma}(x). \tag{2}$$

This follows as (C1) implies invertibility of A. Applying the coordinate transformation $z = Ax$ shows equivalence of the two models. In the sequel we work with the model (2), as it is simpler to analyze.

Other necessary conditions for global asymptotic stability may be derived by showing that the trajectories do not remain in the regions of $\mathbb{R}^n$ where $|x_j| \geq 1, j = 1 \ldots n$ (for the proof we refer to [1]).

Proposition 2 *Given a system(2), if $x_0 = 0$ is globally asymptotically stable, then*

$$\min_j \quad \epsilon_j \sum_{k=1}^{n} \epsilon_k a_{jk} < 0 \tag{C2}$$

for each combination of $\epsilon_1, ..., \epsilon_n = \pm 1$. □

Remark 3 Condition (C2) is equivalent to checking that for each combination $\epsilon_1, ..., \epsilon_n = \pm 1$, there exists a $j = 1, ..., n$ such that $\sum_{k=1}^{n} \epsilon_k a_{jk}$ is not zero and is not of the same sign as ϵ_j.

2.2 Sufficient Conditions for Asymptotic Stability

The next theorem may be proved via a simple Lyapunov argument, giving a sufficient condition for the global asymptotic stability of $x_0 = 0$ in (2). This result was first stated in [10], and has been recently proved for more general models in [5, 7].

Theorem 4
Consider the system (2) and assume that there exists a positive definite diagonal matrix $D = \text{Diag}\,(d_1, \dots, d_n)$*, such that*

$$A^T D + DA = Q < 0 \tag{3}$$

Then $x_0 = 0$ *is globally asymptotically stable.* □

Remark 5 To understand how to apply Theorem 4, one needs to know when, for a given Hurwitz matrix A , there exists a diagonal positive definite D such that equation (3) holds. This problem is considered in [2], where some necessary and sufficient conditions are presented. A straightforward sufficient condition is that

$$A^T + A < 0.$$

This fact holds, for example, when A is symmetric. For two dimensional matrices, it is not difficult to prove (see [1]), that there exists a diagonal D such that inequality (3) holds if and only if both diagonal elements of A are negative.

Other sufficient conditions may be derived for Hurwitz matrices which are M-matrices. A matrix $R \in \mathbb{R}^{q\times q}$ is called an M-matrix if $r_{ij} \leq 0$ for all $i \neq j$, and all the principal minors of R are positive. The following two corollaries follow from well known results about M-matrices, see for example [9].

Corollary 6 *Let* Σ *be a system of type (2), and assume that the matrix* A *is such that:*

$$a_{ii} + \sum_{k\neq i} |a_{ik}| < 0, \qquad \forall\, i = 1, \dots, n. \tag{4}$$

Then $x_0 = 0$ *is globally asymptotically stable.* □

Corollary 7 *Let* Σ *be a system of type (2), and assume that the matrix* A *is such that:*

$$a_{ii} + \sum_{k\neq i} |a_{ki}| < 0, \qquad \forall\, i = 1, \dots, n. \tag{5}$$

Then $x_0 = 0$ *is globally asymptotically stable.* □

Remark 8 Corollaries 6 and 7 may also be proved by more direct methods. If (4) holds, then it may be shown that $W := \{x : |x_j| < 1 \quad j = 1, ..., n\}$ is globally attracting, and thus the system stability is equivalent to that of the linear system $\dot{x} = Ax$. If (5) holds, $V(x) = \sum_{j=1}^{n} |x_j|$ is a global Lyapunov function, proving global asymptotic stability.

3 The Second-Order Case

In the case $A \in \mathbb{R}^{2\times 2}$, the situation is easier to analyze. As the only equilibrium point possible for a system (2) is $x_0 = 0$, it follows from the Poincarè-Bendixon Theorem, see e.g. [12], that if A is Hurwitz and the system is not globally asymptotically stable, there must exist at least one unstable limit cycle which encircles the origin. Furthermore, unstable trajectories lying outside the region enclosed by the limit cycle will either ultimately remain in one of the sectors where both components of $\bar{\sigma}(\cdot)$ are saturating, or circle the origin.

The former case may be easily accounted for by conditions on the coefficients of A, as in Proposition 2.

The latter case is more difficult to handle. One technique we will use is to analyze the system via a Poincaré return map based on the cross section $C = \{(\alpha, 1)^T : \alpha > 1\}$. That is, to consider the map $\alpha \mapsto \tilde{\alpha}$, where $(\tilde{\alpha}, 1)^T$ is the first intersection of the trajectory $\phi(t, \binom{\alpha}{1})$ with the set C and $\alpha \gg 1$ is large enough that the trajectory does not enter the central, linear, region before coming to $(\tilde{\alpha}, 1)^T$. As trajectories outside the central region may be easily integrated, the return map may be constructed, yielding the sequence of points at which the trajectory intersects C. Global stability properties of the system may then be concluded from this return map.

Our main result is the following:

Theorem 9

Let Σ be a system of type (2), with $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}$. Then the system is globally asymptotically stable if and only if the following conditions hold

(i) *A is Hurwitz* (i.e. $\det A > 0$, *and* $\mathrm{Tr}\ A < 0$)

(ii) $S_1 < 0$ *or* $S_2 < 0$

(iii) $D_1 < 0$ *or* $D_2 < 0$

(iv) *One of the following conditions hold*

(a) $S_1 \leq 0$ *and* $D_1 < 0$, *or* $S_1 < 0$ *and* $D_1 = 0$

(b) $S_1 > 0$, $D_1 < 0$, $S_2 < 0$, $D_2 > 0$, $S_1 D_2 \leq S_2 D_1$, $C_1 > 0$, $\phi(t; \binom{1}{1}) \to 0$, *and* $\phi(t; \binom{-1}{1}) \to 0$

(c) $S_2 \leq 0$ *and* $D_2 < 0$, *or* $S_2 < 0$ *and* $D_2 = 0$

(d) $S_1 < 0$, $D_1 > 0$, $S_2 > 0$, $D_2 < 0$, $S_1 D_2 \geq S_2 D_1$, $C_2 > 0$, $\phi(t; \binom{1}{1}) \to 0$, *and* $\phi(t; \binom{-1}{1}) \to 0$

where $S_1 = a_{11} + a_{12}$, $S_2 = a_{21} + a_{22}$, $D_1 = a_{11} - a_{12}$, $D_2 = a_{22} - a_{21}$, *and*

$$C_1 = \frac{D_2}{D_1}\left(\frac{\det A}{a_{22}^2}\ln\left(\frac{-D_2}{S_2}\right) - 2\frac{a_{12}}{a_{22}}\right) - \left(\frac{\det A}{a_{11}^2}\ln\left(\frac{-S_1}{D_1}\right) + 2\frac{a_{21}}{a_{11}}\right),$$

$$C_2 = \frac{D_1}{D_2}\left(\frac{\det A}{a_{11}^2}\ln\left(\frac{-D_1}{S_1}\right) - 2\frac{a_{21}}{a_{11}}\right) - \left(\frac{\det A}{a_{22}^2}\ln\left(\frac{-S_2}{D_2}\right) + 2\frac{a_{12}}{a_{22}}\right).$$

In the definitions of C_1, and C_2, if one of a_{11} or a_{22} is zero, the constant takes on the value as the respective coefficient goes to zero (see equation (10)). □

Remark 10 Note that $S_1 D_2 - S_2 D_1 = 2(a_{22}a_{12} - a_{11}a_{21})$. This may be used to simplify the conditions *(b)* and *(d)*.

Remark 11 It will be clear from the proof of the theorem that, in the two cases *(b)*, and *(d)*, the condition $\phi(t; \binom{1}{1}) \to 0$, and $\phi(t; \binom{-1}{1}) \to 0$, may be replaced by $\phi(t; \binom{x}{1}) \to 0$ for some $x \gg 1$. It is also possible to give a bound, in terms of the entries of the matrix A, on how large the constant x has to be.

Notice that, even if not easily verified analytically, it is relatively easy to check conditions *(b)* and *(d)* via numerical integration for any given matrix A.

The remainder of this section is devoted to exploring the ideas described above and proving Theorem 9.

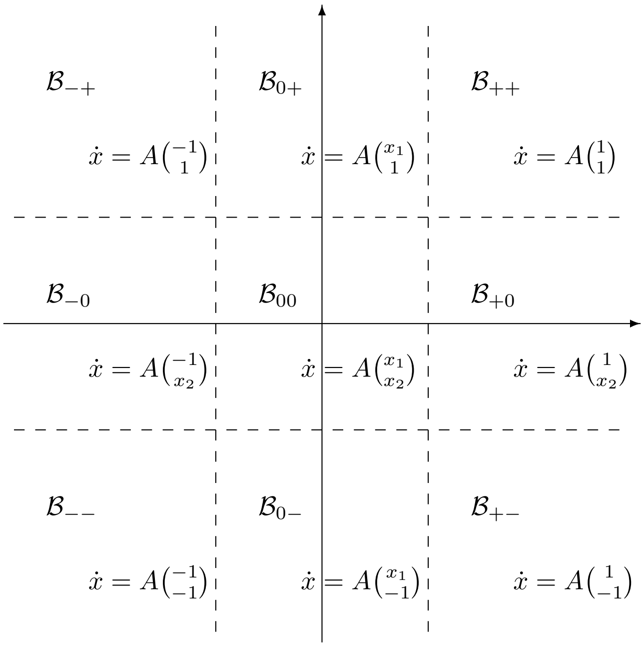

Figure 1: Subdivision of $\mathbb{R}^2$.

3.1 Preliminary Results and Discussion

We first set up some notation and then prove some results regarding the characteristics of the vector field $A\bar{\sigma}(x)$. The main points will be illustrated with examples.

An important feature of the dynamical system (2) is that it is piecewise affine. This means that the plane may be divided into 9 regions in which the system is affine and may be solved exactly. We denote these regions by $\mathcal{B}_{ij}$, where $i, j \in \{+, -, 0\}$. These regions are shown in Figure 1 along with the differential equations giving the flow in each region, we shall denote these regions *sectors*, the sector boundaries are given by dashed lines in the figure.

As in the statement of Theorem 9, we use the following notation throughout:

$$S_1 = a_{11} + a_{12}, \quad D_1 = a_{11} - a_{12}, \quad S_2 = a_{22} + a_{21} \quad D_2 = a_{22} - a_{21}.$$

A. Invariant sectors

A sector, $\mathcal{B}$ is called *invariant* if for all initial conditions $x_0 \in \mathcal{B}$, the solution $\phi(t; x_0)$ remains in $\mathcal{B}$ for all $t > 0$.

A first characterization of the flow is found by considering invariance of the sectors described above. The following results are immediate.

Lemma 12 *If $\mathcal{B}_{00}$ is invariant, A is Hurwitz and $x_0 = 0$ is globally asymptotically stable.* □

Proof. If $\mathcal{B}_{00}$ is invariant, it may be seen that (4) holds, and so, by Corollary 6, A is Hurwitz and $x_0 = 0$ is globally asymptotically stable. ■

Lemma 13 *If one of the sectors $\mathcal{B}_{+0}, \mathcal{B}_{-0}, \mathcal{B}_{0+}, \mathcal{B}_{0-}$ is invariant, then A is not Hurwitz.* □

Figure 2: Periodic Solution and Vector field for A_1.

Proof. Without loss of generality, consider that the sector $\mathcal{B}_{+0}$ is invariant. Then $S_1 \geq 0$, $S_2 \leq 0$, $D_1 \geq 0$ and $D_2 \leq 0$. It follows that $a_{11} \geq 0$, $a_{22} \leq 0$, $|a_{12}| \leq |a_{11}|$ and $|a_{21}| \leq |a_{22}|$. Thus $-a_{11}a_{22} \geq |a_{12}a_{21}|$, and so $\det A \leq 0$. Thus A is not Hurwitz. ■

The following lemma is an obvious consequence of Proposition 2.

Lemma 14 *If $\mathcal{B}_{++}$, $\mathcal{B}_{-+}$, $\mathcal{B}_{+-}$ or $\mathcal{B}_{--}$ is invariant, then the system is not globally stable.* □

It is interesting to note that the property of the matrix A of being Hurwitz gives information about the invariance of the sectors $\mathcal{B}_{+0}, \mathcal{B}_{-0}, \mathcal{B}_{0+}$, and $\mathcal{B}_{0-}$, but it gives no information about the invariance of the corner sectors $\mathcal{B}_{++}, \mathcal{B}_{+-}, \mathcal{B}_{--}$, and $\mathcal{B}_{-+}$. In fact, Lemma 13 implies that if A is a Hurwitz matrix, then none of the sectors $\mathcal{B}_{+0}, \mathcal{B}_{-0}, \mathcal{B}_{0+}$, and $\mathcal{B}_{0-}$ are invariant. In contrast, the next example shows that there are cases in which A is Hurwitz, but two of the corner sectors are invariant. This system is thus not globally asymptotically stable, as implied by Lemma 14.

Example 15 Consider the flow associated with the Hurwitz matrix $A_1 = \begin{bmatrix} 2 & 1 \\ -8 & -3 \end{bmatrix}$, as depicted in Figure 2. The sectors $\mathcal{B}_{+-}$ and $\mathcal{B}_{-+}$ are invariant, and so the system is only locally stable. A numerical treatment of the problem yields the unstable periodic solution, $x_p(t)$, shown in Figure 2. The open set encircled by $x_p(t)$ is the domain of attraction of the origin, and outside of $x_p(t)$ all trajectories are unbounded. □

The next result was proved in [1], and deals with the situation in which at least one of the strips $|x_1| \leq 1$ or $|x_2| \leq 1$ is invariant, *i.e.* one of the unions $\mathcal{B}_{0+} \cup \mathcal{B}_{00} \cup \mathcal{B}_{0-}$, or $\mathcal{B}_{+0} \cup \mathcal{B}_{00} \cup \mathcal{B}_{-0}$ is invariant.

Proposition 16 *Let Σ be a system as in (2). If the matrix A is Hurwitz and either $S_1 < 0$ and $D_1 < 0$ or $S_2 < 0$ and $D_2 < 0$, then $x_0 = 0$ is globally asymptotically stable.* □

It is not difficult to see that, by using the same arguments used in [1] to prove the previous proposition, one may extend this result to the case in which one of the previous quantities is zero. More precisely, the following proposition holds:

Proposition 17 *Let Σ be a system as in (2). If the matrix A is Hurwitz and one of the following conditions hold:*

Figure 3: Periodic Solution and Vector field for A_2.

(i) $S_1 \leq 0$ *and* $D_1 < 0$,

(ii) $S_1 < 0$ *and* $D_1 \leq 0$,

(iii) $S_2 \leq 0$ *and* $D_2 < 0$,

(iv) $S_2 < 0$ *and* $D_2 \leq 0$,

then $x_0 = 0$ *is globally asymptotically stable.* $\square$

B. No Invariant sectors

The following example shows that in the case that none of the sectors is invariant and A is Hurwitz, it is possible that x_0 is not globally asymptotically stable.

Example 18 Consider the flow generated by the matrix $A_2 = \begin{bmatrix} 2 & 4 \\ -8 & -3 \end{bmatrix}$. Then A_2 is Hurwitz, and there are no invariant sectors. Nevertheless there exists an unstable periodic solution and the associated system is only locally stable, as shown in Figure 3 $\square$

The mechanism for instability is simple to understand in this case. When $||x_0|| \gg 1$ we may disregard what happens in the sectors $\mathcal{B}_{+0}, \mathcal{B}_{-0}, \mathcal{B}_{0+}, \mathcal{B}_{0-}$. Thus, the flow is approximately given by

$$\dot{x} = Asgn(x). \tag{6}$$

Assuming that the direction of flow is clockwise, *i.e.* $S_1 > 0,\ S_2 < 0, D_1 < 0,\ D_2 > 0$, the trajectory for the an initial condition $\binom{\alpha}{0}$ may be easily integrated, as in Figure 4, to see that:

$$\tilde{\alpha} = \left(\frac{S_1 D_2}{S_2 D_1}\right)^2 \alpha = \frac{(a_{11} + a_{12})^2}{(a_{22} + a_{21})^2} \frac{(a_{22} - a_{21})^2}{(a_{11} - a_{12})^2} \alpha =: M(A)^2 \alpha \tag{7}$$

Thus the flow associated with A is globally asymptotically stable only if $M(A) < 1$. In the previous example $M(A_2) = 15/11 > 1$.

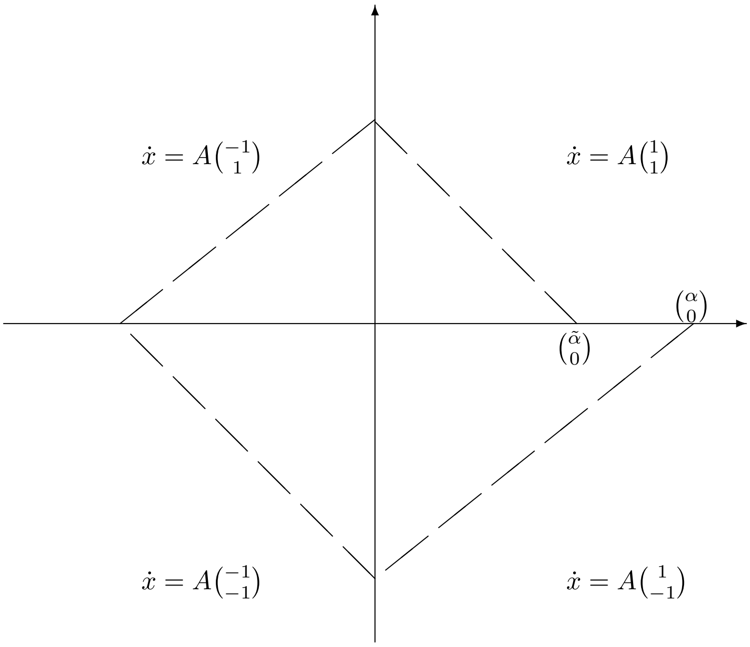

Figure 4: The flow associated with (6).

Remark 19 In performing this calculation we have assumed that the periodic solution encircles the origin with a clockwise orientation. This is true if and only if

$$S_1 > 0, \quad S_2 < 0, \quad D_1 < 0, \quad \text{and} \quad D_2 > 0. \tag{8}$$

In the sequel we assume that (8) holds. Note that if the periodic solution has an anticlockwise orientation we can transform to the case (8) via the co-ordinate transformation $\binom{x_1}{x_2} \to \binom{x_2}{x_1}$.

The following two examples deal with the cases when $M(A) < 1$ and $M(A) = 1$.

Example 20 Consider the flow generated by the matrix $A_3 = \begin{bmatrix} 2 & 4 \\ -4 & -3 \end{bmatrix}$. Then A_3 is Hurwitz, there are no invariant sectors, and $M(A_3) = 3/7$, thus one may expect global asymptotic stability. From the numerical results it is clear that the flow associated with A_3 is indeed globally stable. See Figure 5 for the vector field and a trajectory in the neighborhood of 0. □

Example 21 Now consider the flow generated by the matrix $A_4 = \begin{bmatrix} 2 & 4 \\ -6 & -3 \end{bmatrix}$. Then A_3 is Hurwitz, there are no invariant sectors, and $M(A_3) = 1$. With the analysis performed so far it is unclear whether the system associated with A_4 is globally stable or not, however it appears that it is indeed globally stable, see Figure 6. □

It is thus clear that knowledge of $M(A)$ is not sufficient to conclude global asymptotic stability of the system. We now calculate the return map exactly.

C. Calculation of the Return Map

Assume that (8) holds, so that trajectories circle the origin in a clockwise fashion. Consider the trajectory $\phi(t; \binom{\alpha}{1})$, where $\alpha \gg 1$ is chosen large enough that $\phi(t; \binom{\alpha}{1})$ does not intersect the

Figure 5: Stable Solutions and Vector field for A_3.

Figure 6: Stable Solution and Vector field for A_4.

region $\mathcal{B}_{00}$ before the next intersection of the local cross section $C = \left\{(\alpha, 1)^T : \alpha > 1\right\}$. Denote by $t^* > 0$ the first point in time when this occurs, *i.e.* $\phi(t^*; \binom{\alpha}{1}) = \binom{\alpha^*}{1}$. We now calculate an expression for α^* in terms of α and the matrix A which generates the flow.

Due to the symmetry of the system, we only explicitly calculate the maps from $\binom{\alpha}{1}$ to $\binom{\alpha_1}{-1}$ and the map from $\binom{\alpha_1}{-1}$ to $\binom{1}{\alpha_2}$ and then state the final result.

The map $r_1 : \alpha \mapsto \alpha_1$. In the sector $\mathcal{B}_{+0}$ the differential equations are

$$\begin{aligned} \dot{x}_1 &= a_{11} + a_{12}x_2, & x_1(0) = \alpha \\ \dot{x}_2 &= a_{21} + a_{22}x_2, & x_2(0) = 1 \end{aligned}$$

Thus for $a_{22} \neq 0$:

$$\begin{aligned} x_2(t) &= \left(1 + \frac{a_{21}}{a_{22}}\right) e^{a_{22}t} - \frac{a_{21}}{a_{22}} \\ x_1(t) &= \alpha + \left(a_{11} - \frac{a_{21}a_{12}}{a_{22}}\right) t + \left(e^{a_{22}t} - 1\right) \frac{a_{12}}{a_{22}} \left(1 + \frac{a_{21}}{a_{22}}\right) \end{aligned}$$

Solving $x_2(t_2) = -1$ for t_2 and then evaluating $\alpha_1 = x_1(t_2)$ yields

$$r_1(\alpha) = \alpha - 2\frac{a_{12}}{a_{22}} + \frac{\det A}{a_{22}^2} \ln\left(\frac{a_{21} - a_{22}}{a_{21} + a_{22}}\right) \tag{9}$$

If $a_{22} = 0$ the result is:

$$\begin{aligned} x_2(t) &= 1 + a_{21}t \\ x_1(t) &= \alpha + (a_{11} + a_{12})t + \frac{a_{12}a_{21}}{2}t^2 \end{aligned}$$

so that

$$r_1(\alpha) = \alpha - 2\frac{a_{11}}{a_{21}}$$

It is not difficult to show that:

$$\lim_{a_{22} \to 0} 2\frac{a_{12}}{a_{22}} - \frac{\det A}{a_{22}^2} \ln\left(\frac{a_{21} - a_{22}}{a_{21} + a_{22}}\right) = 2\frac{a_{11}}{a_{21}} \tag{10}$$

so without loss of generality we may use the expression (9).

The map $r_2 : \alpha_1 \mapsto \alpha_2$. In this case the vector field is constant, and it is straightforward to see that

$$r_2(\alpha_2) = -1 + (\alpha_2 - 1)\frac{(a_{22} - a_{21})}{(a_{11} - a_{12})} \tag{11}$$

The map $r : \alpha \mapsto \alpha^*$. It is important to note that each return map is affine. Additionally, the α_i is only offset when passing through the sectors $\mathcal{B}_{+0}, \mathcal{B}_{0-}, \mathcal{B}_{-0}, \mathcal{B}_{0+}$, and rescaled and offset when passing through the

sectors $\mathcal{B}_{++}, \mathcal{B}_{+-}, \mathcal{B}_{--}, \mathcal{B}_{-+}$. The total set of transitions is given by

$$\begin{pmatrix}\alpha\\1\end{pmatrix} \overset{r_1}{\mapsto} \begin{pmatrix}\alpha_1\\-1\end{pmatrix} \overset{r_2}{\mapsto} \begin{pmatrix}1\\\alpha_2\end{pmatrix} \overset{r_3}{\mapsto} \begin{pmatrix}-1\\\alpha_3\end{pmatrix} \overset{r_4}{\mapsto} \begin{pmatrix}\alpha_4\\-1\end{pmatrix} \overset{r_5}{\mapsto} \begin{pmatrix}\alpha_5\\1\end{pmatrix} \overset{r_6}{\mapsto} \begin{pmatrix}-1\\\alpha_6\end{pmatrix} \overset{r_7}{\mapsto} \begin{pmatrix}1\\\alpha_7\end{pmatrix} \overset{r_8}{\mapsto} \begin{pmatrix}\alpha^*\\1\end{pmatrix}$$

and

$$\alpha^* = r(\alpha) = M(A)^2\alpha + c \tag{12}$$

where

$$c = -\left(\frac{D_2S_1}{D_1S_2}+1\right)\left[\frac{D_2S_1}{D_1S_2}-1-\frac{D_2S_1}{D_1S_2}\left(\frac{\det A}{a_{22}^2}\ln\left(\frac{-D_2}{S_2}\right)-2\frac{a_{12}}{a_{22}}\right)+\frac{S_1}{S_2}\left(\frac{\det A}{a_{11}^2}\ln\left(\frac{-S_1}{D_1}\right)+2\frac{a_{21}}{a_{11}}\right)\right]$$

We make the convention that, if $a_{22}=0$, then the quantity $\left(\frac{\det A}{a_{22}^2}\ln\left(\frac{-D_2}{S_2}\right)-2\frac{a_{12}}{a_{22}}\right)$ is substituted with its limit as $a_{22}\to 0$, which is $2a_{11}/a_{21}$, and, if $a_{11}=0$, then the quantity $\left(\frac{\det A}{a_{11}^2}\ln\left(\frac{-S_1}{D_1}\right)+2\frac{a_{21}}{a_{11}}\right)$ is substituted with its limit as $a_{11}\to 0$ which is $-2a_{22}/a_{12}$.

If $M(A)>1$ then the system will be unstable, as $r(\alpha)>\alpha$ for sufficiently large α. Thus, when (8) holds, the condition

$$M(A)\le 1,$$

is necessary for global asymptotic stability. In this case there will exist an α^* for which $r(\alpha^*)=\alpha^*$. If $\alpha^*\ge 1$, then this means that there will exist a limit cycle passing through $(\alpha^*,1)$. To avoid this problem, we need to ensure that $\alpha^*<1$. It is straightforward to see that $\alpha^*<1$ iff $c-1+M(A)^2<0$. Now

$$c-1+M(A)^2 = -\left(\frac{D_2S_1}{D_1S_2}+1\right)\frac{S1}{S2}\left[-\frac{D_2}{D_1}\left(\frac{\det A}{a_{22}^2}\ln\left(\frac{-D_2}{S_2}\right)-2\frac{a_{12}}{a_{22}}\right)+\frac{\det A}{a_{11}^2}\ln\left(\frac{-S_1}{D_1}\right)+2\frac{a_{21}}{a_{11}}\right]$$

Since $\frac{D_2S_1}{D_1S_2}>0$, and $\frac{S_1}{S_2}<0$, it is easy to see that $c-1+M(A)<0$ iff

$$\frac{D_2}{D_1}\left(\frac{\det A}{a_{22}^2}\ln\left(\frac{-D_2}{S_2}\right)-2\frac{a_{12}}{a_{22}}\right)-\left(\frac{\det A}{a_{11}^2}\ln\left(\frac{-S_1}{D_1}\right)+2\frac{a_{21}}{a_{11}}\right)>0. \tag{13}$$

If the system is globally asymptotically stable, there will be no limit cycle, and so (13) gives another necessary condition. However, this does not preclude the existence of a limit cycle for which $\alpha^*=1-\epsilon$, where $0<\epsilon\ll 1$. The following example demonstrates such a limit cycle.

Example 22 Now consider the flow generated by the matrix $A_5=\begin{bmatrix}1 & 3.7\\ -6 & -1.2\end{bmatrix}$, as depicted in Figure 7. There exists a periodic solution which enters and leaves $\mathcal{B}_{00}$ on each of its faces. This demonstrates the possibility that a limit cycle exists for which $\alpha^*<1$, and which would not be detected by the condition (13). However a simple calculation shows that in this case $M(A)^2=1.347$, so existence of a limit cycle was expected. □

Thus it remains to be proven that in the case that $M(A)\le 1$ and $c-1+M(A)^2<0$, all trajectories intersecting $\mathcal{B}_{00}$ converge to $x^*=0$. This will require a further condition on two special trajectories of the system, however, see Conjecture 2 in the Conclusions.

Lemma 23 *Assume that A is a Hurwitz matrix, and that $S_1>0, S_2<0, D_1<0$ and $D_2>0$. Then the system is globally asymptotically stable iff $M(A)\le 1$, $c-1+M(A)^2<0$, $\phi(t;\binom{1}{1})\to 0$, and $\phi(t;\binom{-1}{1})\to 0$.* □

Proof. As $S_1>0, S_2<0, D_1<0$ and $D_2>0$, trajectories circle the origin in a clockwise sense, and we may construct the return map as detailed in this section.

Since $M(A)\le 1$ and $c-1+M(A)^2<0$, we have that any trajectory is bounded. Let $x(t)$ be a fixed trajectory, and Ω be its positive limit set. By classical results for second-order systems (see, for example [6], Theorem 1.3), we know that one of the following holds:

Figure 7: Periodic Solution and Vector field for A_5.

(a) Ω is an equilibrium point;

(b) Ω is a periodic orbit;

(c) Ω contains some equilibrium points together with some trajectories that have among these equilibrium points their positive and negative limit sets.

Since A is Hurwitz, zero is the only equilibrium point, and it is locally asymptotically stable. Thus we may exclude case (c), as otherwise there would exist a trajectory whose negative limit set is zero, contradicting stability of the origin.

Assuming that the model is not globally asymptotically stable, then according to (b), there must exist a limit cycle. Since $M(A) \leq 1$ and $c-1+M(A)^2 < 0$, this limit cycle has to intersect the sector $\mathcal{B}_{00}$. Moreover, this cycle determines two regions in the plane, one compact region inside the cycle and one outside. It is easy to see that necessarily at least one of the points $\binom{1}{1}$ or $\binom{-1}{1}$ lies in the unbounded region. Since its trajectory goes to zero, it must intersect the periodic orbit, yielding a contradiction. Thus there can not exist a limit cycle, and the system must be globally asymptotically stable.

The converse statement is obvious. ■

3.2 Proof of Theorem 9

With the results of the previous sections, we are in a position to prove Theorem 9.

Necessity Suppose that the system is globally asymptotically stable. Then it is also locally asymptotically stable, so A must be Hurwitz. Thus *(i)* is necessary.

Necessity of *(ii)* and *(iii)* follow from Lemma 14.

Now we consider the signs of S_1, S_2, D_1, and D_2. By *(ii)*, we know that S_1 or S_2 is negative, without loss of generality, we may assume that $S_2 < 0$. If $D_2 \leq 0$, then we are in the case *(c)*, thus *(iv)* holds. If $D_2 > 0$, then, by *(iii)*, we know that $D_1 < 0$. If $S_1 \leq 0$, then we are in the case *(a)*, so, again, *(iv)* holds. If $S_1 > 0$, then the sign of these four quantities are as in *(b)* of *(iv)*. Now, from Lemma 23, we have that *(b)* of *(iv)* must hold since $M(A) \leq 1$ is equivalent to $S_1D_2 \leq S_2D_1$, and $c-1+M(A)^2 < 0$ is equivalent to $C_1 > 0$. Thus condition *(iv)* is necessary.

Sufficiency To prove sufficiency of the conditions *(i)–(iv)*, consider that *(i)–(iii)* hold, and then consider the cases *(a), (b), (c)* and *(d)* separately. If *(a)* or *(c)* holds, then we can conclude global asymptotic stability by Proposition 17. If we are in the case *(b)*, then global asymptotic stability holds by Lemma 23. Case *(d)* is the dual of case *(b)*, so also in this case we conclude that the model is global asymptotically stable. Thus the conditions of the theorem are also sufficient.

4 Conclusions

In this paper we have presented necessary conditions and sufficient conditions for the *global asymptotic stability* of the system $\dot{x} = A\sigma(x)$. Our main contribution deals with the second-order case, that is for $A \in I\!R^{2\times 2}$, where necessary and sufficient conditions are presented. Theorem 9 gives our main result, but we expect that it may be simplified. Specifically we suspect that the following conjectures hold, which would lead to a simplification of conditions *(iv)(b)* and *(iv)(d)* of the theorem.

Conjecture 1 Suppose that A is Hurwitz and that $S_1 > 0,\ S_2 < 0,\ D_1 < 0$ and $D_2 > 0$. Then if $S_1 D_2 \le S_2 D_1$, then $C_1 > 0$.
(equivalently, if $S_1 < 0,\ S_2 > 0,\ D_1 > 0$ and $D_2 < 0$; then if $S_1 D_2 \ge S_2 D_1$, then $C_2 > 0$)

Conjecture 2 Suppose that A is Hurwitz and that $S_1 > 0,\ S_2 < 0,\ D_1 < 0$ and $D_2 > 0$. Then if $S_1 D_2 \le S_2 D_1$, then $\phi(t; \binom{1}{1}) \to 0$, and $\phi(t; \binom{-1}{1}) \to 0$ as $t \to \infty$.
(equivalently if $S_1 < 0,\ S_2 > 0,\ D_1 > 0$ and $D_2 < 0$; then if $S_1 D_2 \ge S_2 D_1$ then $\phi(t; \binom{1}{1}) \to 0$, and $\phi(t; \binom{-1}{1}) \to 0$ as $t \to \infty$.)

Conjecture 1 is a nonlinear constrained optimization problem and has been numerically "proven", but no analytical proof has been found yet. Conjecture 2 has held in all examples we have constructed, but is still unproved.

References

[1] F. Albertini and Domenico D'Alessandro, Asymptotic Stability of Continuous Time Systems with Saturation Nonlinearities, Systems and Control Letters, to appear.

[2] G.P. Barker, A. Berman, and R.J. Plemmons, Positive Diagonal Solutions to the Lyapunov Equations, Linear and Multilinear Algebra, Vol. 5, pp. 249-256, 1978.

[3] L.O. Chua and T. Roska, Stability of a class of nonreciprocal cellular neural networks, IEEE Trans. Circuits Syst. vol. 37, pp.1520-1527, Dec. 1990.

[4] L.O. Chua and C.W. Wu, On the universe of stable cellular neural networks, Int. J. Circuit Theory and App., vol. 20, pp. 497-572, 1992.

[5] M.Forti, "On global asymptotic stability of a class of nonlinear systems arising in neural networks theory", *J. Differential Equations*, vol.113, no. 1, pp. 246-264, 1994.

[6] J.K. Hale, Ordinary Differential Equation, Robert E. Krieger Publishing Company, Malabar, Florida, 1980.

[7] E.Kaszkurewicz and A. Bhaya, "Robust stability and diagonal Lyapunov functions", *SIAM J.Matrix Anal.Appl.*, vol 14, No.2, pp.508-520, 1993.

[8] D.Liu and A.N.Michel, Sparsely Interconnected Neural Networks for Associative Memories With Applications to Cellular Neural Networks, IEEE Trans. Circuits Syst. vol. 41, N. 4, pp.295-307, April 1994.

[9] A.N. Michel, J.A. Farrell and W.Porod, Qualitative Analysis of Neural Networks, IEEE Trans. Circuits Syst. vol 36, N. 2, pp 229-243, Febr. 1989.

[10] S.K.Persidskii, "Problem of Absolute Stability", *Automat. Remote Control*, 12 (1969), pp. 1889-1895.

[11] W.Tang and A.N.Michel, Qualitative analysis of Dynamical Systems Determined by Differential Inequalities with Applications to RoBust Stability, IEEE Trans. Circuits Syst. vol 41, N. 2, pp 377-386, May 1994.

[12] M.Vidyasagar, *Nonlinear Systems Analysis.* Prentice Hall, New Jersey, 2nd edition, 1993.