Quantum Symmetries and Cartan Decompositions in Arbitrary Dimensions

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Abstract

We investigate the relation between Cartan decompositions of the unitary group and discrete quantum symmetries. To every Cartan decomposition there corresponds a quantum symmetry which is the identity when applied twice. As an application, we describe a new and general method to obtain Cartan decompositions of the unitary group of evolutions of multipartite systems from Cartan decompositions on the single subsystems. The resulting decomposition, which we call of the *odd-even type*, contains, as a special case, the *concurrence canonical decomposition* (CCD) presented in [5],[6],[7] in the context of entanglement theory. The CCD is therefore extended from the case of a multipartite system of n qubits to the case where the component subsystems have arbitrary dimension.

Keywords: Lie groups decompositions, Quantum symmetries, Quantum multipartite systems.

1 Introduction

Decompositions of Lie groups have been extensively used in control theory to design control algorithms for bilinear, right invariant, systems with state varying on a Lie group. Once it is known how to factorize a target final state X_f as the product

$$X_f = X_1 X_2 \cdots X_r, \tag{1}$$

then the task of controlling to X_f can be reduced to the (simpler) task of controlling to the factors $X_1, ..., X_r$. In quantum information theory, a factorization of the type (1) can be interpreted as the implementation of a quantum logic operation with a sequence of elementary operations. In this case, the relevant Lie group is the Lie group of unitary matrices of dimensions n, $U(n)$. In general, a decomposition of the unitary evolution operator of the form (1) is useful to determine several aspects of the dynamics of quantum systems including the degree of entanglement (see e.g. [16]), time optimality of the evolution [11] and constructive controllability (see e.g. [8], [14]).

Most of the studies presented so far, which involve Lie group decompositions applied to the quantum systems, are concerned with low dimensional systems. For these systems, several

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complete and elegant results can be obtained, which also have important physical implications. Decompositions of the unitary group $U(n)$ for large n exist and can be applied to the dynamical analysis of high dimensional quantum systems. However, the information obtained with this study is rarely as useful and of direct physical interpretation as in the low dimensional cases. For multipartite systems, this motivates the search for Lie group decompositions constructed in terms of decompositions on the single subsystems. We shall construct such type of decomposition in the present paper.

The main motivation for the study presented here was given by the recent papers [5] [6], [7]. In these papers, a decomposition of $U(2^n)$ called the Concurrence Canonical Decomposition was obtained for a quantum system of n two level systems (qubits). Such a decomposition has the above mentioned feature of being expressed in terms of elementary decompositions on the single qubit subsystems. It is related to time reversal symmetry and this raises the question of what in general the relation is between quantum mechanical symmetries and decompositions. As we shall see here, the answer to this fundamental question is instrumental in developing a general method to construct decompositions of multipartite systems from elementary decompositions of the single subsystems. We shall develop a decomposition which we call of the 'odd-even type' that contains the concurrence canonical decomposition as a special case.

The paper is organized as follows. In Section 2 we review the basic definitions and results concerning discrete quantum symmetries and Cartan decompositions of the Lie algebra $su(n)$ and therefore the Lie group $SU(n)$. We shall stress the important result that, up to conjugacies, there are only three types of Cartan decompositions which are usually labeled as **AI**, **AII** and **AIII**. In Section 3, we investigate the relation between Cartan decompositions and quantum symmetries and establish a one to one correspondence between Cartan decompositions and a subclass of symmetries which we call Cartan symmetries. To every Cartan decomposition of the Lie algebra $u(n)$ and corresponding Cartan symmetry there corresponds a decomposition of the Jordan algebra of Hermitian matrices of dimension n, $iu(n)$ equipped with the anticommutator operation. This is described in Section 4. This is also the crucial fact used to develop the general decomposition of the odd-even type for multipartite systems in arbitrary dimensions in Section 5. This decomposition is a Cartan decomposition and, in Section 6, we show how to determine its type (**AI** or **AII**). The Cartan decomposition also leads to a decomposition of the evolution of any quantum system into the product of an evolution with antisymmetric Hamiltonian and one with symmetric Hamiltonian with respect to a Cartan symmetry. This result is discussed in Remark 4.1.

2 Background material

2.1 Discrete Symmetries in Quantum Mechanics

Given a quantum system with underlying Hilbert space $\mathcal{H}$, a quantum mechanical *symmetry* is defined (see e.g. [9] Chapter 7, [13] Chapter 4) as a one to one and onto map $\Theta : \mathcal{H} \rightarrow \mathcal{H}$ such that physically indistinguishable states are also mapped into physically indistinguishable states i.e. for every $|\psi> \in \mathcal{H}$ and $\phi_1 \in \mathbf{R}$,

$$\Theta(e^{i\phi_1}|\psi>) = e^{i\phi_2}\Theta(|\psi>), \tag{2}$$

for some $\phi_2 \in \mathbf{R}$. Moreover Θ preserves the inner product of two states, namely if $|\tilde{\alpha}>:= \Theta|\alpha>$, then, for any two states $|\alpha>$ and $|\beta>$

$$|<\tilde{\alpha}|\tilde{\beta}>| = |<\alpha|\beta>|. \tag{3}$$

In this definition, we omit for simplicity the consideration of selection rules and assume that all the states are physically realizable. A detailed discussion of this point can be found in [9].

According to Wigner's theorem [15], every such operation Θ can be represented as

$$\Theta = e^{i\phi}U, \tag{4}$$

where ϕ is a constant, physically irrelevant, real parameter, and U is either a *unitary* operator or an *anti-unitary* one. Recall that an anti-unitary operator U, $|\alpha> \rightarrow |\tilde{\alpha}>:= U|\alpha>$ is defined as satisfying

$$<\tilde{\beta}|\tilde{\alpha}>=<\beta|\alpha>^*, \tag{5}$$

$$U(c_1|\alpha> + c_2|\beta>) = c_1^*U|\alpha> + c_2^*U|\beta> . \tag{6}$$

Once a basis of the Hilbert space $\mathcal{H}$ is chosen, an anti-unitary operator U can always be written as

$$U|\alpha>= XK|\alpha>, \tag{7}$$

where K is the operation which conjugates all the components of the vector $|\alpha>$ and X is unitary.

A symmetry Θ, whether unitary or anti-unitary, induces a transformation on the space of Hermitian operators A as

$$A \rightarrow \Theta A \Theta^{-1} := \bar{\theta}(A). \tag{8}$$

It is in fact easily verified that $\bar{\theta}(A)$ is a linear and Hermitian operator. Moreover the eigenvalues of $\bar{\theta}(A)$ are the same as those of A and a set of orthonormal eigenvectors are given by $\Theta|\alpha_j>$, where $|\alpha_j>$ is an orthonormal basis of eigenvectors of A. It can be proved [9], [13], that, up to a phase factor, $\bar{\theta}(A) := \Theta A \Theta^{-1}$ is the only choice that guarantees

$$| < \tilde{\alpha} | \bar{\theta}(A) | \tilde{\beta} >= | < \alpha | A | \beta > |. \tag{9}$$

Description of the symmetry Θ is usually done by specifying how $\bar{\theta}$ acts on Hermitian operators rather than how Θ acts on states. This is because, Hermitian operators represent physical observables and therefore the action of θ on observables is typically suggested by physical considerations. For example, the *space translation symmetry* has to be such that

$$\bar{\theta}(\hat{x}) = \hat{x} - a, \tag{10}$$

for some constant a where $\hat{x}$ is the position operator. As another example, the *parity* or *space inversion symmetry* is defined such that

$$\bar{\theta}(\hat{x}) = -\hat{x}. \tag{11}$$

On the other hand, specification of $\bar{\theta}$ on an irreducible set of observables uniquely determines Θ up to a phase factor [9]. Recall that an irreducible set of observables $\{A_j\}$ is defined such that if an observable B commutes with all of the $\{A_j\}$, then B is a multiple of the identity.

An observable H is said to *satisfy a symmetry* Θ or *to be symmetric with respect to* Θ if

$$\bar{\theta}(H) = H, \tag{12}$$

or equivalently

$$\Theta H = H \Theta. \tag{13}$$

It is said to be *antisymmetric with respect to* Θ if

$$\bar{\theta}(H) = -H \leftrightarrow \Theta H = -H \Theta. \tag{14}$$

A special type of symmetry is the *time reversal symmetry.* In classical mechanics a time reversal symmetry changes a system into one which evolves with time reversal trajectories. This suggests to define a time reversal symmetry in quantum mechanics so that $\bar{\theta}$ acts on the position $\hat{x}$ and momentum operator $\hat{p}$ according to

$$\bar{\theta}(\hat{x}) = \hat{x}, \tag{15}$$

$$\bar{\theta}(\hat{p}) = -\hat{p}. \tag{16}$$

This implies that the corresponding Θ transforms momentum eigenvectors $|p>$ as

$$\Theta |p >= | - p > . \tag{17}$$

If the system under consideration has no spin degree of freedom, then $\hat{x}$ and $\hat{p}$ form an irreducible set of observables and therefore (15) and (16) uniquely specify the transformation Θ on the state. Moreover, from the definition of angular momentum $\hat{L} := \hat{x} \times \hat{p}$, we obtain

$$\bar{\theta}(\hat{L}) = \bar{\theta}(\hat{x}) \times \bar{\theta}(\hat{p}) = -\hat{L}. \tag{18}$$

For a system with spin angular momentum $\hat{S}$, we impose by definition, according to (18)

$$\bar{\theta}(\hat{S}) = -\hat{S}, \tag{19}$$

and $\hat{x}$, $\hat{p}$, $\hat{S}$ form an irreducible set of observables. If $|m>$ is an eigenvector of the (spin) angular momentum corresponding to eigenvalue m, we have

$$\Theta|m>=|-m>. \tag{20}$$

From these specifications, it is possible to obtain an explicit expression of the time reversal symmetry for a system of N particles with spin operators $\hat{S}_1$,...,$\hat{S}_N$. It is given (in a basis of tensor products of the eigenstates of the $z-$ component of the spin operators) by (see [9])

$$\Theta = e^{-\frac{i\pi}{\hbar}(\hat{S}_{1,y}+\hat{S}_{2,y}+...+\hat{S}_{N,y})}K, \tag{21}$$

where $\hat{S}_{j,y}$ is the y component of the spin operator corresponding to the $j-$th particle, $j = 1, ..., N$, and K is the conjugation operator (same as in (7)).

2.2 Cartan involutions and decompositions of $su(n)$

We discuss next the Cartan decompositions for the Lie algebra $su(n)$. What we say could be generalized to general semisimple Lie algebras. We refer to [10] for more details.

A *Cartan decomposition* of $su(n)$ is a vector space decomposition

$$su(n) = \mathcal{K} \oplus \mathcal{P}, \tag{22}$$

where the subspaces $\mathcal{K}$ and $\mathcal{P}$ satisfy the commutation relations

$$[\mathcal{K}, \mathcal{K}] \subseteq \mathcal{K}, \tag{23}$$

$$[\mathcal{K}, \mathcal{P}] \subseteq \mathcal{P}, \tag{24}$$

$$[\mathcal{P}, \mathcal{P}] \subseteq \mathcal{K}. \tag{25}$$

In particular, notice that $\mathcal{K}$ is a subalgebra of $su(n)$. A Cartan decomposition of $su(n)$ induces a factorization of the elements of the Lie groups $SU(n)$. Let us denote by $e^{\mathcal{L}}$ the connected Lie group associated to a generic Lie algebra $\mathcal{L}$. Then, given a Cartan decomposition (22), every element X in $SU(n)$ can be written as

$$X = KP, \tag{26}$$

where $K \in e^{\mathcal{K}}$ and P is the exponential of an element of $\mathcal{P}$. Moreover if $\mathcal{A}$ is a maximal Abelian subalgebra of $su(n)$, with $\mathcal{A} \subseteq \mathcal{P}$, then one can prove that

$$\cup_{K \in e^{\mathcal{K}}} K\mathcal{A}K^* = \mathcal{P}. \tag{27}$$

This implies that one can write P in (26) as $P = K_1 A K_1^*$ with $K_1 \in e^{\mathcal{K}}$ and $A \in e^{\mathcal{A}}$. Therefore every element X in $SU(n)$ can be written as

$$X = K_1 A K_2, \tag{28}$$

with $K_1, K_2 \in e^{\mathcal{K}}$ and $A \in e^{\mathcal{A}}$. This is often referred to as KAK *decomposition.*

A *Cartan involution* of $su(n)$ is a homomorphism $\theta : su(n) \to su(n)$ such that θ^2 is equal to the identity on $su(n)$. Associated to a Cartan decomposition (22) is a Cartan involution which is equal to the identity on $\mathcal{K}$ and multiplies by -1 the elements of $\mathcal{P}$, i.e.

$$\theta(K) = K, \qquad \forall K \in \mathcal{K}, \tag{29}$$

$$\theta(P) = -P, \qquad \forall P \in \mathcal{P}. \tag{30}$$

Therefore, given a Cartan decomposition, relations (29) and (30) determine a Cartan involution θ. Viceversa given a Cartan involution θ, the $+1$ and -1 eigenspaces of θ determine a Cartan decomposition.

According to a theorem of Cartan [10], there exist only three types of Cartan decompositions for $su(n)$ up to conjugacy. More specifically, given a Cartan decomposition (22) there exists a matrix $H \in SU(n)$ such that $\mathcal{K}' := H\mathcal{K}H^*$, $\mathcal{P}' := H\mathcal{P}H^*$, where $\mathcal{K}'$ and $\mathcal{P}'$ fall in one of the following cases labeled **AI**, **AII** and **AIII**[3] .

AI

$$\mathcal{K}' = so(n), \quad \mathcal{P}' = so(n)^{\perp}, \tag{31}$$

where $so(n)$ is the Lie algebra of real skew-Hermitian matrices of dimension n, $so(n)^{\perp}$ is the vector space over the reals of purely imaginary skew-Hermitian matrices. The corresponding Cartan involution, which we denote by θ_I, returns the complex conjugate of a matrix, i.e.

$$\theta_I(A) := \bar{A}. \tag{32}$$

[3]In the following definitions and in the rest of the paper the inner product $< A, B >$ in $su(n)$ is defined as $< A, B > := Tr(AB^*)$ and it is proportional to the Killing form (see e.g. [10])

AII

$$\mathcal{K}' = sp(\frac{n}{2}), \quad \mathcal{P}' = sp(\frac{n}{2})^{\perp}, \tag{33}$$

where, we are assuming n even, and $sp(\frac{n}{2})$ is the Lie algebra of symplectic $n \times n$ matrices i.e. the Lie algebra of skew-Hermitian matrices A satisfying

$$AJ + JA^T = 0. \tag{34}$$

The matrix J is defined as

$$J := \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ -I_{\frac{n}{2}} & 0 \end{pmatrix}. \tag{35}$$

The corresponding Cartan involution θ_{II} is given by

$$\theta_{II}(A) := J\bar{A}J^{-1} = -J\bar{A}J. \tag{36}$$

AIII. In this case $\mathcal{K}'$ is the set of all the skew-Hermitian matrices A of the form

$$A = \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}, \tag{37}$$

where $R \in u(p)$, $S \in u(q)$, $p, q > 0$, $p + q = n$ and $Tr(R) + Tr(S) = 0$. $\mathcal{P}'$ is equal to $\mathcal{K}'^{\perp}$. The corresponding Cartan involution is given by

$$\theta_{III}(A) := I_{p,q} A I_{p,q}, \tag{38}$$

where the matrix $I_{p,q}$ is defined as the block matrix $I_{p,q} := \begin{pmatrix} I_{p\times p} & 0 \\ 0 & -I_{q\times q} \end{pmatrix}$.

Several authors have proposed Lie algebra decompositions for $su(n)$ that, although special cases of the general Cartan decomposition, are of particular significance in some contexts. For example, Khaneja and Glaser [12] (see also [4] for the relation of this decomposition with Cartan decomposition) have factorized unitary evolutions in $SU(2^n)$, namely unitary evolution of n two level quantum systems (*qubits*), into local operations i.e. operations on only one qubit and two-qubits operations. This result has consequences both in the study of universality of quantum logic gates and in control theory. In the latter context, one would like to decompose the task of steering the evolution operator to a prescribed target into a sequence of steering problems to intermediate targets with a determined structure.

Another decomposition which is of particular interest to us is the *Concurrence Canonical Decomposition* (CCD) of $su(2^n)$ which was studied in [5], [6], [7] in the context of *entanglement* and *entanglement dynamics*. In this decomposition, $\mathcal{K}'$ and $\mathcal{P}'$ are real span of tensor products, multiplied by i, of n 2×2 matrices chosen in the set $\{I_{2\times 2}, \sigma_x, \sigma_y, \sigma_z\}$, where $\sigma_{x,y,z}$ are the x, y, z Pauli matrices. In particular, $\mathcal{K}'$ is spanned by tensor products with an *odd* number of

Pauli matrices and $\mathcal{P}'$ is spanned by tensor products with an *even* number of Pauli matrices. It was shown in [5], [6] that for n even this decomposition is a Cartan **AI** decomposition and for n odd is a Cartan **AII** decomposition. One of the primary goals of the present paper is to extend the CCD to the case of multipartite systems of arbitrary dimensions. The CCD was also used in [1], [2] to characterize the input-output equivalent models of networks spin $\frac{1}{2}$, in a problem motivated by parameter identification for spin Hamiltonians. Generalizations of these results for networks of spins of any value, in view of the results presented here, will be given in a forthcoming paper [3].

3 Relation between Cartan decompositions and symmetries

The results of [5], [6] associate to the Concurrence Canonical Decomposition a time reversal symmetry. In particular, there is a relation between the involution θ corresponding to the CCD and the time reversal symmetry Θ in (21) (with $N = n$ the number of spin assumed all equal to $\frac{1}{2}$). This relation is given by

$$\theta(A) = \Theta A \Theta^{-1}, \quad \forall A \in su(2^n), \tag{39}$$

where the right hand side needs to be interpreted as composition of operators. It is also easily seen, using only the fact that the time reversal symmetry is antiunitary and the general formula (7), that, if $\bar{\theta}$ is the time reversal symmetry on observables iA, we have

$$\bar{\theta}(iA) := \Theta i A \Theta^{-1} = -i\Theta A \Theta^{-1} = -i\theta(A). \tag{40}$$

This rises the question of whether there is in general a one to one correspondence between symmetries Θ, $\bar{\theta}$ (8), and Cartan involutions θ and therefore Cartan decompositions. Also the question arises on whether formula (see (40))

$$\bar{\theta}(iA) = -i\theta(A), \forall A \in u(n) \tag{41}$$

is always valid. We shall investigate these issues in this section. We shall see that only a particular class of symmetries, which we call *Cartan symmetries* give rise to Cartan involutions.

Definition 3.1 A symmetry Θ is called a Cartan symmetry if and only if Θ^2 is equal to the identity up to a phase factor.

Cartan symmetries have the property that applied two times to any state return the physical state unchanged. For example the time reversal symmetry and the parity (11) are Cartan symmetries while the space translation symmetry (10) is not a Cartan symmetry.

Whether or not a symmetry is a Cartan symmetry can be verified once we have its representation in a given basis i.e. (cf. (7))

$$\Theta|\alpha>= XK|\alpha>, \tag{42}$$

where X is unitary and K is the identity if Θ is a unitary symmetry and is the conjugation of all the components of $|\alpha>$ if Θ is antiunitary. Θ is a Cartan symmetry if and only if

$$X\bar{X} = e^{i\phi} I_{n\times n}, \tag{43}$$

for some $\phi \in \mathbf{R}$ in the antiunitary case and $X^2 = e^{i\phi} I_{n\times n}$ for some $\phi \in \mathbf{R}$ in the unitary case. This is clearly of the particular orthonormal basis chosen. If Θ is antiunitary and T is a unitary transformation which transforms one orthonormal basis into another and XK describes the action of the symmetry in one basis then $TX\bar{T}^*K$ describes the action of the symmetry in the new basis. It is easily seen that if X satisfies (43) so does $TX\bar{T}^*$ and an analogous fact holds for unitary symmetries.

Generalizing the approach in [5] [6] we now give the following definition.

Definition 3.2 The *transformation induced by a symmetry* Θ on $su(n)$ is defined as

$$\theta(A) := \Theta A \Theta^{-1}. \tag{44}$$

Notice this definition is analogous to the one of symmetries $\bar{\theta}$ on observables (8) which we repeat here with different notations:

$$\bar{\theta}(iA) := \Theta i A \Theta^{-1}, \qquad \forall A \in su(n). \tag{45}$$

In order to give an expression of the induced transformation in a given basis, we consider the antiunitary and the unitary case separately. In the antiunitary case, if K is the conjugation $\Theta = XK$, $\Theta^{-1} = \bar{X}^* K = KX^*$, θ which gives

$$\theta(A) = X\bar{A}X^*. \tag{46}$$

Analogously, one obtains

$$\theta(A) = XAX^*, \tag{47}$$

in the unitary case.

Theorem 1 *The transformation θ on $su(n)$ induced by a symmetry $(\Theta, \bar{\theta})$ is a Cartan involution if and only if $(\Theta, \bar{\theta})$ is a Cartan symmetry. Moreover, if $(\Theta, \bar{\theta})$ is antiunitary, we have $\forall A \in su(n)$,*

$$\bar{\theta}(iA) := -i\theta(A). \tag{48}$$

Moreover, if $(\Theta, \bar{\theta})$ *is unitary, we have* $\forall A \in su(n)$,

$$\bar{\theta}(iA) := i\theta(A). \tag{49}$$

Proof. It is easily verified that θ defined in (46) or (47) is a homomorphism. Moreover, assume Θ is a Cartan symmetry. Then we calculate (in the antiunitary case and analogously in the unitary case)

$$\theta^2(A) = X(X\bar{\bar{A}}X^*)X^* = X\bar{X}A\bar{X}^*X^* = A, \tag{50}$$

where in the last equality we have used the fact that Θ is a Cartan symmetry. Therefore the associated θ is a Cartan involution.

Conversely consider a Cartan involution θ on $su(n)$, induced by a symmetry Θ. Then we want to show that Θ is a Cartan symmetry.

Since θ must be of the type **AI**, **AII** or **AIII**, we must be able to write it as (32), (36) or (38) up to conjugacy. In particular there exists a unitary T such that (case **AI**)

$$\theta(B) = T\bar{T}^*\bar{B}\bar{T}T^*, \forall B \in su(n), \tag{51}$$

or such that (case **AII**)

$$\theta(B) = TJ\bar{T}^*\bar{B}\bar{T}J^{-1}T^*, \forall B \in su(n), \tag{52}$$

or such that (case **AIII**)

$$\theta(B) = TI_{p,q}T^*BTI_{p,q}T^*, \tag{53}$$

in the $AIII$ case [4] We take Θ in the cases **AI**, **AII** and **AIII** given by (cf. (46) and (47))

$$\Theta = T\bar{T}^*K, \tag{54}$$

$$\Theta = TJ\bar{T}^*K, \tag{55}$$

and

$$\Theta = TI_{p,q}T^*, \tag{56}$$

respectively. It is easily verified that these are all Cartan symmetries, i.e. $X\bar{X} = I$ with $X = T\bar{T}^*$, $X = TJ\bar{T}^*$ and $X^2 = I_{n\times n}$ with $X = TI_{p,q}T^*$. Moreover the choice is unique, up to a phase factor which does not change the property of the symmetry of being a Cartan symmetry, as the set of matrices $su(n)$ is an irreducible set of skew-Hermitian operators. This concludes the proof of the theorem. □

[4] In the case AI in appropriate coordinates the involution is equal to conjugation. If T is the matrix that makes the change of coordinates, every $B \in su(n)$ can be written as $B = TAT^*$ for a unique A in $su(n)$ and therefore $A = T^*BT$. Now $\theta_I(B) = T\bar{A}T^*$ and replacing $A = T^*BT$, one obtains (51). The other cases are analogous.

Remark 3.3 The theorem could have been stated in a somewhat stronger form. In fact, the proof shows not only that the symmetry corresponding to a Cartan involution is a Cartan symmetry by also that it exists and is unique up to a phase factor. Therefore there is a one to one correspondence given by (44) (45) between Cartan symmetries and Cartan involutions and therefore decompositions.

Remark 3.4 It follows from the proof of the theorem that antiunitary Cartan symmetries correspond to Cartan involutions of the type **AI** and **AII** while unitary ones give rise to Cartan involutions of type **AIII.**

4 Dual structures of $u(n)$ and $iu(n)$; Commutation and anticommutation relations

In this section, we study the dual structure of the Lie algebra $u(n)$ of skew-Hermitian matrices and the Jordan algebra $iu(n)$ of Hermitian matrices equipped with the anticommutator operation. We shall see that to a Cartan decomposition of $u(n)$ there correspond a decomposition of $iu(n)$ which we also call 'Cartan' where the role of the subspaces are possibly reversed. This correspondence is crucial in the development of general decompositions for multipartite systems developed in the next section. The situation is somehow different if we consider decompositions of the type **AI** and **AII** and if we consider decompositions of the type **AIII**. Therefore we shall consider the two cases separately. Only the case **AI** and **AII** will in fact be used in the next section.

Consider a Cartan decomposition of $su(n)$ (22), (23), (24), (25) of the type **AI** or **AII**, its corresponding Cartan involution θ and Cartan symmetry $\bar{\theta}$ related through (48). This decomposition naturally extends to a decomposition of $u(n)$ by replacing $\mathcal{P}$ with $\mathcal{P} \oplus span\{iI_{n\times n}\}$. We shall denote this subspace, with some abuse of notation, again by $\mathcal{P}$. So that

$$u(n) = \mathcal{K} \oplus \mathcal{P}, \tag{57}$$

$\mathcal{P} = \mathcal{K}^{\perp}$, where the orthogonal complement is now taken in $u(n)$, and the commutation relation (23), (24) and (25) also holds, within $u(n)$. The Cartan involution θ of the type **AI** and **AII,** is naturally extended to $u(n)$ and $span\{iI_{n\times n}\}$ will belong to the -1 eigenspace of θ so that the new definition of $\mathcal{P}$ is consistent with the fact that $\mathcal{P}$ is the -1 eigenspace of θ. The corresponding symmetry on $iu(n)$ will be given by (48) or equivalently by (45).

Consider now $iu(n)$[5] which has a structure of a Jordan algebra when equipped with the

[5]From now on in this section all the matrices are assumed skew-Hermitian so that matrices multiplied by i are Hermitian. An exception is in Remark 4.1 below where the H's denote Hermitian operators.

anti-commutator operation

$$\{iA, iB\} := (iA)(iB) + (iB)(iA). \tag{58}$$

Associated to a Cartan decomposition of $u(n)$ (57) is a decomposition of $iu(n)$, which we also call Cartan decomposition, given by

$$iu(n) = i\mathcal{K} \oplus i\mathcal{P}. \tag{59}$$

Moreover it follows from (45) that $\bar{\theta}$ is a homomorphism on the Jordan algebra $iu(n)$. It is in fact an involution as $\bar{\theta}^2$ is equal to the identity map. It follows from (48) that $i\mathcal{P}$ and $i\mathcal{K}$ are, respectively, the $+1$ and -1 eigenspaces of $\bar{\theta}$ and therefore we have

$$\{i\mathcal{P}, i\mathcal{P}\} \subseteq i\mathcal{P}, \qquad \{i\mathcal{P}, i\mathcal{K}\} \subseteq i\mathcal{K}, \qquad \{i\mathcal{K}, i\mathcal{K}\} \subseteq i\mathcal{P}. \tag{60}$$

So the roles of the subspaces $\mathcal{K}$ and $\mathcal{P}$ are somehow reversed when going from $u(n)$ to $iu(n)$.

In the case of a Cartan decomposition of the type **AIII** of $su(n)$, the construction is similar. In this case, we extend the Cartan decomposition to $u(n)$ by incorporating $span\{iI_{n\times n}\}$ into $\mathcal{K}$ rather than into $\mathcal{P}$. The commutation relations (23) (24) and (25) are still valid with this modified definition. The induced decomposition on $iu(n)$ given in (59) is such that $i\mathcal{K}$ and $i\mathcal{P}$ are the $+1$ and -1 eigenspaces of the involution $\bar{\theta}$. This follows from the correspondence between θ and $\bar{\theta}$ which in this case is given by (49). We have

$$\{i\mathcal{K}, i\mathcal{K}\} \subseteq i\mathcal{K}, \qquad \{i\mathcal{P}, i\mathcal{K}\} \subseteq i\mathcal{P}, \qquad \{i\mathcal{P}, i\mathcal{P}\} \subseteq i\mathcal{K}. \tag{61}$$

Remark 4.1 (Decomposition of dynamics) It was pointed out in ([6]) that every evolution of a finite dimensional quantum system $U := e^{iH}$ can be decomposed as

$$e^{iH} = e^{iH_a} e^{iH_s}, \tag{62}$$

where the Hamiltonian H_s is symmetric with respect to time reversal symmetry and the Hamiltonian H_a is antisymmetric (cf. (12)-(14)) with respect to time symmetry, i.e. $\bar{\theta}(H_s) = H_s$ and $\bar{\theta}(H_a) = -H_a$. Therefore every evolution can be decomposed into a time symmetric one and a time antisymmetric one. In view of the above treatment such a decomposition can be extended to any Cartan symmetry. For Cartan symmetries of the type **AI** and **AII**, iH_a and iH_s are in the $\mathcal{K}$ and $\mathcal{P}$ subspace of the associated decomposition so that the decomposition (62) is Cartan decomposition (26). To this purpose also notice that the $\mathcal{K}$ Lie algebras in all the three types of Cartan decompositions correspond to semisimple compact Lie groups so that the exponential map is surjective [10]. The same argument can be repeated for Cartan symmetries of the type **AIII** with the only change that this time $iH_s \in \mathcal{K}$ and $iH_a \in \mathcal{P}$.

5 Cartan decompositions for multipartite systems in arbitrary dimensions; Decompositions of the odd-even type

In this section we shall generalize the Concurrence Canonical Decomposition to the general case i.e. to the case of a multipartite system consisting of any number of quantum systems of *any* dimension. We shall call the general decomposition a *decomposition of the even-odd type* because the two subspaces in the Cartan decomposition consist of elements which are tensor products of an odd or even number of elements in appropriate subspaces. In doing this, we shall make use of the correspondence between decompositions in $u(n)$ and decompositions in $iu(n)$ described in the previous section. In particular, we shall consider decompositions of $u(n)$-$iu(n)$ associated to antiunitary Cartan symmetries. To this correspond Cartan decompositions and involutions which have the property to extend to Cartan decompositions and involutions for multipartite systems as we shall now describe.

Consider a multipartite quantum system composed of N quantum systems of dimensions n_1, n_2,...,n_N and with Hilbert spaces $\mathcal{H}_1$,...,$\mathcal{H}_N$. The space of skew-Hermitian (Hermitian) operators acting on the space $\mathcal{H}_j$, $j = 1, ..., N$ is $u(n_j)$ ($iu(n_j)$). The space of skew-Hermitian (Hermitian) operators acting on the total Hilbert space $\mathcal{H}_{TOT} := \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$ is $u(n_1 n_2 \cdots n_N)$ (iu$(n_1 n_2 \cdots n_N)$). Consider now Cartan decompositions of $u(n_j)$, not necessarily all of the same type but all of the type **AI** or **AII**,

$$u(n_j) = \mathcal{K}_j \oplus \mathcal{P}_j, \tag{63}$$

and the corresponding decompositions for $iu(n_j)$

$$iu(n_j) = i\mathcal{K}_j \oplus i\mathcal{P}_j, \tag{64}$$

satisfying, with obvious modification of the notations, the commutation relations (23), (24), (25) and anticommutation relations (60). Let us denote by σ_j a generic element of an orthonormal basis in $i\mathcal{K}_j$ which is an Hermitian matrix. Also let us denote by S_j a generic element of an orthonormal basis in $i\mathcal{P}_j$ which is also an Hermitian matrix. An orthonormal basis in $iu(n_1 n_2 \cdot \cdot \cdot n_N)$ is given by tensor products of the form

$$F := T_1 \otimes T_2 \otimes \cdots \otimes T_N, \tag{65}$$

where $T_j = \sigma_j$ or $T_j = S_j$, with all the possible combinations of σ's and S's in the N places. We define $\mathcal{I}_o$ ($\mathcal{I}_e$) the subspace of $iu(n_1 n_2 \cdots n_N)$ spanned by tensor products which display an odd (even) number of elements σ, so that we write

$$iu(n_1 n_2 \cdots n_N) = \mathcal{I}_o \oplus \mathcal{I}_e. \tag{66}$$

We shall call this decomposition, along with the corresponding decomposition of $u(n_1 n_2 \cdots n_N)$

$$u(n_1 n_2 \cdots n_N) = i\mathcal{I}_o \oplus i\mathcal{I}_e, \tag{67}$$

a decomposition of the *odd-even type*. We have the following result

Theorem 2 *The decomposition of the odd-even type (66) (67) is a Cartan decomposition which is associated to a antiunitary Cartan symmetry, i.e.*

$$[i\mathcal{I}_o, i\mathcal{I}_o] \subseteq i\mathcal{I}_o, \qquad [i\mathcal{I}_o, i\mathcal{I}_e] \subseteq i\mathcal{I}_e, \qquad [i\mathcal{I}_e, i\mathcal{I}_e] \subseteq i\mathcal{I}_o, \tag{68}$$

$$\{\mathcal{I}_o, \mathcal{I}_o\} \subseteq \mathcal{I}_e, \qquad \{\mathcal{I}_o, \mathcal{I}_e\} \subseteq \mathcal{I}_o, \qquad \{\mathcal{I}_e, \mathcal{I}_e\} \subseteq \mathcal{I}_e \tag{69}$$

Proof. The proof is by induction on the number of systems N. For $N = 1$, $\mathcal{I}_o = i\mathcal{K}_1$ and $\mathcal{I}_e = i\mathcal{P}$ so that the commutation and anticommutation relations (68) and (69) are the same as (23)-(25) and (60), respectively. Assuming now (68) and (69) true for every number of subspaces strictly less then N we can verify (68) for N by using the formula

$$[A \otimes B, C \otimes D] = \frac{1}{2}([A, C] \otimes \{B, D\} + \{A, C\} \otimes [B, D]), \tag{70}$$

and considering all the subcases. For example, to show the first one of (68) one considers the four cases, by indicating with the superscript the number of factors in the tensor products:

C1: $\{ A \in \mathcal{I}_o^{N-1}, B \in i\mathcal{I}_e^1, C \in \mathcal{I}_o^{N-1}, D \in i\mathcal{I}_e^1 \}$
C2: $\{ A \in \mathcal{I}_o^{N-1}, B \in i\mathcal{I}_e^1, C \in \mathcal{I}_e^{N-1}, D \in i\mathcal{I}_o^1 \}$
C3: $\{ A \in \mathcal{I}_e^{N-1}, B \in i\mathcal{I}_o^1, C \in \mathcal{I}_o^{N-1}, D \in i\mathcal{I}_e^1 \}$
C4: $\{ A \in \mathcal{I}_e^{N-1}, B \in i\mathcal{I}_o^1, C \in \mathcal{I}_e^{N-1}, D \in i\mathcal{I}_o^1 \}$

Analogously, one can verify (69) by using induction along with the formula

$$\{A \otimes B, C \otimes D\} = \frac{1}{2}([A, C] \otimes [B, D] + \{A, C\} \otimes \{B, D\}). \tag{71}$$

$\square$

Associated to a decomposition of the odd-even type is a Cartan involution on $u(n_1 n_2 \cdots n_N)$, θ^{TOT}, and the corresponding Cartan symmetry on the space $iu(n_1 n_2 \cdots n_N)$, $\bar{\theta}^{TOT}$. If $\theta_1, ..., \theta_N$ and $\bar{\theta}_1, ..., \bar{\theta}_N$ are the Cartan involutions and symmetries associated to the $1, 2, ..., N$-th decomposition, θ^{TOT} and $\bar{\theta}^{TOT}$ can be described as follows.

Let A be an element of the orthonormal basis of $i\mathcal{I}_o$, i.e. it can be written as

$$A = T_1 \otimes \cdots \otimes (iT_k) \otimes \cdots \otimes T_N, \tag{72}$$

where $T_j = \sigma_j$ or $T_j = S_j$, with an odd number of σ's. Then

$$\theta^{TOT}(A) = \bar{\theta}_1(T_1) \otimes \cdots \otimes \theta_k(iT_k) \otimes \cdots \otimes \bar{\theta}_N(T_N) = \pm A, \tag{73}$$

since $\bar{\theta}_j(\sigma_j) = -\sigma_j$, $\bar{\theta}_j(S_j) = S_j$, $\theta_k(i\sigma_k) = i\sigma_k$, and $\theta_k(iS_k) = -iS_k$.

In general an element of the orthonormal basis in $u(n_1 n_2 \cdots n_N)$ is a tensor product of σ and S elements, with i multiplying one of the elements. θ^{TOT} is obtained by applying $\bar{\theta}_j$ in all the positions j without i and θ_k in the $k-$th position where there is the factor i. If in the $k-$th position there is a factor of the type σ this gives a $+1i\sigma$ factor when transformed. In the remaining terms, all the factors S are transformed into S while factors σ give other factors of the type σ and a collective factor $(-1)^{p-1}$. Here p is the total number of σ's and this is 1 if p is odd and -1 if p is even, so that $\theta^{TOT}(A) = A$ in one case and $\theta^{TOT}(A) = -A$ in the other case.

Analogously, one can treat the case where in the $k-$th position there is a factor of the type iS. With a similar argument, one shows that $\bar{\theta}^{TOT}$ can be defined on tensor products by applying $\bar{\theta}_j$ in every $j-$th position which clearly gives a factor $(-1)^p$ where p is the number of factors σ. This shows θ^{TOT} and $\bar{\theta}^{TOT}$ are the involution and symmetry associated with the odd-even Cartan decomposition.

An alternative treatment could have been to first define the involutions and symmetries and then to obtain the decomposition (66)-(69) in terms of eiegenspaces of these homomorphisms.

Remark 5.1 The Concurrence Canonical Decomposition is obtained as a special case of the odd-even decomposition when the N systems are all two level systems and the Cartan decomposition chosen on each of them is of the type **AII**. This gives $\mathcal{K} = su(2) = sp(1)$ and $\mathcal{P} = \{0\}$ in the decomposition of $su(2)$ (22). It corresponds to a time reversal symmetry ((21) with $N = 1$ and spin $\frac{1}{2}$) which is indeed a Cartan symmetry. Notice that $n = 2$ is the only case where in the Cartan decompositions **AI**, **AII** and **AIII**, we can take $\mathcal{K}$ equal to the whole Lie algebra $su(n)$. This fact makes it difficult, in higher dimensions, to obtain natural decompositions of dynamics into local and entangling parts as it was done for the 2-qubits case for example in [16].

6 The nature of the odd-even decomposition

It follows (for instance) from formulas (68) (69) that the odd-even decomposition is a decomposition of the type **AI** or **AII**, namely a decomposition corresponding to a Cartan symmetry. It is interesting to know how the choice of the single decompositions on the various subsystems determines whether the odd-even decomposition is of the type **AI** or **AII**. One reason for that is that one may want to further decompose the Lie algebra $i\mathcal{I}_o$ and therefore would like to know its nature. For example, it was shown in [6] that the Concurrence Canonical Decomposition is **AI** in the case of even number of qubit subsystems and **AII** in the case of odd qubits. In our notation $i\mathcal{I}_o$ is (conjugate to) $so(2^N)$ for N even and $sp(2^{N-1})$ for N odd.

In general this information can be obtained by a simple count of the dimensions. Recall that in a **AI** decomposition of $u(n)$ the dimension of the Lie algebra $\mathcal{K}$ in (57) is the dimension of $so(n)$ i.e. $d_I := \frac{n(n-1)}{2}$ while in a **AII** decomposition of $u(n)$ the dimension of the Lie algebra $\mathcal{K}$ is the dimension of $sp(\frac{n}{2})$ i.e. $d_{II} := \frac{n(n+1)}{2}$. These numbers are never the same and therefore they uniquely identify the type of decomposition obtained. We have the following result.

Theorem 3 *Consider an odd-even decomposition on N subsystem obtained by performing* **AII** *decompositions on r subsystems and* **AI** *decompositions on $N-r$ subsystems. Then the resulting decomposition is of type* **AII** *if r is odd and of type* **AI** *if r is even.*

Proof. The proof is by induction on N. If N is equal to 1 the result is obvious. Consider now N subsystems and consider first the case r odd. Assume, without loss of generality, that an **AII** decomposition is performed on the $N-$th subsystem. Let n_1 (n_2) denote the dimension of the vector composed of the first $N-1$ systems (of the $N-$th subsystem). By the inductive assumption the odd-even decomposition on the first $N-1$ system is of the **AI** type. Therefore there are $\frac{n_1(n_1-1)}{2}$ elements of the odd type i.e. tensor products containing an odd number of σ matrices spanning the associated subalgebra $\mathcal{K}$. A remaining orthonormal set of $n_1^2 - \frac{n_1(n_1-1)}{2} = \frac{n_1(n_1+1)}{2}$ even type elements span the orthogonal complement in $u(n_1)$. The basis for the Lie algebra $\mathcal{K}$ for the system composed of all the N subsystems is obtained by tensor products of the $\frac{n_1(n_1-1)}{2}$ odd elements with the $n_2^2 - \frac{n_2(n_2+1)}{2} = \frac{n_2(n_2-1)}{2}$ even type elements on the $N-$th subsystem or by products of the $\frac{n_1(n_1+1)}{2}$ even elements with the $\frac{n_2(n_2+1)}{2}$ even type elements on the $N-$th subsystem. The dimension of the $\mathcal{K}$ Lie algebra in the resulting odd-even decomposition is therefore

$$\frac{n_1(n_1-1)}{2} \times \frac{n_2(n_2-1)}{2} + \frac{n_1(n_1+1)}{2} \times \frac{n_2(n_2+1)}{2} = \frac{n_1 n_2(n_1 n_2 + 1)}{2}, \tag{74}$$

which indicates a decomposition of the type **AII** for the total system which has dimension $n_1 n_2$. An analogous reasoning proves the case where r is even. □

The result of [5] is obtained as a special case of the above theorem as in the case treated all the decompositions applied are of type **AII** and therefore the Concurrence Canonical Decomposition is of type **AI** on an even number of subsystems and of type **AII** on an odd number.

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