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DEFORMATION OF COMPLEX MANIFOLDS AND APPLICATIONS

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Introduction

The aim of this thesis is to give an overview of classical deformation theory, mainly of compact complex manifolds, and to present some of its many applications on problems of classifications in geometry. The theory represents an active area of research nowadays, being important in many fields, especially in algebraic geometry and mathematical physics.

Roughly speaking, infinitesimal deformation theory is the study of infinitesimal variations of structures of certain mathematical objects. In fact, one can picture this as having an object (possibly rigid) in space whose shape is deformed under the application of some external forces, in such a way that at each instant t, one would get a different shape of the same object. In particular, given a compact complex manifold M, deformation theory allows us to study the variations of different complex manifold structures on M which depend (either continuously, smoothly or holomorphically) on a variable t. We recall that a compact complex manifold M can be constructed by gluing a finite number of domains in $\mathbb{C}^n$ via holomorphic functions called the transition functions. The idea is to make these transition functions depend on an additional variable t such that, for each t, we get a different way of gluing the same domains. Thus we get a different compact complex manifold M_t which can be considered as a deformation of M.

This approach, referred to as "shift of the patches", was due to K.Kodaira and D.Spencer, who developed their theory of deformation of compact complex manifolds during the period between 1958 and 1960. They were inspired by the important result published in 1957 by Frolicher and Nijenhuis, which said that if the first cohomology group with values in a sheaf of germs of vector fields of a compact complex manifold vanishes, then its complex structure does not change under small variations (this will be explained in chapter 2, theorem 2.1.2). In fact, the theory focuses on the study of infinitesimal deformations of the complex structure of compact complex manifold M, which can be identified with elements of the first cohomology group with values in the sheaf of germs of vector fields $H^1(M,\Theta)$, and are obtained by deriving the transition functions of M with respect to the variable t. Intuitively, an infinitesimal deformation of M may be thought of as a derivative of the complex structure on M, therefore one may wonder whether there exists a process similar to integration to get a family of deformations of M from an element of $H^1(M,\Theta)$. This is discussed in one of the fundamental theorems called the "theorem of existence". One of the goals of Kodaira and Spencer was to study the relationship between the dimension of $H^1(M,\Theta)$ as a $\mathbb{C}$-vector space and the number of effective parameters involved in the definition of M, which is called the number of moduli. The theory was developed even further by Kuranishi, Grauert and others. In modern times, the theory of deformations is presented using functorial language and is used on more abstract mathematical objects such as schemes.

This thesis is divided into three chapters: the first chapter contains basic definitions about complex manifolds and vector bundles, as well as the necessary tools of sheaf cohomology needed in chapter 2. We also define in a rigorous way the notion of a family of compact complex manifolds and we define the deformation of such manifolds. The second chapter deals with the construction of infinitesimal deformations of compact complex manifolds; the main result being the theorem of existence. Finally, in the third chapter, we present some applications of infinitesimal deformations, for instance the study of local properties of moduli spaces. We will give definitions of a classification problem and of a moduli space, then we proceed with some examples about curves and vector bundles.

Chapter 1

Complex geometry gadgetry

In this chapter, we briefly introduce the central geometric objects that we will apply deformations on : complex manifolds.

We assume that the reader is familiar with some basic notions of Complex Analysis.

1.1 Complex manifolds

First of all, given a point $c = (c_1, \dots, c_n) \in \mathbb{C}^n$ and $r = (r_1, \dots, r_n)$ where $r_1, \dots, r_n$ are positive real numbers, we define the **polydisc with center** c:

$$U_r = \{z = (z_1, \dots, z_n) \mid |z_k - c_k| < r_k,\ k = 1, \dots, n\}.$$

Let us generalize the notion of holomorphic functions of one variable into multiple variables. Let $f : D \longrightarrow D'$ be a complex valued function, $D, D' \subset \mathbb{C}^n$, $n \geq 1$. We say that $f = (f_1, \dots, f_n)$ is holomorphic at $z = (z_1, \dots, z_n)$ if all the f_k $(k = 1, \dots, n)$ are holomorphic as a one-variable function on each of the z_k, $(k = 1, \dots, n)$.

Now, we introduce the necessary tools for the construction of a complex manifold. Let M be a set and $U \subset M$. First, we say that the pair (U, φ) is a n-**dimensional chart** if the map $\varphi : U \longrightarrow \mathbb{C}^n$ is injective and $\varphi(U)$ is an open set in the usual topology of $\mathbb{C}^n$ (which is the euclidean topology, since $\mathbb{C}^n$ can be identified with $\mathbb{R}^{2n}$).

We say that two n-dimensional charts (U,φ) and (V,ψ) are compatible if:

_ either $U \cap V = \emptyset$,

_ or the transition map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$ is a biholomorphism.

Then we define the notion of a **n-dimensional maximal atlas** $\mathfrak{A}$ on a set M (also called a system of local coordinates), which is the collection of all compatible n-dimensional charts (U_i,φ_i) of M.

A maximal atlas $\mathfrak{A}$ of dimension n defined on a topological space M is called a **complex analytic structure**.

Now we are able to introduce the definition of a complex manifold, which is a connected, Hausdorff space M, on which is defined a complex analytic structure.

Definition 1.1.1 *Let M be a connected Hausdorff space, and $\mathfrak{A} = \{(U_i,\varphi_i) | i \in I\}$ a complex analytic structure on M. Then the pair $(M,\mathfrak{A})$ is called a complex manifold of dimension n. It has a topology verifying the following properties:*

1. *U_i are open for all $i \in I$.*
2. *$\varphi_i : U_i \longrightarrow \varphi_i(U_i)$ is a homeomorphism.*

M is called the underlying topological space of the complex manifold $(M,\mathfrak{A})$. However, for simplicity, $(M,\mathfrak{A})$ will also be denoted by M. Furthermore, we will use the notation M^n if we want to specify the dimension of M.

Examples 1.1.2

- *A domain (connected open subset) D of $\mathbb{C}^n$ is a complex manifold. In fact, it has an atlas consisting of the chart $id_D : z \longmapsto z$.*
- *A Riemann surface is a complex manifold endowed with a 1-dimensional atlas (U_i,φ_i), i.e., $\varphi_i : U_i \longrightarrow \mathbb{C}$ for each $i \in I$.*
- *If M and N are two complex manifolds with atlases $\{(U_i,\varphi_i)\}_{i\in I}$ and $\{(V_j,\psi_j)\}_{j\in J}$ respectively, then the cartesian product $M \times N$ of the underlying sets of M and N, endowed with the atlas $\{(U_i \times V_j, \varphi_i \times$*

$\psi_j)\}_{(i,j)\in I\times J}$ is a complex manifold called the product manifold of M and N.

Remark 1.1.3

- *By using the same construction as above we can define a smooth differentiable manifold of dimension n. In this case the charts are maps with values in $\mathbb{R}^n$ and the transition maps are C^∞. It is clear that a complex manifold of dimension n is also a smooth differentiable manifold of dimension $2n$.*
- *If $\{(U_i,\varphi_i)\}$ is an atlas for a complex manifold M of dimension n, then we can assume without loss of generality that the open subset $\varphi_i(U_i)\subset\mathbb{C}^n$ is a polydisc.*

Definition 1.1.4 *Let M and N be two complex manifolds given with the charts (U_i,φ_i) and (V_j,ψ_j) respectively. A map $f: M\longrightarrow N$ is holomorphic (resp. biholomorphic) if the map $\psi_j\circ f\circ\varphi_i^{-1}$ is a holomorphic function (resp. a biholomorphic function) for each $i\in I$ and $j\in J$.*

A complex manifold M is **compact** if its underlying topological space is compact. Hence a compact complex manifold can be covered by a finite number of charts (U_i,φ_i).

In fact for a compact complex manifold M^n, we can consider a system of local coordinates $\{(U_i,z_i)\}_{i\in I}$ such that I is a finite set. Let $\mathcal{U}_i=z_i(U_i)\subset\mathbb{C}^n$ and $f_{ji}=z_j\circ z_i^{-1}$ be the transition map from $\mathcal{U}_{ij}=z_i(U_i\cap U_j)$ to $\mathcal{U}_{ji}=z_j(U_i\cap U_j)$ such that $f_{ii}=id_{\mathcal{U}_i}$ and $f_{jk}\circ f_{ki}=f_{ji}$. Since $U_i\cong\mathcal{U}_i$, then $M\cong\bigcup\limits_{i\in I}\mathcal{U}_i$, thus we can obtain M by glueing a finite number of $\mathcal{U}_i$ using the identification $f_{ji}(x_i)=x_j\in\mathcal{U}_{ji}$ for every $x_i\in\mathcal{U}_{ij}$.

From now on, given a chart (U,φ) of a complex manifold M, we will denote it by φ (if there is no risk of confusion) and we shall call it a local coordinate. Also, φ_q denotes a local coordinate centered at q, i.e., $\varphi_q(q)=0$, for some $q\in M$.

A submanifold S of a complex manifold M^n is a subset of M, defined locally at each of its points by a system of holomorphic equations which verify certain conditions. More precisely:

Definition 1.1.5 *Let S be a connected closed subset of M^n. S is a submanifold of dimension m of M if at each point $q\in S$, there is a holomorphic* *function* $f_q = (f_q^1, \ldots, f_q^{\nu})$ *defined on a neighborhood* $U(q)$ *of* q *such that* $f_q(p) = 0$ *for each* $p \in S \cap U(q)$*, and verifying the following conditions:*

1. *For a fixed local coordinate* $\varphi_q = (\varphi_q^1, \ldots, \varphi_q^n)$, $\operatorname{rank}\left(\dfrac{\partial f_q}{\partial \varphi_q}\right) = \nu$.
2. *In a small neighborhood* $U(q)$ *of* q*, the map* $\tilde{\varphi}_q = (\varphi_q^1, \ldots, \varphi_q^m, f_q^1, \ldots, f_q^{\nu})$ *is biholomorphic, where* $m = n - \nu$.

Remark 1.1.6 *A submanifold of a complex manifold is also a complex manifold.*

Definition 1.1.7 *Let* B *be a domain of* $\mathbb{C}^m$. *A complex analytic family of compact complex manifolds is a family of compact complex manifolds* $\{M_t | t \in B\}$ *such that there is a complex manifold* $\mathcal{M}^{\eta}$ *and a holomorphic map* ϖ *from* $\mathcal{M}$ *onto* B *satisfying the following conditions:*

1. *The rank of the Jacobian of* ϖ *is equal to* m *at every point of* $\mathcal{M}$.
2. *For every* $t \in B$, $\varpi^{-1}(t)$ *is a compact submanifold of* $\mathcal{M}$.
3. $\varpi^{-1}(t) = M_t$ *for every* $t \in B$.

The triple $(\mathcal{M}, B, \varpi)$ is used to denote the complex analytic family $\{M_t | t \in B\}$. $\mathcal{M}$ is called the total space of the family and B is called the parameter space of the family.

Condition (1) ensures, thanks to the implicit function theorem, that $\varpi^{-1}(t)$ is a closed complex submanifold of $\mathcal{M}$ of dimension $n = \eta - m$ for each $t \in B$. Condition (2) tells us that $\varpi^{-1}(t)$ are compact complex manifolds for all $t \in B$ and finally condition (3) requires that for each $t \in B$ the submanifold $\varpi^{-1}(t)$ is biholomorphic to the compact complex manifold M_t.

Using the implicit function theorem on ϖ, we can construct a system of local coordinates $(U_i, z_i)_{i \in I}$ for $\mathcal{M}^{\eta}$ such that $\{U_i\}_{i \in I}$ is a locally finite covering and

$$z_i : U_i \longrightarrow z_i(U_i) = \mathcal{U}_i \subset \mathbb{C}^{\eta}$$

is a holomorphic map which takes an element $p \in U_i$ and maps it to

$$z_i(p) = (z_i^1(p), \ldots, z_i^n(p), t^1, \ldots, t^m),$$

with $\varpi(p) = (t^1, \dots, t^m) = t \in B$. Thus we see that each z_i maps U_i into the product complex manifold $\mathbb{C}^n \times B \subset \mathbb{C}^\eta$.

The compact complex manifold $M_t = \varpi^{-1}(t)$, for each $t \in B$, has a system of local coordinates $(U'_j, z_j)_{j \in J}$ where U'_j belongs to the open cover $\{U_j \cap \varpi^{-1}(t) | U_j \cap \varpi^{-1}(t) \neq \emptyset\}$, and z_j is a holomorphic map on each $U_j \cap \varpi^{-1}(t) \neq \emptyset$ given by $p \longmapsto z_j(p) = (z_j^1(p), \dots, z_j^n(p))$. In terms of these local coordinates, ϖ is just the projection map ρ given by

$$(z_j^1(p), \dots, z_j^n(p), t^1, \dots, t^m) \longmapsto (t^1, \dots, t^m).$$

The open subsets $\mathscr{U}_i \subset \mathbb{C}^\eta$ are glued by the holomorphic transition functions $f_{ij} = z_i \circ z_j^{-1}$ for each $U_i \cap U_j \neq \emptyset$. Then f_{ij} is a holomorphic function of the variables t and $z_j(p)$, and is hence written coordinatewise as

$$f_{ij}^\alpha(z_j, t) = f_{ij}^\alpha(z_j^1, \dots, z_j^n, t^1, \dots, t^m)$$

for each $\alpha = 1, \dots, \eta$. Therefore, the compact complex manifold M_{t_0}, for a fixed $t_0 \in B$, is obtained by gluing a finite number of open subsets

$$\mathscr{U}_{i,t_0} = \{\rho^{-1}(t_0) \cap \mathscr{U}_i | i \in I\}$$

in $\mathbb{C}^n$ via the transition functions $f_{ij}(z_j, t_0)$.

It turns out in general that for any complex analytic family $(\mathscr{M}, B, \varpi)$ all the fibers of ϖ are diffeomorphic to one another. Therefore we may think of a complex analytic family of compcat complex manifolds as a family of varying complex structures on the same underlying differentiable manifold.

Definition 1.1.8 *Let M and N be two compact complex manifolds, and B a domain of $\mathbb{C}^m$. N is called a deformation of M if there is a complex analytic family $(\mathscr{M}, B, \varpi)$ such that $M = \varpi^{-1}(t_0)$ and $N = \varpi^{-1}(t_1)$, for some $t_0, t_1 \in B$.*

The idea here is that a deformation of a compact complex manifold M, given with a system of local coordinates $(U_i, z_i)_{i \in I}$, is obtained by gluing the same polydiscs $\{z_i(U_i) = \mathscr{U}_i\}_{i \in I}$ using different transition maps which depend on a parameter $t \in B \subset \mathbb{C}^m$ in the following way:

$$f_{ij}(z_j, t) = f_{ij}(z_j, t_1, \dots, t_m)$$

with the condition $f_{ij}(z_j, t_0) = f_{ij}(z_j)$. This condition ensures that the deformation of M that we obtain for $t_0 \in B$ is just M itself. Now if we denote by M_t the compact complex manifold obtained by gluing $\{\mathscr{U}_i\}_{i \in I}$ using the transition maps $f_{ji}(z_i, t)$, then M_t is a deformation of M according to the definition above. Furthermore, this point of view of a deformation is the link to complex analytic families which we will explain a bit later.

Definition 1.1.9 *Let $(\mathscr{M}, B, \phi)$ and $(\mathscr{N}, B, \psi)$ be two complex analytic families. We say that they are equivalent if there is a biholomorphic map $\Phi : \mathscr{M} \longrightarrow \mathscr{N}$ such that for each $t \in B$, Φ maps $M_t = \phi^{-1}(t)$ biholomorphically into $N_t = \psi^{-1}(t)$.*

Now let M be a compact complex manifold with a system of local coordinates $\{(U_i, z_i)\}_{i\in I}$. Let $\varpi : M \times B \longrightarrow B$ be the projection map from the complex manifold $M \times B$ to $B \subset \mathbb{C}^m$. Then $(M \times B, B, \varpi)$ is a complex analytic family of compact complex manifolds. It is clear that the complex structure of $M = \varpi^{-1}(t) = M \times \{t\}$ does not vary with t, for each $t \in B$.

Definition 1.1.10 *Let $(\mathscr{M}, B, \varpi)$ be a complex analytic family of compact manifolds and $M = \varpi^{-1}(t_0)$, for a fixed $t_0 \in B$. Then $(\mathscr{M}, B, \varpi)$ is called **trivial** if it is equivalent to $(M \times B, B, \varpi)$.*

Suppose $(\mathscr{M}, B, \varpi)$ is a complex analytic family of compact complex manifolds and $h : D \longrightarrow B$ a holomorphic map of complex manifolds. Define the fiber product

$$\mathscr{M} \times_B D = \{(m, s) \in \mathscr{M} \times D \mid \varpi(m) = h(s)\} \subset \mathscr{M} \times D,$$

and consider the image of this fiber product by the map

$$\varpi \times id_D : \mathscr{M} \times D \longrightarrow B \times D.$$

The image $(\varpi \times id_D)(\mathscr{M} \times_B D)$ is the graph G_h of the map h, which is a submanifold of $B \times D$ and is biholomorphic to D via the second projection $p : B \times D \longrightarrow D$. Since ϖ is a holomorphic map of maximal rank, then so is the map $\varpi \times id_D$. Therefore $(\varpi \times id_D)^{-1}(G_h)$ acquires naturally the structure of a complex submanifold of $\mathscr{M} \times B$, and is precisely the fiber product $\mathscr{M} \times_B D$. Therefore, the fiber product is in a natural way the deformation space of a complex analytic family $(\mathscr{M} \times_B D, D, p \circ (\varpi \times id_D))$. This family is called the **induced family** from $(\mathscr{M}, B, \varpi)$ via the map h, it is also called the pullback of $(\mathscr{M}, B, \varpi)$ via h and is denoted for simplicity by $(h^*\mathscr{M}, D, h^*\varpi)$.

Let $(\mathscr{M}, B, \varpi)$ be a complex analytic family of compact complex manifolds. Let $I \subset B$ be a domain in B, therefore I has a natural structure of submanifold of B. Then the pullback of the family $(\mathscr{M}, B, \varpi)$ via the inclusion map $i: I \hookrightarrow B$ which is $(i^*\mathscr{M}, I, i^*\varpi)$ is called the restriction of $(\mathscr{M}, B, \varpi)$ to $I \subset B$. We simplify the notation of $(i^*\mathscr{M}, I, i^*\varpi)$ to $(\mathscr{M}_I, I, \varpi)$.

We notice that the deformation space of the restricted family $\mathscr{M}_I$ is actually $\varpi^{-1}(I)$.

Definition 1.1.11 *A complex analytic family $(\mathscr{M}, B, \varpi)$ is called locally trivial if for each $t \in B$, there is an open neighborhood $I \subset B$ of t such that $(\mathscr{M}_I, I, \varpi)$ is trivial.*

From now on when we say that the complex structure of $M_t = \varpi^{-1}(t)$ does not depend on $t \in B$, we mean that the complex analytic family $(\mathscr{M}, B, \varpi)$ is locally trivial.

We will now talk about the relationship between a complex analytic family of compact complex manifolds and the deformation of compact complex manifolds. In fact, if we are given a complex manifold M which occurs as the fiber over some $t_0 \in B$ of a complex analytic family $(\mathscr{M}, B, \varpi)$, then the question that arises is for which $t \in B$ the compact complex manifold $M_t = \varpi^{-1}(t)$ is a deformation of M. To answer this question, we use the following theorem:

Theorem 1.1.12 *Let $(\mathscr{M}, B, \varpi)$ be a complex analytic family of compact complex manifolds and $M_{t_0} = \varpi^{-1}(t_0)$ a compact complex manifold for a fixed $t_0 \in B$, obtained by gluing the polydiscs $\{\mathscr{U}_j\}_{j \in J}$ via the transition functions f_{jk} defined on $U_j \cap U_k \neq \emptyset$ as above. Then there exists a domain $I \subset B$ such that for each $t \in I$, $M_t = \varpi^{-1}(t)$ is a compact complex manifold obtained by gluing the same polydiscs $\{\mathscr{U}_j\}_{j \in J}$ using different transition functions also defined above $f^{\alpha}_{jk}(z_k, t)$, with the initial condition $f^{\alpha}_{jk}(z_k, t_0) = f^{\alpha}_{jk}(z_k)$.*

Proof.
Given a complex analytic family $(\mathscr{M}, B, \varpi)$, we can find a locally finite open cover $\{U_j\}_{j \in J}$ of $\mathscr{M}$ and a system of local coordinates $\{(U_j, z_j)\}_{j \in J}$ with $z_j(U_j) = \mathscr{U}_j \times I_j$, where $\mathscr{U}_j$ can be assumed to be a polydisc of $\mathbb{C}^n$ and $I_j \subset B$ which can be identified with a polydisc in $\mathbb{C}^m$ by using a local coordinate.

Let $\{f_{jk}\}$ be the transition functions which glue the domains $\{\mathscr{U}_j \times I_j\}$ to give the complex manifold $\mathscr{M}$. For a fixed $t_0 \in B$, $M_{t_0} = \varpi^{-1}(t_0)$ is the compact complex manifold obtained by gluing the polydiscs $\{\mathscr{U}_j \times \{t_0\} \cong \mathscr{U}_j, U_j \cap M_{t_0} \neq \emptyset\}$ via the transition functions $\{f_{jk}(z_k, t_0)\}$.

By the compactness of the fibers of ϖ and the fact that the open cover is locally finite, there exists an open neighborhood I of $t_0 \in B$ which can be identified with a polydisc in $\mathbb{C}^m$, such that $\varpi^{-1}(I)$ is contained in the union of a finite number of U_j which can be assumed to be $U_1, \ldots, U_l$. We put $U'_j = U_j \cap M_{t_0}$.

Then by construction, for each $t \in I$ we have that $\varpi^{-1}(t) = M_t$ is a compact complex manifold obtained by gluing the $\{\mathscr{U}_j \times \{t\} \cong \mathscr{U}_j \mid j = 1, \ldots, l\}$ via the transition functions $f_{jk}(z_k, t)$.

Therefore we see that the restriction $(\mathscr{M}_I, I, \varpi)$ of $(\mathscr{M}, B, \varpi)$ to I consists of fibers which are all obtained by gluing the same polydiscs $\{\mathscr{U}_1, \ldots, \mathscr{U}_l\}$ with transition functions $f_{jk}(z_k, t)$, depending on the variable $t \in I$. ■

1.2 Family of elliptic curves

We shall give an example of a complex analytic family of Elliptic curves. In this section, K will denote a field of characteristic zero. Recall that

$$\mathbb{P}^n_K := K^{n+1} - \{0\} / \sim$$

where $(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n)$ if and only if $b_i = \lambda a_i$ for every non zero $\lambda \in K$ and $i = 0, \ldots, n$. The equivalence class of $(a_0, \ldots, a_n)$ is denoted by $[a_0, \ldots, a_n]$.

Definition 1.2.1 *An elliptic curve E over K (denoted by E/K) is a pair (E, O), where E is a smooth curve of genus one in the projective plane $\mathbb{P}^2_K$ and $O \in E$. O is called the base point of E.*

In fact, for a suitable choice of coordinates in $\mathbb{P}^2_K$, any elliptic curve E/K can be given by a Weierstrass equation

$$Y^2 + a_1 XY + a_3 Y = X^3 + a_2 X^2 + a_4 X + a_6$$

with coefficients $a_1, \ldots, a_6 \in K$, (see [8],chapter III section 3).

We will show that an elliptic curve $E/\mathbb{C}$ is a Riemann surface biholomorphic to a torus, and that its complex structure depends on one complex parameter ω.

First of all, let us define the complex torus. Let $\omega \in \mathbb{C}$, with $Im(\omega) > 0$. We consider the lattice

$$\Lambda = \{n + m\omega \mid n, m \in \mathbb{Z}\}$$

with basis $(1,\omega)$. It is clearly an additive subgroup of $\mathbb{C}$. The quotient group $\mathbb{C}/\Lambda$ can be identified with the parallelogram with vertices 0, 1 ,$\omega+1$ and ω. By identifying x, $0 \le x \le 1$, with $x+\omega$ and identifying $t\omega$, $0 \le t \le 1$, with $t\omega+1$, we obtain the torus $\mathbb{T}^2$, and hence $\mathbb{C}/\Lambda = \mathbb{T}^2$. We can define a structure of Riemann surface on the torus : in fact, taking the continuous projection map $\pi : \mathbb{C} \longrightarrow \mathbb{C}/\Lambda$, we take a small enough disc D centered in $z \in \mathbb{C}$, such that $\pi_{|D}$ is injective. Then we define a local coordinate on $\mathbb{T}^2$ by $(\pi(D), \pi_{|D}^{-1})$, for each $z \in \mathbb{C}$. The transition maps are the translations of z by $n+m\omega$. Note that the torus is a compact complex manifold.

We shall use some well known results of the classical theory of elliptic functions to show that $\mathbb{C}/\Lambda$ is biholomorphic to an elliptic curve given by a Weierstrass equation in $\mathbb{P}^2_{\mathbb{C}}$ (for details, one can see [8], chapter VI section 3).

Let us put $\omega_{mn} = n+m\omega$. The Weierstrass $\wp$-function is defined by the series

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z-\omega_{mn})^2} - \frac{1}{\omega_{mn}^2} \right).$$

$\wp$ is a meromorphic function on $\mathbb{C}$, verifying $\wp(z+\omega_{mn}) = \wp(z)$, for any ω_{mn}. Hence it can be considered as a meromorphic function on $\mathbb{C}/\Lambda$. It has a pole of order 2 at $0 \in \mathbb{C}/\Lambda$ and it is holomorphic elsewhere. $\wp$ satisfies the following differential equation:

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 \tag{1.1}$$

where g_2 and g_3 are given by

$$g_2 = 60 \sum_{(m,n)\neq(0,0)} \frac{1}{\omega_{mn}^4}, \quad g_3 = 140 \sum_{(m,n)\neq(0,0)} \frac{1}{\omega_{mn}^6}.$$

In regard of equation 1.1, we put

$$P(\xi) = \xi_2\xi_1^2 - 4\xi_0^3 + g_2\xi_2^2\xi_0 + g_3\xi_2^3$$

where $\xi = [\xi_0, \xi_1, \xi_2] \in \mathbb{P}^2_{\mathbb{C}}$. Since P has discriminant $g_2^3 - 27g_3^2 \neq 0$, we have that for every $\xi \in \mathbb{P}^2_{\mathbb{C}}$, at least one of the partial derivatives of P does not vanish, so $P(\xi) = 0$ defines a smooth algebraic curve Γ of degree 3 in $\mathbb{P}^2_{\mathbb{C}}$. Now we define the following map

$$\Phi : \mathbb{C}/\Lambda - \{0\} \longrightarrow \Gamma \subset \mathbb{P}^2_{\mathbb{C}}$$

by $z \longmapsto \xi = [\wp(z), \wp'(z), 1]$. By using the properties of $\wp$, we can prove that Φ can be extended to a biholomorphism from $\mathbb{C}/\Lambda$ onto the elliptic curve Γ. Thus an elliptic curve may be identified with a torus $\mathbb{C}/\Lambda$, which is a compact Riemann surface. Let $\mathbb{H}$ denote the upper half-plane of $\mathbb{C}$. Since for each $\omega \in \mathbb{H}$ we get a different lattice with basis $(1,\omega)$, therefore for each ω we may obtain non biholomorphic complex structures of an elliptic curve.

Now we consider the family of elliptic curves over $\mathbb{C}$: $\{E_\omega \mid \omega \in \mathbb{C}\}$ and we show that it is indeed a complex analytic family. In the process we shall use without proving it the following theorem (for the proof see [5], chapter 2 section 2.2 (b))

Theorem 1.2.2 *Let G be a group of biholomorphic automorphisms of a complex manifold W. If the action of G on W is fixed point free and G acts discontinuously on W, then the set W/G of orbits of G in W has a canonical complex manifold structure inherited from W.*

We know that the product manifold $\mathbb{C}\times\mathbb{H}$ is a complex manifold of dimension 2. We define the group

$$G = \{g_{mn} : (z,\omega) \longmapsto (z + m\omega + n, \omega) \mid m,\ n \in \mathbb{Z},\ \ \omega \in \mathbb{H}\}$$

of biholomorphic automorphisms of $\mathbb{C}\times\mathbb{H}$, which acts discontinuously on $\mathbb{C}\times\mathbb{H}$ and has no fixed points. Therefore $\mathscr{M} = (\mathbb{C}\times\mathbb{H})/G$ is a complex manifold by the previous theorem. Now we can construct the holomorphic map

$$\phi : \mathscr{M} \longrightarrow \mathbb{H}$$

induced from the projection $p : \mathbb{C}\times\mathbb{H} \longrightarrow \mathbb{H}$ onto the second factor. For each $\omega \in \mathbb{H}$, we see that the fiber $\phi^{-1}(\omega)$ is identified with a torus $\mathbb{C}/\Lambda$, where Λ is the lattice generated by $(1,\omega)$. Therefore it is an elliptic curve. Moreover, it is easy to check, using the local coordinates on $\mathscr{M}$ induced by those of $\mathbb{C}\times\mathbb{H}$, that the rank of the Jacobian matrix of ϕ is maximal and equal to 1. Hence $(\mathscr{M},\mathbb{H},\phi)$ is a complex analytic family of elliptic curves.

1.3 Vector bundles

Definition 1.3.1 *Let M and E be complex manifolds, and $\pi : E \longrightarrow M$ a holomorphic map. E is called a complex vector bundle over M of rank v if it satisfies the following conditions:*

1. *For every $p \in M$, $\pi^{-1}(p)$ is a $\mathbb{C}$-vector space of dimension v.*
2. *For every $q \in M$, there is a neighborhood U of q such that there exists a biholomorphic map $h : \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{C}^v$, and for any $p \in U$, $h|p : \pi^{-1}(p) \xrightarrow{\sim} \{p\}\times\mathbb{C}^v$ is an isomorphism of $\mathbb{C}$-vector spaces.*

We call M the base space of E, π the projection and $\pi^{-1}(p)$ the fiber of E over p. We use the triple (E, M, π) to refer to a complex vector bundle E with base space M and projection π.

Complex vector bundles of rank $v = 1$ are called **line bundles**.

Let (E, M, π) be a complex vector bundle of rank v. If M admits a locally finite open cover $\mathfrak{U} = \{U_i\}_{i \in I}$, then by condition (2) we obtain $h_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{C}^v$ by taking U_i small enough. Now if $U_i \cap U_j \neq \emptyset$, we can define

$$h_{ij} = h_i \circ h_j^{-1} : U_i \cap U_j \times \mathbb{C}^v \longrightarrow U_i \cap U_j \times \mathbb{C}^v$$

such that $h_{ij}(x, v) = (x, g_{ij}(x)v)$. Here $g_{ij} : U_i \cap U_j \longrightarrow GL(v, \mathbb{C})$ such that $g_{ij}(x) = (h_i)_{|x} \circ (h_j)^{-1}_{|x}$ are linear transformations called **transition matrices**. It can be easily verified that $g_{ij} = g_{ji}^{-1}$ and $g_{ij} \circ g_{jk} \circ g_{ki} = id$.

A **smooth section** of (E, M, π) over an open subset U of M is a smooth map $s : U \longrightarrow E$ such that $\pi \circ s = id_U$.

Definition 1.3.2 *(E, M, π) is called a **holomorphic vector bundle** if the transition matrices g_{ij} are holomorphic maps.*

Sections of a holomorphic vector bundle are holomorphic maps.

The most natural vector bundles over a complex manifold M are the tangent bundle TM and the cotangent bundle T^*M. In order to construct them, we shall first define what is a real tangent space:

Definition 1.3.3 *Let M^n be a smooth differentiable manifold. The tangent space at $x \in M$ is the real vector space of all **derivations** (also called **tangent vectors**) from the space of germs of C^∞-functions defined on a neighborhood of x to $\mathbb{R}$. We denote it by T_xM.*

Let us fix a local coordinate $\varphi = (\varphi_1, \ldots, \varphi_n)$, and write $\varphi_i(x) = x_i$, then the family $\left\{\frac{\partial}{\partial x_i}\right\}_{i=1\ldots n}$ of derivations defines a basis of T_xM with respect to the local coordinate φ. Now if M^n is a complex manifold, then it is also a differentiable manifold of dimension $2n$, therefore for $p \in M$, $\varphi_j(p) = z_j = x_j + iy_j$ and $\left\{\frac{\partial}{\partial z_j}\right\}_{j=1\ldots n}$ forms a basis of the complex vector space T_pM.

We will not prove but will use later the fact that if M^n is a smooth manifold, then $\dim(M) = \dim(T_x M)$ for every $x \in M$.

Now we define the **tangent bundle** by:

$$TM = \bigcup_{p\in M} T_p M$$

which is given with the canonical projection $\pi : TM \longrightarrow M$ defined by $(p,\delta) \longmapsto p$, where δ is a derivation from the space of germs of holomorphic functions defined on a neighborhood of p to $\mathbb{C}$.

A **holomorphic vector field** is a section of π, i.e., a holomorphic map σ satisfying $\pi \circ \sigma = id_M$.

TM has a structure of complex manifold which is induced from M. In fact, if $(U_i, \phi_i)_{i\in I}$ is a system of local coordinates of M^n, we can define a system of local coordinates $(\bigcup_{x\in U_i} T_x M, \tilde{\phi}_i)_{i\in I}$ for TM as such

$$\tilde{\phi}_i : \bigcup_{x\in U_i} T_x M \longrightarrow \mathbb{C}^{2n}$$

defined by $(x,\delta) \longmapsto (\phi_i(x), \delta(\phi_i))$. We can easily show that this map is indeed a chart for TM. Now to define the transition maps, let $x \in U_i \cap U_j$ with U_i and U_j domains for the two local coordinates (U_i, ϕ_i) and (U_j, ϕ_j). Let

$$x_i = \phi_i(x), \;\; x_j = \phi_j(x).$$

We know that on the local coordinate (U_j, ϕ_j), we have $\delta = \sum_{\beta=1}^{n} \delta(x_j^\beta) \frac{\partial}{\partial x_j^\beta}$.

Since $\frac{\partial}{\partial x_j^\beta} = \sum_{\alpha=1}^{n} \frac{\partial x_i^\alpha}{\partial x_j^\beta} \frac{\partial}{\partial x_i^\alpha}$, we get the following

$$\begin{aligned}
\delta &= \sum_{\beta=1}^{n} \delta(x_j^\beta) \frac{\partial}{\partial x_j^\beta} \\
&= \sum_{\beta=1}^{n} \delta(x_j^\beta) \left(\sum_{\alpha=1}^{n} \frac{\partial x_i^\alpha}{\partial x_j^\beta} \frac{\partial}{\partial x_i^\alpha} \right) \\
&= \sum_{\alpha=1}^{n} \left(\sum_{\beta=1}^{n} \delta(x_j^\beta) \frac{\partial x_i^\alpha}{\partial x_j^\beta} \right) \frac{\partial}{\partial x_i^\alpha}
\end{aligned}$$

Thus the transition maps between the charts in the tangent bundle are of the form:

$$\tilde{\phi}_i \circ \tilde{\phi}_j^{-1}(x,\delta) = (\phi_i \circ \phi_j^{-1}(x), G(\delta))$$

where $G : \delta = \sum_{\beta=1}^{n} \delta(x_j^\beta) \frac{\partial}{\partial x_j^\beta} \longmapsto \sum_{\alpha=1}^{n} \left(\sum_{\beta=1}^{n} \delta(x_j^\beta) \frac{\partial x_i^\alpha}{\partial x_j^\beta} \right) \frac{\partial}{\partial x_i^\alpha}$.

Next if we consider the dual space $T_x^* M$ of the vector space $T_x M$, we define the **cotangent bundle** as such :

$$T^* M = \bigcup_{x \in M} T_x^* M$$

which is given with the canonical projection $\tau : T^* M \longrightarrow M$ similarly as in the case of the tangent bundle. Also, we can define a structure of complex manifold using a similar method to the one used for the tangent bundle.

1.4 Čech cohomology

The goal here is to define the Čech cohomology group which will play an essential role in the next chapter about infinitesimal deformation. We will introduce some basic definitions about sheaf theory, then construct the Čech cohomology group for a topological space.

First we go rapidly through some basic definitions and properties about presheaves and sheaves:

Let X be a topological space, K a ring. A **presheaf** F on X of K-modules is the data of a K-module $F(U)$ for every open subset $U \subset X$, and for every open subset $V \subset U$, morphisms of K-modules $\rho_{UV} : F(U) \longrightarrow F(V)$ called restriction morphisms and verifying the following conditions:

- $\rho_{UU} = id_{F(U)}$.
- For $W \subset V \subset U$ open subsets, $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

$s \in F(U)$ is called a section over U and we write $\rho_{UV}(s) = s_{|V}$.

Example 1.4.1 1. *The constant presheaf Λ such that $\Lambda(U) = M$ for every non-empty open subset $U \subset X$ where M is a K-module.*

2. *The presheaf C^∞ of smooth real valued functions on X such that $C^\infty(U)$ is the real vector space of all smooth functions defined on $U \subset X$.*

Let $x \in X$ and $U \subset X$ an open neighborhood of x. We define the **stalk** of F at x by

$$F_x = \frac{\coprod_{x \in U} F(U)}{\sim}$$

where for $s \in F(U)$ and $t \in F(V)$, $s \sim t \Leftrightarrow$ there exists a neighborhood $W \subset U \cap V$ of x such that $s_{|W} = t_{|W}$.

Definition 1.4.2 *Let $\{U_i\}_{i \in I}$ be an open covering of X. A **sheaf** on X of K-modules is a presheaf on X of K-modules F verifying the following conditions:*

- *If for $s, t \in F(X)$ we have $s_{|U_i} = t_{|U_i}$ for all $i \in I$, then $s = t$.*
- *Given a family $\{s_i \in F(U_i)\}_{i \in I}$ such that for every $i, j \in I$ we have $s_{i|U_{ij}} = s_{j|U_{ij}}$ $(U_{ij} = U_i \cap U_j)$, then there exists $s \in F(X)$ such that $s_{|U_i} = s_i$ for all $i \in I$.*

For simplicity, we will refer to a sheaf F on X of K-modules as just "a sheaf F".

Example 1.4.3 *Let M be a complex manifold and (E, M, π) a vector bundle. Let $U \subset M$ be an open subset, then the set of sections $\{\sigma : U \longrightarrow E\}$ which we denote by $\Gamma(U, E)$ is an abelian group. Furthermore, for each open subset $V \subset U \subset M$, we define the restriction map $\rho_{UV} : \Gamma(U, E) \longrightarrow \Gamma(V, E)$. Then $U \longmapsto \Gamma(U, E)$ is a sheaf called the sheaf of sections of the vector bundle E over M. In particular, if $E = TM$, then it is called the sheaf of vector fields of TM over M and is denoted by Θ.*

The following is the old definition of a sheaf:

Definition 1.4.4 *A **sheaf over** X is a topological space $\mathscr{F}$ together with a continuous map $\pi : \mathscr{F} \longrightarrow X$ such that:*

1. *π is a local homeomorphism.*
2. *$\mathscr{F}_x = \pi^{-1}(x)$ is a K-module for all $x \in X$.*

$\pi : \mathscr{F} \longrightarrow X$ is called the projection map.

From the following we get back to definition 1.4.2 of a sheaf:

Let $\mathscr{F}$ be a sheaf and $\pi : \mathscr{F} \longrightarrow X$ the projection map. Let $U \subset X$ be an open subset. A continuous map $\sigma : U \longrightarrow \mathscr{F}$ such that $\pi \circ \sigma = id_U$ is called a section of $\mathscr{F}$ over U. By definition, for all $x \in U$ we have $\sigma(x) \in \mathscr{F}_x$. We call the 0-section the section which gives the zero of the K-module $\mathscr{F}_x$ for all $x \in U$.

We denote the K-module of all sections of $\mathscr{F}$ over U by $\Gamma(U, \mathscr{F})$. It can be easily checked that $\Gamma(., \mathscr{F})$ is a sheaf (in the sense of definition 1.4.2).

For the rest of this section, X is a topological space, $\mathscr{F}$ is a sheaf of K-modules over X (K is a ring). Let $\mathscr{U} = \{U_i\}_{i\in I}$ be a locally finite covering of X. We put $\mu = (U_0, \ldots, U_q)$ where U_i are elements of $\mathscr{U}$ such that $U_0 \cap \ldots \cap U_q \neq \emptyset$.

We define a q**-cochain** f^q with respect to $\mathscr{U}$ as an assignment

$$\mu \longmapsto f^q(\mu) \in \Gamma(U_0 \cap \ldots \cap U_q, \mathscr{F})$$

for every μ, and verifying for every permutation ε of the indices $0, \ldots, q$

$$f((U_{\varepsilon(0)}, \ldots, U_{\varepsilon(q)})) = sign(\varepsilon) f((U_0, \ldots, U_q)).$$

Remark 1.4.5 *For an easier notation, if we have a section $\sigma_{0,\ldots,q} \in \Gamma(U_0 \cap \ldots \cap U_q, \mathscr{F})$, then we denote the q-cochain $f^q \in C^q(\mathscr{U}, \mathscr{F})$ such that $f^q((U_0, \ldots, U_q)) = \sigma_{0,\ldots,q}$ by $\{\sigma_{0,\ldots,q}\}$.*

We denote by $C^q(\mathscr{U}, \mathscr{F})$ the set of all q-cochains with respect to $\mathscr{U}$. We endow it with a structure of K-module as follows: For all $k_1, k_2 \in K$ and $f_1^q, f_2^q \in C^q(\mathscr{U}, \mathscr{F})$

$$(k_1.f_1^q + k_2.f_2^q)(\mu) = k_1.f_1^q(\mu) + k_2.f_2^q(\mu)$$

Next we define the coboundary map $\delta : C^q(\mathscr{U}, \mathscr{F}) \longrightarrow C^{q+1}(\mathscr{U}, \mathscr{F})$ as follows:

$$\delta(f^q)(\mu) = \sum_{i=0}^{q+1} (-1)^i f^q(\mu_i)$$

where $\mu_i = (U_0, \ldots, \hat{U}_i, \ldots, U_{q+1})$. We can easily check that $\delta^2 = 0$, therefore we obtain a complex of K-modules

$$0 \longrightarrow C^0(\mathscr{U}, \mathscr{F}) \xrightarrow{\delta} C^1(\mathscr{U}, \mathscr{F}) \xrightarrow{\delta} \ldots$$

called the **Čech complex** associated to $\mathscr{F}$ and the covering $\mathscr{U}$. We define the K-module of q-cocycles

$$Z^q(\mathscr{U},\mathscr{F}) = \{f^q \in C^q(\mathscr{U},\mathscr{F}) | \delta(f^q) = 0\}$$

and we construct the q-th **Čech cohomology group** with respect to $\mathscr{U}$ for $q > 0$

$$H^q(\mathscr{U}) = \frac{Z^q(\mathscr{U},\mathscr{F})}{\delta C^{q-1}(\mathscr{U},\mathscr{F})}$$

for $q = 0$ we put

$$H^0(\mathscr{U},\mathscr{F}) = Z^0(\mathscr{U},\mathscr{F})$$

Proposition 1.4.6 *We have $H^0(\mathscr{U},\mathscr{F}) = \Gamma(X,\mathscr{F})$.*

Proof.
Let $f^0 \in Z(\mathscr{U},\mathscr{F})$, then $\delta(f^0)((U_i,U_j)) = 0$, therefore $f^0(U_i) - f^0(U_j) = 0$ inside $\Gamma(U_i \cap U_j,\mathscr{F})$. Since $f^0(U_i) \in \Gamma(U_i,\mathscr{F})$ and $f^0(U_j) \in \Gamma(U_j,\mathscr{F})$, then by the second property of sheaves we obtain a section $s \in \Gamma(X,\mathscr{F})$ such that $s_{|U_i} = f^0(U_i)$ and $s_{|U_j} = f^0(U_j)$, for every i and j. ■

Definition 1.4.7 *The q-th Čech cohomology group of $\mathscr{F}$ on X is the direct limit of $H^q(\mathscr{U},\mathscr{F})$ over refinements of the cover*

$$H^q(X,\mathscr{F}) = \varinjlim_{\mathscr{U}} H^q(\mathscr{U},\mathscr{F})$$

For a reference on how the projective limit is defined, see [5], chpater 3, section 3.3 (b).

In practice, it is nearly impossible to compute the cohomology group of X as the direct limit of cohomology groups with respect to covers. We shall therefore use Leray's theorem to fix this handicap by considering a special kind of coverings of X.

Definition 1.4.8 *We say that a covering $\mathscr{U} = \{U_i\}_{i\in I}$ of X is* ***acyclic*** *for $\mathscr{F}$ if $H^q(U_{i_1} \cap \ldots \cap U_{i_k},\mathscr{F}) = 0$ for $q > 0$ and for all $i_1,\ldots,i_k \in I$.*

Theorem 1.4.9 (Leray's theorem) *Let $\mathscr{U}$ be an open cover of X. If $\mathscr{U}$ is acyclic then*

$$H^q(\mathscr{U},\mathscr{F}) = H^q(X,\mathscr{F})$$

for all $q > 0$.

The proof of this theorem uses the language of spectral sequences which is a broad subject by itself, therefore we shall not prove this theorem for the general case. However, we will study and concentrate on the case of the first order:

Theorem 1.4.10 (First order case) *Let* $\mathscr{U} = \{U_i\}_{i\in I}$ *be an open cover of* X*. If* $H^1(U_i, \mathscr{F}) = 0$ *for all* $i \in I$*, then*

$$H^1(\mathscr{U}, \mathscr{F}) = H^1(X, \mathscr{F})$$

To prove this particular case of Leray's theorem, we shall discuss some results:

First, we say that a covering $\mathscr{V} = \{V_\lambda\}_{\lambda\in J}$ is a **refinement** of the covering $\mathscr{U} = \{U_i\}_{i\in I}$ if for each $\lambda \in J$ there exists an $i \in I$ such that $V_\lambda \subset U_i$.

Now let's say $\mathscr{V}$ is a refinement of $\mathscr{U}$. For each $\lambda \in J$, fix an arbitrary $\tau(\lambda) \in I$ such that $V_\lambda \subset U_{\tau(\lambda)}$, then we define a map

$$\Pi^{\mathscr{U}}_{\mathscr{V}} : Z^1(\mathscr{U}, \mathscr{F}) \longrightarrow Z^1(\mathscr{V}, \mathscr{F})$$

such that $f^1((U_{i_0}, U_{i_1})) \longmapsto g^1((V_{\lambda_0}, V_{\lambda_1})) = f^1((U_{\tau(\lambda_0)}, U_{\tau(\lambda_1)}))_{|V_{\lambda_0}\cap V_{\lambda_1}}$, which induces a morphism of cohomology groups

$$\Pi^{\mathscr{U}}_{\mathscr{V}} : H^1(\mathscr{U}, \mathscr{F}) \longrightarrow H^1(\mathscr{V}, \mathscr{F}).$$

This morphism does not depend on the choice of $\tau(\lambda)$ for each $\lambda \in J$. It is in fact uniquely determined by $\mathscr{U}$ and $\mathscr{V}$.

Theorem 1.4.11 *The group morphism* $\Pi^{\mathscr{U}}_{\mathscr{V}} : H^1(\mathscr{U}, \mathscr{F}) \longrightarrow H^1(\mathscr{V}, \mathscr{F})$ *is injective.*

Proof.

In order to show the injectivity of $\Pi^{\mathscr{U}}_{\mathscr{V}}$, we need to show that its kernel is zero. Let $f^1 \in Z^1(\mathscr{U}, \mathscr{F})$ such that $\Pi^{\mathscr{U}}_{\mathscr{V}}(f^1((U_i, U_j))) = f^1((U_{\tau(k)}, U_{\tau(l)}))$ is a coboundary in $\delta C^0(\mathscr{V}, \mathscr{F})$, where $V_k \subset U_{\tau(k)}$ and $V_l \subset U_{\tau(l)}$. Thus there exists $g^0 \in C^0(\mathscr{V}, \mathscr{F})$ such that $f^1((U_{\tau(k)}, U_{\tau(l)})) = g^0(V_k) - g^0(V_l)$ on $V_k \cap V_l$. Now on $U_i \cap V_k \cap V_l$, we have that

$$\delta f^1((U_i, U_{\tau(k)}, U_{\tau(l)})) = f^1((U_{\tau(k)}, U_{\tau(l)})) - f^1((U_i, U_{\tau(l)})) + f^1((U_i, U_{\tau(k)})) = 0$$

therefore we have

$$g^0(V_k) - g^0(V_l) = f^1((U_i, U_{\tau(l)})) - f^1((U_i, U_{\tau(k)}))$$

which gives

$$g^0(V_k) + f^1((U_i, U_{\tau(k)})) = f^1((U_i, U_{\tau(l)})) + g^0(V_l).$$

Now by considering the family $\{g^0(V_k) + f^1((U_i, U_{\tau(k)})) \in \mathscr{F}(U_i \cap V_k)\}$, using the second axiom of sheaves we obtain $h^0(U_i) \in F(U_i)$ such that

$$h^0(U_i) = g^0(V_k) + f^1((U_i, U_{\tau(k)}))$$

on $U_i \cap V_k$. Now on $U_i \cap U_j \cap V_k$ we get

$$\begin{aligned} f^1((U_i, U_j)) =& f^1((U_i, U_{\tau(k)})) + f^1((U_{\tau(k)}, U_j)) \\ &= f^1((U_i, U_{\tau(k)})) + g^0(V_k) + f^1((U_{\tau(k)}, U_j)) - g^0(V_k) \\ &= h^0(U_i) - h^0(U_j). \end{aligned}$$

Since k is arbitrary, by using the first axiom of sheaves we see that this last equality holds on $U_i \cap U_j$, thus f^1 is a coboundary. ■

Corollary 1.4.12 $H^1(X, \mathscr{F}) = \bigcup_{\mathscr{U}} H^1(\mathscr{U}, \mathscr{F})$.

Since $H^1(X, \mathscr{F}) = \varinjlim_{\mathscr{U}} H^1(\mathscr{U}, \mathscr{F})$, by lemma 1.4.11 and using the properties of the direct limit, we get that the morphism $\Pi^{\mathscr{U}} : H^1(\mathscr{U}, \mathscr{F}) \longrightarrow H^1(X, \mathscr{F})$ is injective.

Proof of theorem 1.4.10.

We assume that $H^1(U_i, \mathscr{F}) = 0$ for all $i \in I$. We want to show that the morphism $\Pi^{\mathscr{U}} : H^1(\mathscr{U}, \mathscr{F}) \longrightarrow H^1(X, \mathscr{F})$ is an isomorphism. By theorem (1.4.11)we have injectivity, thus it is enough to prove that for any refinement $\mathscr{V}$ of the open cover $\mathscr{U}$ the morphism $\Pi^{\mathscr{U}}_{\mathscr{V}} : H^1(\mathscr{U}, \mathscr{F}) \longrightarrow H^1(\mathscr{V}, \mathscr{F})$ is surjective. In other words, for any 1-cocycle $f^1 \in H^1(\mathscr{V}, \mathscr{F})$, we must show that there exists a 1-cocycle $c^1 \in H^1(\mathscr{U}, \mathscr{F})$ such that

$$\Pi^{\mathscr{U}}_{\mathscr{V}}(c^1) - f^1 \in \delta C^0(\mathscr{V}, \mathscr{F}).$$

We have that $\{U_i \cap V_\alpha\}_{\alpha \in J}$ is an open cover of U_i which we denote by $U_i \cap \mathscr{V}$. The assumption implies that $H^1(U_i \cap \mathscr{V}, \mathscr{F}) = 0$, so there exists an element $g_{i\alpha} \in F(U_i \cap v_\alpha)$ such that

$$f^1(V_\alpha, V_\beta) = g_{i\alpha} - g_{i\beta}$$

on $U_i \cap V_\alpha \cap V_\beta$. Furthermore, on $U_i \cap U_j \cap V_\alpha \cap V_\beta$ we have

$$g_{j\alpha} - g_{j\beta} = g_{i\alpha} - g_{i\beta}$$

which implies that

$$g_{j\alpha} - g_{i\alpha} = g_{j\beta} - g_{i\beta}$$

therefore by the second axiom of sheaves there exists elements c_{ij} on $U_i \cap U_j$ satisfying

$$c_{ij} = g_{j\alpha} - g_{i\alpha}$$

on $U_i \cap U_j \cap V_\alpha$. Thus $\{c_{ij}\} \in Z^1(\mathscr{U}, \mathscr{F})$ is a 1-cocycle. Setting

$$h_\alpha = g_{\tau(\alpha)\alpha} | V_\alpha \in F(V_\alpha)$$

on $V_\alpha \cap V_\beta$ we have that

$$\begin{aligned} c_{\tau(\alpha)\tau(\beta)} - f^1((V_\alpha, V_\beta)) &= (g_{\tau(\beta)\alpha} - g_{\tau(\alpha)\alpha}) - (g_{\tau(\alpha)\alpha} - g_{\tau(\beta)\beta}) \\ &= g_{\tau(\beta)\beta} - g_{\tau(\alpha)\alpha} = h_\beta - h_\alpha. \end{aligned}$$

Thus $\{c_{\tau(\alpha)\tau(\beta)}\} - f^1 \in \delta C^0(\mathscr{V}, \mathscr{F})$. ■

Chapter 2

Infinitesimal deformations

2.1 Construction

In general, by considering the partial derivative of a function with respect to a certain variable, we get information on whether the function depends on that particular variable or not. Our aim is to have a similar concept for complex analytic families. More precisely, given a complex analytic family of compact complex manifolds $(\mathcal{M}, B, \varpi)$, with $B \subset \mathbb{C}^m$ and $\varpi^{-1}(t_0) = M_{t_0}$ for some $t_0 \in B$, we look for a criterium to tell if the complex structure of the underlying manifold of M_{t_0} varies for each parameter $t \in B$.

Recall that for each $t \in B$, a deformation $M_t = \varpi^{-1}(t)$ of a compact complex manifold $M^n = \varpi^{-1}(t_0)$ given with a system of local coordinates $(U_i, z_i)_{i \in I}$ (I is finite) does not depend on the open subsets used to construct it, but rather on the transition functions $\{f_{ij}(z_j, t)\}$, $z_j \in \mathcal{U}_{ij} \subset \mathbb{C}^n$ and $t \in B$. Now following the same idea above, we want to differentiate the transition functions $\{f_{ij}(z_j, t)\}$ with respect to $t = (t_1, \ldots, t_m)$, and then write the resulting derivatives as coefficients of vector fields, which are elements of the first Čech cohomology group with coefficients in the sheaf of holomorphic vector fields.

By differentiating a holomorphic function with respect to $t \in B \subset \mathbb{C}^m$ we mean taking a tangent vector $\dfrac{\partial}{\partial t} \in T_t B$ such that

$$\frac{\partial}{\partial t} = \sum_{\lambda=1}^{m} c_\lambda \frac{\partial}{\partial t_\lambda},$$

where $\left\{\dfrac{\partial}{\partial t_\lambda}\right\}_{\lambda=1,\ldots,m}$ is the basis of $T_t B$ with respect to the local coordinate $t = (t_1, \ldots, t_m)$ of B, then applying it on the holmorphic function.

Let $p \in U_i \cap U_j \cap U_k$, putting $z_i(p) = z_i$, $z_j(p) = z_j$, $z_k(p) = z_k$ we have $z_i = f_{ij}(z_j, t) = f_{ik}(z_k, t)$, and $z_j = f_{jk}(z_k, t)$, therefore on each non empty intersection $\mathscr{U}_i \cap \mathscr{U}_j \cap \mathscr{U}_k$ the transition functions verify the following condition:

$$f^{\alpha}_{ik}(z_k, t) = f^{\alpha}_{ij}(f^1_{jk}(z_k, t), \dots, f^n_{jk}(z_k, t), t), \ \alpha = 1, \dots, n \tag{2.1}$$

Considering this equation and putting $z^{\beta}_j = f^{\beta}_{jk}(z_k, t)$ we get

$$\frac{\partial f^{\alpha}_{ik}(z_k, t)}{\partial t} = \frac{\partial [f^{\alpha}_{ij}(z^1_j, \dots, z^n_j, t)]}{\partial t} = \frac{\partial f^{\alpha}_{ij}(z_j, t)}{\partial t} + \sum_{\beta=1}^{n} \frac{\partial f^{\alpha}_{ij}(z_j, t)}{\partial z^{\beta}_j} \frac{\partial f^{\beta}_{jk}(z_k, t)}{\partial t} \tag{2.2}$$

Now we introduce the following vector fields

$$\theta_{jk}(t) = \sum_{\beta=1}^{n} \frac{\partial f^{\beta}_{jk}(z_k, t)}{\partial t} \frac{\partial}{\partial z^{\beta}_j}, \ \ z_k = f_{kj}(z_j, t) \tag{2.3}$$

which are defined on each non-empty intersection $U_j \cap U_k$.

From equation (2.2) and using the equality (2.1) we deduce that these vector fields verify the following condition

$$\begin{aligned} \theta_{ik}(t) &= \sum_{\alpha=1}^{n} \frac{\partial f^{\alpha}_{ik}(z_k, t)}{\partial t} \frac{\partial}{\partial z^{\alpha}_i} \\ &= \sum_{\alpha=1}^{n} \frac{\partial f^{\alpha}_{ij}(z_j, t)}{\partial t} \frac{\partial}{\partial z^{\alpha}_i} + \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \frac{\partial f^{\beta}_{jk}(z_k, t)}{\partial t} \frac{\partial f^{\alpha}_{ij}(z_j, t)}{\partial z^{\beta}_j} \frac{\partial}{\partial z^{\alpha}_i} \\ &= \theta_{ij}(t) + \theta_{jk}(t) \end{aligned} \tag{2.4}$$

which holds on each non-empty open set $U_i \cap U_j \cap U_k$. For $i = k$, $f^{\alpha}_{kk} = z^{\alpha}_k$ thus $\theta_{kk}(t) = 0$. Consequently, these vector fields also verify

$$\theta_{jk}(t) = -\theta_{kj}(t). \tag{2.5}$$

Now considering Θ_t to be the sheaf of holomorphic vector fields over M_t, we have that $\theta_{jk}(t) \in \Gamma(U_j \cap U_k, \Theta_t)$, so from (2.4) and (2.5) there is a 1-cocycle $\{\theta_{jk}(t)\} \in Z^1(\mathscr{U}_t, \Theta_t)$, where $\mathscr{U}_t = \{U_i \times \{t\}\}_{i \in I} \simeq \{U_i\}_{i \in I}$ is a finite covering of polydiscs of M_t as explained after definition 1.1.7. Let $\theta(t)$ be the class corresponding to $\{\theta_{jk}(t)\}$ in $H^1(\mathscr{U}_t, \Theta_t)$. Since $H^1(\mathscr{U}_t, \Theta_t) \subset H^1(M_t, \Theta_t)$ from corollary 1.4.12, then $\theta(t)$ is an element of the first Čech cohomology group.

$\theta(t)$ **is called the infinitesimal deformation of the compact manifold** M_t, we denote it by $\dfrac{d(M_t)}{dt}$.

At this point we have to show that this $\theta(t)$ does not depend on the choice of the system of local coordinates on M_t.

First, fixing a complex analytic family $(\mathscr{M}, B, \varpi)$ with a system of local coordinates $(U_i, z_i)_{i\in I}$ (as defined in 1.1.7), we show that $\theta(t)$ does not change if we consider a refinement of the open covering $\mathscr{U} = \{U_i\}_{i\in I}$ of $\mathscr{M}$:

Let $\mathscr{V} = \{V_j\}_{j\in J}$ be any refinement of $\mathscr{U} = \{U_i\}_{i\in I}$, such that $V_j \subset U_{\tau(j)}$ for all $j \in J$. Let $z'_j : V_j \longrightarrow \mathbb{C}^n$ be a local coordinate defined on V_j for each $j \in J$, which is the restriction of the local coordinate $z_{\tau(j)} : U_{\tau(j)} \longrightarrow \mathbb{C}^n$. Thus the holomorphic vector field ${\theta'}_{jk}(t)$ as constructed in (2.3) is the restriction of $\theta_{\tau(j)\tau(k)}(t)$ to $V_j \cap V_k \cap \varpi^{-1}(t)$. In fact, putting $\mathfrak{B}_t = \{V_{jt}\}_{j\in J}$ where $V_{jt} = V_j \cap \varpi^{-1}(t) \neq \emptyset$, we have

$$ {\theta'}_{jk}(t) = {\theta'}_{\tau(j)\tau(k)}(t)_{|V}, \;\; V = V_{jt} \cap V_{kt}. $$

Hence the 1-cocycles $\{\theta_{jk}(t)\} \in Z^1(\mathscr{U}, \Theta)$ and $\{{\theta'}_{jk}(t)\} \in Z^1(\mathscr{V}, \Theta)$ verify $\{{\theta'}_{jk}(t)\} = \Pi^{\mathscr{U}_t}_{\mathfrak{B}_t}(\{\theta_{jk}(t)\})$.

Next, we consider two systems of local coordinates $(V_l, z_l)_{l\in L}$ and $(V'_\lambda, z'_\lambda)_{\lambda\in\Lambda}$. Since two locally finite coverings have a common refinement and since $\theta(t)$ does not depend on the choice of a refinement, we can work with a system of local coordinates $(U_i, z_i)_{i\in I}$ comprised of open subsets belonging to a refinement of both $\{V_l\}_{l\in L}$ and $\{V'_\lambda\}_{\lambda\in\Lambda}$, as well as the restrictions of the local coordinates z_l and z'_λ to those open subsets.

Therefore given z_i and w_i defined on each U_i, we want to show that the infinitesimal deformation $\eta(t)$ defined with respect to w_i coincides with $\theta(t)$. In fact, if $h_{jk} : w_k \longmapsto w_j$ are transition maps for the local coordinates $\{w_i\}$, then similarly to (2.3) we obtain the holomorphic vector field

$$ \eta_{jk}(t) = \sum_{\beta=1}^{n} \frac{\partial h^\beta_{jk}(w_k, t)}{\partial t} \frac{\partial}{\partial w^\beta_j}, \qquad w_k = h_{kj}(w_j, t). \tag{2.6} $$

Then we consider the cohomology class $\eta(t) \in H^1(M_t, \Theta_t)$ corresponding to the 1-cocycle $\{\eta_{jk}(t)\} \in Z^1(\mathscr{U}_t, \Theta_t)$. We have that $w^\alpha_j = g^\alpha_j(z_j, t)$ where g^α_j is holomorphic function on the variables $z^1_j, \ldots, z^n_j, t$. Since

$$ g^\alpha_j(z_j, t) = w^\alpha_j = h^\alpha_{jk}(w_k, t) = h^\alpha_{jk}(g_k(z_k, t), t) $$

and $z_j = f_{jk}(z_k, t)$, we have

$$g_j^\alpha(f_{jk}(z_k,t),t) = h_{jk}^\alpha(g_k(z_k,t),t) \tag{2.7}$$

on $U_j \cap U_k \neq \emptyset$. Now we differentiate the equality (2.7) with respect to t

$$\sum_{\beta=1}^{n} \frac{\partial g_j^\alpha}{\partial z_j^\beta}\frac{\partial f_{jk}^\beta}{\partial t} + \frac{\partial g_j^\alpha}{\partial t} = \sum_{\beta=1}^{n} \frac{\partial h_{jk}^\alpha}{\partial w_k^\beta}\frac{\partial g_k^\beta}{\partial t} + \frac{\partial h_{jk}^\alpha}{\partial t}$$

i.e.,

$$\sum_{\beta=1}^{n} \frac{\partial g_j^\alpha}{\partial t} + \frac{\partial w_j^\alpha}{\partial z_j^\beta}\frac{\partial f_{jk}^\beta}{\partial t} = \sum_{\beta=1}^{n} \frac{\partial w_j^\alpha}{\partial w_k^\beta}\frac{\partial g_k^\beta}{\partial t} + \frac{\partial h_{jk}^\alpha}{\partial t}$$

Multiplying by $\dfrac{\partial}{\partial w_j^\alpha}$ from the right and taking the summation $\sum_{\alpha=1}^{n}$, we obtain the following equality

$$\sum_{\beta=1}^{n} \frac{\partial f_{jk}^\beta}{\partial t}\frac{\partial}{\partial z_j^\beta} + \sum_{\alpha=1}^{n} \frac{\partial g_j^\alpha}{\partial t}\frac{\partial}{\partial w_j^\alpha} = \sum_{\beta=1}^{n} \frac{\partial g_k^\beta}{\partial t}\frac{\partial}{\partial w_k^\beta} + \sum_{\alpha=1}^{n} \frac{\partial h_{jk}^\alpha}{\partial t}\frac{\partial}{\partial w_j^\alpha},$$

where we use the equalities

$$\frac{\partial}{\partial z_j^\beta} = \sum_{\alpha=1}^{n} \frac{\partial w_j^\alpha}{\partial z_j^\beta}\frac{\partial}{\partial w_j^\alpha}$$

and

$$\frac{\partial}{\partial w_k^\beta} = \sum_{\alpha=1}^{n} \frac{\partial w_j^\alpha}{\partial w_k^\beta}\frac{\partial}{\partial w_j^\alpha}.$$

Now putting $\theta_j(t) = \sum_{\alpha=1}^{n} \dfrac{\partial g_j^\alpha(z_j,t)}{\partial t}\dfrac{\partial}{\partial w_j^\alpha}$, $w_j^\alpha = g_j^\alpha(z_j,t)$, we have

$$\theta_{jk}(t) - \eta_{jk}(t) = \theta_k(t) - \theta_j(t). \tag{2.8}$$

Since $\theta_j(t)$ is a holomorphic vector field on $U_{jt} = U_j \cap M_t$, there exists a coboundary $\{\theta_j(t)\} \in C^0(\mathscr{U}_t, \Theta_t)$, moreover (2.8) tells us that

$$\{\theta_{jk}(t)\} - \{\eta_{jk}(t)\} = \delta(\{\theta_j(t)\}),$$

where $\{\theta_{jk}(t)\} \in Z^1(\mathscr{U}_t, \Theta_t)$. Therefore $\eta(t)$ coincides with $\theta(t)$, and the infinitesimal deformation $\theta(t)$ does not depend on the choice of system of local coordinates. ■

Now we want to test if the infinitesimal deformation $\theta(t) = \dfrac{d(M_t)}{dt}$ truly plays the role of a derivative for the complex structure of M_t. We fix a complex analytic family of compact complex manifolds $(\mathscr{M}, B, \varpi)$ and put $M_t = \varpi^{-1}(t)$ for each $t \in B$ in all of the following.

Proposition 2.1.1 *If $(\mathscr{M}, B, \varpi)$ is locally trivial, then the infinitesimal deformation of M_t vanishes, i.e., $\theta(t) = \dfrac{d(M_t)}{dt} = 0$.*

Proof.

Let us suppose that $(\mathscr{M}, B, \varpi)$ is locally trivial. Then there exists an open subset $I \subset B$ such that $(\mathscr{M}_I, I, \varpi)$ is equivalent to $(M \times I, B, \varpi)$, where $M = \varpi^{-1}(t_0)$ for a fixed $t_0 \in B$. Let $\{w_j\}_{j \in J}$ be a system of local coordinates of M. Since $\theta(t) = \dfrac{d(M_t)}{dt}$ does not depend on the choice of systems of local coordinates, we can construct $\theta(t)$ for $t \in I$ in terms of local coordinates $\{u_j\}$, $u_j = (w_j, t)$ of $M \times I$. Let h_{jk} be the transition map from w_k to w_j. Since it is independent from t then $\dfrac{\partial h^\beta_{jk}(w_k)}{\partial t} = 0$ for all $j, k \in J$, therefore

$$\theta_{jk}(t) = \sum_{\beta=1}^{n} \frac{\partial h^\beta_{jk}(w_k)}{\partial t} \frac{\partial}{\partial u^\beta_j} = 0,$$

hence $\theta(t) = 0$. ■

At this point, we have defined the infinitesimal deformation $\theta(t)$ of the compact complex manifold M_t to be the derivative of its complex structure and we have proven in proposition 2.1.1 that if $(\mathscr{M}, B, \varpi)$ does not vary with $t \in B$ (in the sense that it is locally trivial), then $\theta(t) = 0$. Now we ask whether the converse of proposition 2.1.1 is also true, in other words if $\theta(t) = 0$ implies that $(\mathscr{M}, B, \varpi)$ is locally trivial. In fact, it is not true in general, unless we impose a condition on the dimension of $H^1(M_t, \Theta_t)$.

Theorem 2.1.2 *Let $(\mathscr{M}, B, \varpi)$ be a complex analytic family and $M = \varpi^{-1}(t_0)$ be a compact complex manifold for some $t_0 \in B$. If $\dim(H^1(M_t, \Theta_t))$ (as a $\mathbb{C}$-vector space) does not depend on t and $\dfrac{d(M_t)}{dt} = 0$ identically, then $(\mathscr{M}, B, \varpi)$ is locally trivial.*

Proof.

We shall only give the idea behind the proof (for the whole proof, see [5], section 4.2(c), theorem 4.6).

The aim is to show that, for a polydisc $\Delta \subset B$, $\mathcal{M}_\Delta = \varpi^{-1}(\Delta)$ is biholomorphic to $M \times \Delta$. Without loss of generality, we can assume that

$$\Delta = \{(t_1, \dots, t_m) \mid |t_1| < r, \dots, |t_m| < r\}.$$

We put

$$\Delta^{m-1} = \{(t_1, \dots, t_{m-1}) \mid |t_1| < r, \dots, |t_{m-1}| < r\}$$

and

$$\Delta_m = \{t_m \mid |t_m| < r\}.$$

Since $\Delta^{m-1} = \Delta^{m-1} \times \{t_0\} \subset \Delta^{m-1} \times \Delta_m = \Delta$, we easily get that $(\varpi^{-1}(\Delta^{m-1}), \Delta^{m-1}, \varpi)$ is a complex analytic family. Furthermore, we can show relatively easily that the theorem is true for $m = 1$, then by induction on m we may assume that $\varpi^{-1}(\Delta^{m-1})$ is locally trivial. $(\varpi^{-1}(\Delta^{m-1}) \times \Delta_m, \Delta, \varpi)$ is also a complex analytic family, thus we get that $\varpi^{-1}(\Delta^{m-1}) \times \Delta_m \cong M \times \Delta^{m-1} \times \Delta_m = M \times \Delta$. Hence it suffices to show that $\mathcal{M}_\Delta$ is biholomorphic to $\varpi^{-1}(\Delta^{m-1}) \times \Delta_m$.

In order to construct the biholomorphism, we consider

$$\theta_{jk}(t) = \sum_{\alpha=1}^{n} \frac{\partial f_{jk}^{\alpha}(z_k, t_1, \dots, t_m)}{\partial t} \frac{\partial}{\partial z_j^{\alpha}},$$

Then the cohomology class $\theta(t) = \rho_t(\frac{\partial}{\partial t})$ corresponding to the 1-cocycle $\{\theta_{jk}(t)\}$ is zero. Thus there exists a coboundary $\{\theta_j(t)\} \in C^0(\mathcal{U}_t, \Theta_t)$ such that $\{\theta_{jk}(t)\} = \delta(\{\theta_j(t)\})$ (also using the fact that $H^1(\mathcal{U}_t, \Theta_t) \subset H^1(M_t, \Theta_t)$). We put

$$\theta_j(t) = \sum_{\alpha=1}^{n} \theta_j^{\alpha}(z_j, t_1, \dots, t_m) \frac{\partial}{\partial z_j^{\alpha}}.$$

At this point, in order to construct the biholomorphism, it suffices to verify that we can choose $\{\theta_j(t)\}$ such that the coefficients $\theta_j^{\alpha}(z_j, t_1, \dots, t_m)$ are holomorphic functions in the variables $z_j^1, \dots, z_j^n, t_1, \dots, t_m$, where the hypothesis about the $\dim(H^1(M_t, \Theta_t))$ is used. We can then prove that the map

$$\Phi : \varpi^{-1}(\Delta^{m-1}) \times \Delta_m \longrightarrow \mathcal{M}_\Delta$$

defined by $(x, t_m) \longmapsto (z_j(x, t_m), t_1, \dots, t_{m-1}, t_m)$ is indeed biholomorphic. ∎

Definition 2.1.3 *The $\mathbb{C}$-linear map $\rho_t : T_t B \longrightarrow H^1(M_t, \Theta_t)$ given by $\rho_t(\frac{\partial}{\partial t}) = \frac{d(M_t)}{dt}$ is called the* ***infinitesimal Kodaira-Spencer*** *map at $t \in B$ for the family $(\mathcal{M}, B, \varpi)$.*

Now given a compact complex manifold M and an element $\theta \in H^1(M,\Theta)$, one may wonder if it is possible to find a complex analytic family $(\mathscr{M},B,\varpi)$ such that $M = \varpi^{-1}(t_0)$, for some $t_0 \in B$. Intuitively, the process is similar to finding a primitive of a real function. We begin by studying the case where such a family indeed exists.

Let M be a compact complex manifold, $(\mathscr{M},B,\varpi)$ a complex analytic family of compact manifolds such that $0 \in B \subset \mathbb{C}^m$ and $\varpi^{-1}(0) = M$. Let us consider Δ to be a small polydisc in B such that $0 \in \Delta \subset B$, then we put

$$\mathscr{M}_\Delta = \varpi^{-1}(\Delta) = \bigcup_{j=1}^{l} (U_j \times \Delta)/\sim_\Delta$$

where the U_j are polydiscs used to construct the manifold M via the transition functions $\{f_{jk}(z_k,0)\}$. Furthermore, the equivalence relation is defined as such: $(z_j,t) \in U_j \times \Delta$ and $(z_k,t') \in U_k \times \Delta$ are equivalent by $\sim_\Delta$ if and only if $t = t'$ and $f_{jk}(z_k,t) = z_j$. Each f_{jk} is holomorphic in the variables $z_k^1,\dots,z_k^n,t$ defined on $U_j \times \Delta \cap U_k \times \Delta \neq \emptyset$.

We have for a tangent vector $\dfrac{\partial}{\partial t} \in T_tB$, $\rho_t(\dfrac{\partial}{\partial t}) = \theta(t) = \dfrac{d(M_t)}{dt}$, where $\theta(t)$ is the cohomology class of the 1-cocycle $\{\theta_{jk}(t)\} \in Z^1(\mathscr{U}_t,\Theta_t)$, $\mathscr{U}_t = \{U_j \times \{t\} \mid j = 1,\dots,l\}$, and the holomorphic vector field $\theta_{jk}(t)$ is defined as follows

$$\theta_{jk}(t) = \sum_{\beta=1}^{n} \frac{\partial f_{jk}^\beta(z_k,t)}{\partial t}\frac{\partial}{\partial z_j^\beta},\ \ z_k = f_{kj}(z_j,t)$$

where n is the dimension of M. Furthermore, on $U_i\times\Delta\cap U_j\times\Delta\cap U_k\times\Delta \neq \emptyset$ we have the following condition on the transition functions

$$f_{ik}^\alpha(z_k,t) = f_{ij}^\alpha(f_{jk}^1(z_k,t),\dots,f_{jk}^n(z_k,t),t),\ \ \alpha = 1,\dots,n.$$

Differentiating both sides of the equality with respect to t and putting

$$\theta_{jk}^\alpha(z_j,t) = \frac{\partial f_{jk}^\alpha(z_k,t)}{\partial t},\ \ z_k = f_{kj}(z_j,t),$$

we obtain the cocycle condition on $\{\theta_{jk}(t)\}$:

$$\theta_{ik}^\alpha(z_i,t) = \theta_{ij}^\alpha(z_i,t) + \sum_{\beta=1}^{n} \frac{\partial z_i^\alpha}{\partial z_j^\beta}\theta_{jk}^\beta(z_j,t),\ \ \alpha = 1,\dots,n.$$

Now we define the **Lie bracket** of two local vector fields on M_t given over the local coordinate $(U_j \times \{t\}, z_j)$ in the form

$$v = \sum_{\alpha=1}^{n} v_j^{\alpha}(\frac{\partial}{\partial z_j^{\alpha}}),\ u = \sum_{\alpha=1}^{n} u_j^{\alpha}(\frac{\partial}{\partial z_j^{\alpha}})$$

by the formula

$$[v,u] = \sum_{\alpha=1}^{n} (v \cdot u_j^{\alpha} - u \cdot v_j^{\alpha}) \frac{\partial}{\partial z_j^{\alpha}}.$$

It has the following properties:

1. $[v,u]$ is $\mathbb{C}$-bilinear in u and v.
2. $[v,u] = -[u,v]$.

Now we set

$$\dot{\theta}_{jk}^{\alpha}(z_j,t) = \frac{\partial \theta_{jk}^{\alpha}(z_j,t)}{\partial t},\ \dot{\theta}_{jk}(t) = \sum_{\alpha=1}^{n} \dot{\theta}_{jk}^{\alpha}(z_j,t) \frac{\partial}{\partial z_j^{\alpha}},$$

then we differentiate the cocycle condition on $\{\theta_{jk}(t)\}$ with respect to t, and using the Lie bracket we get the following relations:

$$\dot{\theta}_{ij}(t) - \dot{\theta}_{ik}(t) + \dot{\theta}_{jk}(t) = [\theta_{ij}(t), \theta_{jk}(t)]. \tag{2.9}$$

For each 1-cocycle $\{\psi_{jk}(t)\} \in Z^1(\mathscr{U}_t, \Theta_t)$, we define for $U_i \times \{t\} \cap U_j \times \{t\} \cap U_k \times \{t\} \neq \emptyset$

$$\zeta_{ijk}(t) = [\psi_{ij}(t), \psi_{jk}(t)],$$

then $\zeta_{ijk}(t)$ is a holomorphic vector field, and we obtain through direct computations that $\{\zeta_{ijk}(t)\} \in Z^2(\mathscr{U}_t, \Theta_t)$ is a 2-cocycle . We can extend this definition to the case where we are given two 1-cocycles $\{\psi_{jk}(t)\}$ and $\{\eta_{jk}(t)\}$ instead of a single one, in fact we have

$$\zeta_{ijk}(t) = \frac{1}{2}([\psi_{ij}(t), \eta_{jk}(t)] + [\eta_{ij}(t), \psi_{jk}(t)])$$

and we still have that $\{\zeta_{ijk}(t)\} \in Z^2(\mathscr{U}_t, \Theta_t)$.

Let $\psi(t)$, $\eta(t) \in H^1(\mathscr{U}_t, \Theta_t)$ and $\zeta(t) \in H^2(\mathscr{U}_t, \Theta_t)$ be the cohomology classes determined respectively by the cocycles $\{\psi_{jk}(t)\}$, $\{\eta_{jk}(t)\}$ and $\{\zeta_{ijk}(t)\}$. Then we can extend the definition of the Lie bracket to cohomology, i.e., $[\psi(t), \eta(t)] = \zeta(t)$. It is easily checked that it is still bilinear in $\psi(t)$ and $\eta(t)$, and also still satisfies $[\psi(t), \eta(t)] = -[\eta(t), \psi(t)]$.

Now if we put $\{\theta_{jk}(t)\} = \{\psi_{jk}(t)\} = \{\eta_{jk}(t)\}$ and put $t = 0$, we get

$$\zeta_{ijk}(0) = [\theta_{ij}(0), \theta_{jk}(0)]$$

furthermore if we put $t = 0$ in the equation (2.9) and replace i by k there, we obtain

$$\dot{\theta}_{jk}(0) = -\dot{\theta}_{kj}(0)$$

which implies that $\{\dot{\theta}_{jk}(0)\} \in C^1(\mathscr{U}_0, \Theta_0)$ is a 1-cochain , where $\mathscr{U}_0$ is an open cover of $M_0 = M$.

For the coboundary map $\delta^1 : C^1(\mathscr{U}_0, \Theta_0) \longrightarrow C^2(\mathscr{U}_0, \Theta_0)$ we have $\delta^1(\{\dot{\theta}_{jk}(0)\}) = \{\tau_{ijk}(0)\}$, where

$$\tau_{ijk}(0) = \dot{\theta}_{ij}(0) - \dot{\theta}_{ik}(0) + \dot{\theta}_{jk}(0).$$

We then see from equation (2.9) that $\tau_{ijk}(0) = \zeta_{ijk}(0)$, therefore $\zeta_{ijk}(0)$ is a coboundary, hence its cohomology class $\zeta(0)$ is zero. In other words

$$[\theta(0), \theta(0)] = 0.$$

From these results, we obtain easily the following theorem:

Theorem 2.1.4 *Let M be a compact complex manifold, and $\theta \in H^1(M, \Theta)$. If there exists a complex analytic family of compact complex manifolds $(\mathscr{M}, B, \varpi)$ such that $M = \varpi^{-1}(t_0)$ for a fixed $t_0 \in B \subset \mathbb{C}^m$ and $\theta = \frac{dM_t}{dt}|_{t=0}$, then $[\theta, \theta]$ must necessarily be zero in $H^2(M, \Theta)$.*

2.2 Theorem of existence

In view of the previous theorem, the question which arises is whether the converse also holds, that is, under the same hypothesis in the theorem above, if $[\theta, \theta] = 0$ then is there a complex analytic family $(\mathscr{M}, B, \varpi)$ such that $M = \varpi^{-1}(t_0)$ and $\theta = \frac{dM_t}{dt}|_{t=0}$?

We will introduce the technique used in [5] which Kodaira and Spencer elaborated to give an appropriate solution to the question. In fact, this approach reflects in a good way the original idea of solving this precise problem. Moreover, a similar method is used to prove a very important theorem later in this section, which is the theorem of completeness.

We suppose that we are given a compact complex manifold M and a cohomology class $\theta \in H^1(M, \Theta)$. Let $B \subset \mathbb{C}^m$ and $\Delta \subset B$ a polydisc containing 0, our objective is to construct a complex analytic family $(\mathscr{M}, B, \varpi)$ such that

$$\mathcal{M} = \bigcup_{j \in J} (U_j \times \Delta) / \sim_\Delta$$

where J is a finite set and each U_j is a polydisc in $\mathbb{C}^n$, the equivalence relation is given by : $(z_j, t) \in U_j \times \Delta$ and $(z_k, t') \in U_k \times \Delta$ are equivalent by $\sim_\Delta$ if and only if $t = t'$ and $f_{jk}(z_k, t) = z_j$.

During the rest of this part, the transition functions $f_{jk}(z_k, t) = z_j$ defined on $U_j \times \Delta \cap U_k \times \Delta$ which will be used to glue together the polydiscs $\mathcal{U} = \{U_j\}_{j \in J}$ to obtain the complex manifold $\mathcal{M}$, will be called the **defining functions of** $\mathcal{M}$, and on $U_i \times \Delta \cap U_j \times \Delta \cap U_k \times \Delta \neq \emptyset$ they must satisfy the compability condition

$$f_{ik}^\alpha(z_k, t) = f_{ij}^\alpha(f_{jk}(z_k, t), t), \ \alpha = 1, \ldots, n. \tag{2.10}$$

Our task is then to determine these defining functions, which are holomorphic in the variable t, therefore they can be expanded in power series in $t = (t_1, \ldots, t_m)$: First we consider

$$f_{jk}^\alpha(z_k, t_1, \ldots, t_m) = \sum_{\nu=0}^{\infty} f_{jk|\nu}^\alpha(z_k)\, t_1^\nu \ldots t_m^\nu$$

For simplicity, we put $t = t_1 \ldots t_m$, therefore we obtain

$$f_{jk}^\alpha(z_k, t) = \sum_{\nu=0}^{\infty} f_{jk|\nu}^\alpha(z_k)\, t^\nu, \alpha = 1, \ldots, n, \tag{2.11}$$

where the coefficients $f_{jk|\nu}^\alpha(z_k)$ are holomorphic functions defined on the open subsets $U_j \cap U_k \neq \emptyset$ on M. Furthermore, on $U_j \cap U_k = U_j \times \{0\} \cap U_k \times \{0\}$ we have that $z_j^\alpha = f_{jk}^\alpha(z_k, 0) = f_{jk|0}^\alpha(z_k)$, thus $\{f_{jk|0}(z_k)\}$ are precisely the transition functions of M. Therefore we may consider the coefficients $f_{jk|0}^\alpha(z_k)$ of the term in t^0 are given with M.

We know that the defining functions satisfy:

$$f_{ik}^\alpha(z_k, t) = f_{ij}^\alpha(f_{jk}(z_k, t), t), \ \alpha = 1, \ldots, n$$

so the coefficients $f_{jk|\nu}^\alpha(z_k)$, $\nu = 1, 2, \ldots$ must be determined such that

$$f_{jk}^\alpha(z_k, t) = f_{jk|0}^\alpha(z_k) + \sum_{\nu=1}^{\infty} f_{jk|\nu}^\alpha(z_k)\, t^\nu$$

satisfy these equalities. We will start with the coefficients of the term in t^1, then determine the other coefficients by induction. But before, let us fix some notations:

For simplicity, we put

$$f_{jk}(z_k,t) = (f^1_{jk}(z_k,t),\dots,f^n_{jk}(z_k,t))$$

and

$$f_{jk|\nu}(z_k) = (f^1_{jk|\nu}(z_k),\dots,f^n_{jk|\nu}(z_k)),$$

then we write (2.11) using vector notations as such:

$$f_{jk}(z_k,t) = \sum_{\nu=0}^{\infty} f_{jk|\nu}(z_k)t^\nu,\ \alpha = 1,\dots,n.$$

For a power series $P(t) = \sum_{\nu=0}^{\infty} P_\nu t^\nu$, we use the following notation

$$P^\nu(t) = P_0 + P_1 t + \ldots + P_\nu t^\nu.$$

Now for two power series $P(t)$ and $Q(t)$ in t, we write $P(t) \equiv_\nu Q(t)$ if $P(t) \equiv Q(t)\ mod\,(t^{\nu+1})$. Therefore we can reduce (2.10) to the system of infinitely many congruences

$$f^\nu_{ik}(z_k,t) \equiv_\nu f^\nu_{ij}(f^\nu_{jk}(z_k,t),t),\ \ \nu = 1,2,\dots \tag{2.12}$$

At this point, using the fact that $\theta \in H^1(M,\Theta)$ is verifying $\theta = \left(\frac{d(M_t)}{dt}\right)_{t=0}$, we would like to construct the next coefficient $f^\alpha_{jk|1}(z_k)$ of the term in t^1. Let $\{\theta_{jk}\} \in Z^1(\mathscr{U},\Theta)$ be an arbitrary representative of the class θ. We write

$$\theta_{jk} = \sum_{\alpha=1}^{n} \theta^\alpha_{jk}(z_j)\frac{\partial}{\partial z^\alpha_j},\ \ \theta^\alpha_{jk}(z_j) = \frac{\partial f^\alpha_{jk}(z_k,t)}{\partial t}$$

and then we put

$$f^\alpha_{jk|1}(z_k) = \theta^\alpha_{jk}(z_j),\ \alpha = 1,\dots,n.$$

As we have already stated, $f_{jk|0}(z_k)$ are given with M. Furthermore we have put $f_{jk|1}(z_k) = \theta_{jk}(z_j)$ with $z_j = f_{jk|0}(z_k)$. On each $U_j \cap U_k \neq \emptyset$, we determine $f_{jk|\nu}(z_k)$, $\nu = 2,3,\dots$ by induction on ν. In fact, for $\nu = 1$, we show that (2.12) is verified for $f_{jk|1}(z_k) = \theta_{jk}(z_j)$. First, we know that

$$f_{ik}(z_k,t) = f_{ik|0}(z_k) + f_{ik|1}(z_k)t,$$

then we have

$$\begin{aligned} f^1_{ij}(f^1_{jk}(z_k,t),t) &\equiv_1 f^1_{ij}(f_{jk|0}(z_k) + f_{jk|1}(z_k)t,t) \\ &\equiv_1 f_{ij|0}(z_j + f_{jk|1}(z_k)t) + f_{ij|1}(z_j + f_{jk|1}(z_k)t)t \\ &\equiv_1 f_{ij|0}(z_j) + \sum_{\beta=1}^{n}\frac{\partial}{\partial z^\beta_j} f_{ij|0}(z_j).f^\beta_{jk|1}(z_k) + f_{ij|1}(z_j)t. \end{aligned}$$

We have that $z_i = f_{ij|0}(z_j) = f_{ik|0}(z_k)$. Hence if we put $z_i^\alpha = f_{ij|0}^\alpha(z_j)$, (2.12) would be equivalent to the following equality

$$f_{ik|1}^\alpha(z_k) = f_{ij|1}^\alpha(z_j) + \sum_{\beta=1}^n \frac{\partial z_i^\alpha}{\partial z_j^\beta} f_{jk|1}^\beta(z_k).$$

namely,

$$\sum_{\alpha=1}^n f_{ik|1}^\alpha(z_k)\frac{\partial}{\partial z_i^\alpha} = \sum_{\alpha=1}^n f_{ij|1}^\alpha(z_j)\frac{\partial}{\partial z_i^\alpha} + \sum_{\beta=1}^n f_{jk|1}^\beta(z_k)\frac{\partial}{\partial z_j^\beta},$$

this last equality follows from the fact that $\{\theta_{jk}(z_j)\}$ is a 1-cocycle.

Now we assume that on each $U_j \cap U_k \neq \emptyset$,

$$f_{jk}^{\nu-1}(z_k, t) = f_{jk|0}(z_k) + \ldots + f_{jk|\nu-1}(z_k)t^{\nu-1}$$

are already determined such that the congruence (2.12)

$$f_{ik}^{\nu-1}(z_k, t) \equiv_{\nu-1} f_{ij}^{\nu-1}(f_{jk}^{\nu-1}(z_k, t), t)$$

holds and consider:

$$f_{jk}^\nu(z_k, t) \equiv_\nu f_{ij}^\nu(f_{jk}^\nu(z_k, t), t). \tag{2.13}$$

Then the right hand side of the equation is written as

$$\begin{aligned} f_{ij}^\nu(f_{jk}^\nu(z_k, t), t) &\equiv_\nu f_{ij}^{\nu-1}(f_{jk}^\nu(z_k, t), t) + f_{ij|\nu}(f_{jk}^\nu(z_k, t))t^\nu \\ &\equiv_\nu f_{ij}^{\nu-1}(f_{jk}^{\nu-1}(z_k, t) + f_{jk|\nu}(z_k, t)t^\nu, t) + f_{ij|\nu}(f_{jk}^\nu(z_k, t))t^\nu. \end{aligned}$$

After some computations similar to the case of $\nu = 1$, we obtain that (2.13) is reduced to

$$f_{ik}^{\nu-1}(z_k, t) - f_{ij}^{\nu-1}(f_{jk}^{\nu-1}(z_k, t), t) \equiv_\nu \sum_{\beta=1}^n \frac{\partial z_i}{\partial z_j^\beta} f_{jk|\nu}^\beta(z_k)t^\nu - f_{ik|\nu}(z_k)t^\nu + f_{ij|\nu}(z_j)t^\nu,$$

and by the induction hypothesis, we have that $f_{ik}^{\nu-1}(z_k, t) - f_{ij}^{\nu-1}(f_{jk}^{\nu-1}(z_k, t), t) \equiv 0 \; mod\,[t^\nu]$, so its power series expansion begins in the term of degree ν of t. Now putting the coefficient of t^ν as $\Gamma_{ijk|\nu}(z_k)$, we have

$$\Gamma_{ijk|\nu}(z_k) = \sum_{\beta=1}^n \frac{\partial z_i}{\partial z_j^\beta} f_{jk|\nu}^\beta(z_k) - f_{ik|\nu}(z_k) + f_{ij|\nu}(z_j). \tag{2.14}$$

Each component $\Gamma_{ijk|\nu}^\alpha(z_k)$ of the vector $\Gamma_{ijk|\nu}(z_k)$ is a holomorphic function defined on $U_i \cap U_j \cap U_k \neq \emptyset$. Writing (2.14) in terms of the components and using vector fields notation, we get the following

$$\sum_\alpha \Gamma_{ijk|\nu}^\alpha(z_k)\frac{\partial}{\partial z_i^\alpha} = \sum_\beta f_{jk|\nu}^\beta(z_k)\frac{\partial}{\partial z_j^\beta} - \sum_\alpha f_{ik|\nu}^\alpha(z_k)\frac{\partial}{\partial z_i^\alpha} + \sum_\alpha f_{ij|\nu}^\alpha(z_j)\frac{\partial}{\partial z_i^\alpha}. \tag{2.15}$$

Therefore putting

$$\Gamma_{ijk|\nu} = \sum_{\alpha} \Gamma^{\alpha}_{ijk|\nu}(z_k)\frac{\partial}{\partial z_i^{\alpha}}, \quad f_{jk|\nu} = \sum_{\beta} f^{\beta}_{jk|\nu}(z_k)\frac{\partial}{\partial z_j^{\beta}}, \dots,$$

we can write (2.15) as

$$\Gamma_{ijk|\nu} = f_{jk|\nu} - f_{ik|\nu} + f_{ij|\nu}. \tag{2.16}$$

Thus if the induction hypothesis holds, then (2.12) is equivalent to (2.16).

In our case, $\{f_{jk|\nu}\}$ is not exactly a 1-cocycle. In this regard, we extend the definitions of 1-cochains, 2-cocycles, etc (see [5]).

We denote the $\mathbb{C}$-vector spaces of 1-cochains, 2-cocycles and 2-nd cohomology group with respect to this new definition by $\hat{C}^1(\mathscr{U},\Theta)$, $\hat{Z}^2(\mathscr{U},\Theta)$ and $\hat{H}^2(\mathscr{U},\Theta)$ respectively.

Now let $\{f_{jk|\nu}\} \in \hat{C}^1(\mathscr{U},\Theta)$ and $\{\Gamma_{ijk|\nu}\} \in \hat{Z}^2(\mathscr{U},\Theta)$. Then from equality (2.16) we have that $\{\Gamma_{ijk|\nu}\} = \delta(\{f_{jk|\nu}\})$, thus in order to determine $f^{\nu}_{jk}(z_k, t)$ such that (2.12) holds, it suffices to determine the 1-cochain $\{f_{jk|\nu}\} \in \hat{C}^1(\mathscr{U},\Theta)$ such that $\{\Gamma_{ijk|\nu}\} = \delta(\{f_{jk|\nu}\})$. Let $\Gamma_\nu \in \hat{H}^2(\mathscr{U},\Theta)$ be the cohomology class of the 2-cocycle $\{\Gamma_{ijk|\nu}\}$, therefore if $\Gamma_\nu \neq 0$ then there exists no 1-cochain $\{\sigma_{jk|\nu}\} \in \hat{C}^1(\{U_j\},\Theta)$ such that $\{\Gamma_{ijk|\nu}\} = \delta(c^1)$, therefore we cannot construct $\{f_{jk|\nu}\}$ such that (2.12) holds. Following this logic, we shall define the notion of obstructions:

Definition 2.2.1 Γ_ν *(as constructed above) is called the* ***obstruction*** *to deformation of* M*. Moreover,* Γ_2 *is called the first obstruction, and* $\Gamma_{\nu+1}$ *the* ν*-th obstruction for* $\nu \geq 2$.

Remark 2.2.2 *Notice that we always have* $\Gamma_1 = 0$*, since*

$$\Gamma_{ijk|1} = f_{ik|0}(z_k) - f_{ij|0}(f_{jk|0}(z_k)) = 0$$

always holds. This also explains why the first obstruction corresponds to $\nu =$ 2.

It is now apparent that unless $\Gamma_2 = 0, \dots, \Gamma_\nu = 0$, $\Gamma_{\nu+1}$ is not defined. Moreover, even if this condition is satisfied, $\Gamma_{\nu+1}$ depends in general on the choice of $\{f_{jk|\nu}\}$ with $\delta\{f_{jk_\nu}\} = \{\Gamma_{ijk|\nu}\}$, hence, it may be that $\Gamma_{\nu+1} \neq 0$ for one choice of $\{f_{jk|\nu}\}$, and $\Gamma_{\nu+1} = 0$ for another.

However, in case $\hat{H}^2(\mathscr{U},\Theta) = 0$, we can determine $\{f_{jk|\nu}\}$ successively for $\nu = 2, 3, \dots$ such that $\delta\{f_{jk|\nu}\} = \{\Gamma_{ijk|\nu}\}$. Thus we construct power series

$$f^{\alpha}_{jk}(z_k, t) = \sum_{\nu=0}^{\infty} f^{\alpha}_{jk|\nu}(z_k)t^{\nu}$$

satisfying the condition

$$f_{ik}^{\alpha}(z_k,t) = f_{ij}^{\alpha}(f_{jk}(z_k,t)),\ \ \alpha = 1,\dots,n.$$

Now it only remains to show that this power series converges on a sufficiently small disk if $\{f_{jk_\nu}\}$ are suitably chosen. Unfortunately, the convergence could not be proven using an elementary method, it actually requires advanced analytic tools (see [5]).

Through this construction, one is able to explicitly see the condition which will answer the question about the existence of a complex analytic family that corresponds to an element of the first cohomology group. Hence the following fundamental theorem:

Theorem 2.2.3 (Theorem of existence) *Let M be a compact complex manifold and suppose that $H^2(\mathscr{U},\Theta) = 0$. Then there exists a compact complex analytic family $(\mathscr{M},B,\varpi)$ with $t_0 \in B \subset \mathbb{C}^m$ verifying the following conditions:*

1. $\varpi^{-1}(t_0) = M$.
2. *The Kodaira-Spencer map $(\rho_t)_{|t=t_0}$ is an isomorphism from $T_{t_0}B$ to $H^1(M,\Theta)$.*

The proof of this theorem requires the use of some rather technical tools from analysis, therefore we invite the reader to see [5] for details.

2.3 Number of moduli

At the end of this section, we shall define the moduli number of a compact complex manifold M , which is basically the number of effective independent parameters on which the infinitesimal deformations of M depend. First, we shall introduce complete families and effectively parametrized families.

Let M be a compact complex manifold and $\theta \in H^1(M,\Theta)$. If we could find a complex analytic family $(\mathscr{M},B,\varpi)$ where $M = \varpi^{-1}(t_0)$ for some $t_0 \in B$, such that the Kodaira-Spencer map $\rho_{t_0} : T_{t_0}B \longrightarrow H^1(M,\Theta)$ is surjective, then we can solve the problem mentioned above since each θ will correspond to some infinitesmal deformation of M along some tangent vector $\dfrac{\partial}{\partial t} \in T_tB$. It turns out that families with this property are locally universal in the sense that they induce locally every infinitesimal deformation of M in any complex analytic family. We then give the following definition:

Definition 2.3.1 *Let $(\mathcal{M}, B, \varpi)$ be a complex analytic family of compact complex manifolds and let $t_0 \in B$. Let $(\mathcal{N}, D, \pi)$ be another complex analytic family such that $N_{s_0} = \pi^{-1}(s_0)$ is biholomorphic to M_{t_0}, for some $s_0 \in D$. We say that the complex analytic family $(\mathcal{M}, B, \varpi)$ is* ***complete at*** *$t_0 \in B$ if for any such a family $(\mathcal{N}, D, \pi)$, there exists a small neighborhood E of s_0 and a holomorphic map $h : E \longrightarrow B$ such that $h(s_0) = t_0$ and the induced family $(h^*\mathcal{M}, E, h^*\varpi)$ is equivalent to the restriction $(\mathcal{N}_E, E, \pi)$ of $(\mathcal{N}, E, \pi)$ to $E \subset D$.*

We can then extend this definition and say that a complex analytic family $(\mathcal{M}, B, \varpi)$ is complete (globally) if it is complete at t for every $t \in B$.

Following the motivation given above, we will state the following theorem:

Theorem 2.3.2 (Theorem of completeness) *Let $(\mathcal{M}, B, \varpi)$ be a complex analytic family of compact complex manifolds. Then for some $t_0 \in B$, if the Kodaira-Spencer map $\rho_{t_0} : T_{t_0} \longrightarrow H^1(\mathcal{M}_{t_0}, \Theta)$ is surjective, then the family is complete at t_0.*

We will explain the philosophy behind the proof. First, let $(\mathcal{M}, B, \varpi)$ be a complex analytic family of compact complex manifolds and $h : D \longrightarrow B$ a holomorphic map of complex manifolds. Considering the induced family $(h^*\mathcal{M}, D, h^*\varpi)$, there exists a holomorphic map $g : h^*\mathcal{M} \longrightarrow \mathcal{M}$ such that it maps $(h^*\varpi)^{-1}(s)$ biholomorphically to $\varpi^{-1}(h(s)) = M_{h(s)}$ for all $s \in D$ such that the following diagram commutes:

$$\begin{array}{ccc} h^*\mathcal{M} & \xrightarrow{g} & \mathcal{M} \\ {\scriptstyle h^*\varpi}\downarrow & & \downarrow{\scriptstyle \varpi} \\ D & \xrightarrow{h} & B \end{array}$$

If we identify $(h^*\varpi)^{-1}(s_0) = M_{h(s_0)} \times \{s_0\}$ with $M_{h(s_0)} = \varpi^{-1}(h(s_0))$ then we can say that g extends the identity map on $M_{h(s_0)}$.

The converse of this statement is a crucial step for the proof of the theorem and will be stated in the following lemma:

Lemma 2.3.3 *Let $(\mathcal{M}, B, \varpi)$ and $(\mathcal{N}, D, \pi)$ be two complex analytic families of compact complex manifolds with $t_0 \in B$ and $s_0 \in D$ such that $M =$*

$\varpi^{-1}(t_0) = \pi^{-1}(s_0)$. *Suppose that we have a holomorphic map* $h : \Delta \longrightarrow B$ *with* $h(s_0) = t_0$, *where* $\Delta \subset D$ *is a sufficiently small neighborhood of* s_0. *If we can find a holomorphic map* $g : \mathcal{N}_\Delta = \pi^{-1}(\Delta) \longrightarrow \mathcal{M}$ *such that:*

1. *g maps* $\pi^{-1}(s)$ *biholomorphically onto* $\varpi^{-1}(h(s))$ *for all* $s \in \Delta$.

2. *g restricted to* $\pi^{-1}(s_0)$ *is the identity map on* M,

then the induced family $(h^*\mathcal{M}, \Delta, h^*\varpi)$ *is exactly the restriction* $(\mathcal{N}_\Delta, \Delta, \pi)$ *of* $(\mathcal{N}, D, \pi)$ *to* Δ.

Proof.
Let us take the family $(h^*\mathcal{M}, \Delta, h^*\varpi)$ induced from $(\mathcal{M}, B, \varpi)$ by the holomorphic map $h : \Delta \longrightarrow B$. Then $(h^*\varpi)^{-1}(s) = M_{h(s)} \times \{s\}$ for each $s \in \Delta$ and

$$h^*\mathcal{M} = \bigcup_{s\in\Delta} M_{h(s)} \times \{s\} \subset \mathcal{M} \times \Delta$$

We have $g : \mathcal{N}_\Delta \longrightarrow \mathcal{M}$. We then define the map $\phi : \mathcal{N}_\Delta \longrightarrow \mathcal{M} \times \Delta$ which takes a point q and maps it to $(g(q), \pi(q))$. Since g maps N_s biholomorphically to $M_{h(s)}$, ϕ maps N_s biholomorphically to $\phi(N_s) = M_{h(s)} \times \{s\}$. Hence ϕ maps $\mathcal{N}_\Delta = \bigcup_{s\in\Delta} N_s$ biholomorphically onto $\mathcal{N} = \bigcup_{s\in\Delta} M_{h(s)} \times \{s\}$. Moreover, $\pi = (h^*\varpi) \circ \phi$ since we have

$$(h^*\varpi)(\phi(N_s)) = (h^*\varpi)(M_{h(s)} \times \{s\}) = s = \pi(N_s).$$

Hence $(\mathcal{N}_\Delta, \Delta, \pi)$ and $(h^*\mathcal{M}, \Delta, h^*\varpi)$ are equivalent, which proves the lemma. ∎

In order to prove the theorem using the lemma above, it suffices that for any given complex analytic family $(\mathcal{N}, D, \pi)$ with $\pi^{-1}(s_0) = M_{s_0}$, if we take Δ a small enough neighborhood of s_0, then we can construct the holomorphic maps h and g defined in the lemma and satisfying the same conditions. In fact, these maps are constructed as power series (see [5], chapter 6 section (a)) in a similar way to how we constructed the transition functions of a deformation space as power series in the previous section. The hypothesis of surjectivity of the Kodaira-Spencer map is also used in this construction. It is then shown that these power series indeed converge to give us the desired holomorphic maps, thus proving the theorem of completeness.

Let $(\mathcal{M}, B, \varpi)$ be a complex analytic family. It is complete at t_0 if the Kodaira-Spencer map ρ_{t_0} is surjective. In this case $(\mathcal{M}, B, \varpi)$ contains all infinitesimal deformations of M_{t_0}. However, it may happen that two distinct tangent vectors in T_tB give rise to the same infinitesimal deformation. To avoid this situation, we would like ρ_{t_0} to be injective. This naturally leads to the following definition:

Definition 2.3.4 *Let $(\mathcal{M}, B, \varpi)$ be a complex analytic family of compact complex manifolds and $t_0 \in B$. We say that the family $(\mathcal{M}, B, \varpi)$ is **effective** at t_0 if the Kodaira-Spencer map $\rho_{t_0} : T_{t_0}B \longrightarrow H^1(M_{t_0}, \Theta)$ is injective. We say that the family $(\mathcal{M}, B, \varpi)$ is **effectively parametrized** if it is effective at each $t \in B$.*

Now we introduce the following theorem which combines both the notion of completeness and effectiveness of a complex analytic family.

Theorem 2.3.5 *Let M be a compact complex manifold and $(\mathcal{M}, B, \varpi)$ a complete and effectively parametrized complex analytic family of compact complex manifolds such that $\varpi^{-1}(t_0) = M$, for some $t_0 \in B$. Then we have the following:*

1. *Let $(\mathcal{N}, D, \pi)$ be any complete effectively parametrized complex analytic family with $s_0 \in D$ such that $\pi^{-1}(s_0) = M$. Then B and D have the same dimension as manifolds.*
2. *Let $(\mathcal{N}, D, \pi)$ be any complex analytic family satisfying the same conditions as in (1), then there exists a domain $E \subset D$ with $s_0 \in E$ and an injective holomorphic map $h : E \longrightarrow B$ with $h(s_0) = t_0$ such that the family induced by h is equivalent to the restriction of the family $(\mathcal{N}, D, \pi)$ to E. In this sense, an effectively parametrized complete complex analytic family with fiber M is uniquely determined, when it exists, locally at the point over which M is the fiber.*

Proof.

Proof of (1): In order to prove the first statement, we will consider the dimension of B to be m, and that of D to be l, then we show that we have both $l \geq m$ and $m \geq l$.

We have that $\varpi^{-1}(t_0) = M = \pi^{-1}(s_0)$. Since $(\mathcal{M}, B, \varpi)$ is complete, then by the definition of a complete family at t_0, we get that that there exists an open neighborhood $E \subset D$ of s_0 and a holomorphic map $h : E \longrightarrow B$ such that $(h^*\mathcal{M}, E, h^*\varpi))$ is equivalent to $(\mathcal{N}_E, E, \pi)$. Let dimension of E be l. Since for any choice of local coordinate containing s on E we get a basis for T_sE, we choose E to be small enough so that it is covered by a single local coordinate, and hence we get a basis of tangent vectors $\left(\dfrac{\partial}{\partial s_1}, \ldots, \dfrac{\partial}{\partial s_l}\right)$ over $\mathbb{C}$ of $T_sE = T_sD$.

Since $(\mathcal{N}, D, \pi)$ is effectively parametrized, then so is $(\mathcal{N}_E, E, \pi)$, therefore $\rho_s : T_s E \longrightarrow H^1(N_s, \Theta_s)$ is injective for all $s \in E$, $N_s = \pi^{-1}(s)$. But $(h^*\mathcal{M}, B, h^*\varpi)$ is equivalent to $(\mathcal{N}_E, E, \pi)$, we have that N_s is biholomorphic to $M_{h(s)}$ for all $s \in E$. Thus $\left(\frac{\partial N_s}{\partial s_1}, \ldots, \frac{\partial N_s}{\partial s_l}\right)$ are linearly independent over $\mathbb{C}$ in $H^1(N_s, \Theta_s) = H^1(M_{h(s)}, \Theta_{h(s)})$, where $\frac{\partial N_s}{\partial s_i} = \rho_s\left(\frac{\partial}{\partial s_i}\right)$ for all $i = 1, \ldots, l$.

Next let the dimension of B be equal to m and $t = (t_1, \ldots, t_m)$ be a local coordinate in a sufficiently small neighborhood of $t_0 = h(s_0) \in B$. Then by choosing E to be smaller (if necessary), we get that $t = h(s)$ and $\left(\frac{\partial M_t}{\partial t_1}, \ldots, \frac{\partial M_t}{\partial t_m}\right)$ are linearly independent over $\mathbb{C}$ in $H^1(M_{h(s)}, \Theta_{h(s)}) = H^1(M_t, \Theta_t)$, where we have used the fact that $\rho_t(T_t B)$ inside $H^1(M_t, \Theta_t)$ has dimension m because the family $(\mathcal{M}, B, \varpi)$ is effectively parametrized. We have the following commutative diagram :

$$\begin{tikzcd} H^1(M_{h(s)}, \Theta_{h(s)}) \arrow[r, Rightarrow, "="] & H^1(M_t, \Theta_t) \\ T_s D \arrow[u] \arrow[r, "dh_s"] & T_t B \arrow[u] \end{tikzcd}$$

moreover, using theorem (3.3.1.1) in [9], we obtain that $\mathrm{Im}(\rho_s) \subset Im(\rho_t)$ for each $s \in E$. Thus we get that $l \leq m$ by using the fact that $\left(\frac{\partial N_s}{\partial s_1}, \ldots, \frac{\partial N_s}{\partial s_l}\right)$ are linearly independent over $\mathbb{C}$.

By repeating the same argument again and reversing the roles of $(\mathcal{M}, B, \varpi)$ and $(\mathcal{N}, D, \pi)$ m, we get that $m \leq l$, therefore $m = l$.

Proof of (2): We have shown that for all $s \in E$, $\rho_s(T_s D) = \rho_t(T_t B) \in H^1(M_t, \Theta_t)$ where $t = h(s)$. Thus the determinant of the matrix $\left(\frac{\partial t_j}{\partial s_i}\right)_{i=j=1,\ldots,l}$ is non zero. But this matrix is precisely the Jacobian matrix of the holomorphic map $h : E \longrightarrow B$ (written in terms of the local coordinates at $s_0 \in E$ and $t_0 = h(s_0) \in B$). Hence h has to be injective and thus $\Delta = h(E)$ is a domain in B containing t_0. By taking E small enough, so that Δ becomes small enough as well, in such a way that we can write the deformation space of the restricted family $(\mathcal{M}_\Delta, \Delta, \varpi)$ of $(\mathcal{M}, B, \varpi)$ to Δ as

$$\mathcal{M}_\Delta = \bigcup_{j=1}^{k} (U_j \times \Delta)/ \sim_\Delta$$

where the U_j are polydiscs used to construct the manifold M_{t_0} via the transition functions $\{f_{jk}(z_k, t_0)\}$. Furthermore, the equivalence relation is defined as such: $(z_j, t) \in U_j \times \Delta$ and $(z_k, t') \in U_k \times \Delta$ are equivalent by $\sim_\Delta$ if and only if $t = t'$ and $f_{jk}(z_k, t) = z_j$. Then we may write

$$\mathscr{N}_E = \bigcup_{j=1}^{k} (U_j \times \Delta) / \sim_E$$

where the equivalence relation $\sim_E$ is defined as such: $(z_j, s) \in U_j \times E$ and $(z_k, s') \in U_k \times E$ are equivalent by $\sim_E$ if and only if $s = s'$ and $f_{jk}(z_k, h(s)) = z_j$.

Now since $h : E \longrightarrow \Delta$ is biholomorphic, we can identify E and Δ as biholomorphic structures on the same differentiable manifold. Then the complex analytic families $(\mathscr{M}_\Delta, \Delta, \varpi)$ and $(\mathscr{N}_E, E, \pi)$ can be considered the same complex analytic families but with different choices of coordinate systems on $\mathscr{M}_\Delta$ and $\mathscr{N}_E$. ■

Motivated by the above theorem, we state the following definition which introduces the number of moduli of a compact complex manifold M:

Definition 2.3.6 *Let M be a compact complex manifold. Let $(\mathscr{M}, B, \varpi)$ be a complete and effectively parametrized complex analytic family of compact complex manifolds with $t_0 \in B$ such that $\varpi^{-1}(t_0) = M$. We define the number of moduli of M to be the dimension of B. We denote it by $m(M)$.*

Chapter 3

Moduli spaces

3.1 Construction

Closely related with deformation theory is the problem of classification of geometric or algebraic objects, where the question of existence of a space parametrizing these objects arises. This space, when it exists, is called a moduli space. During this chapter, we define in the first part what do we mean by a classification problem and give examples of the construction of the moduli spaces. In the latter part, we give examples of how infinitesimal deformation can be applied to get local properties of moduli spaces, specifically we use the existence theorem introduced in the previous chapter to determine the number of moduli as well as other properties of the moduli space.

First, we shall fix some notations that we will use throughout the section. Let $\mathscr{C}$ be a category, if X is an object in $\mathscr{C}$, then we write $X \in ob(\mathscr{C})$. A set with only one point will be denoted by $\{pt\}$.

In order to illustrate the steps to solve a classification problem, as well as the purpose of a moduli space, we will explicitly solve the classification problem for the set of equivalence classes of all quartic surfaces in $\mathbb{P}^3_{\mathbb{C}}$.

A quartic surface K in $\mathbb{P}^3_{\mathbb{C}}$ is the set of roots of a homogeneous polynomial $F(x_0, x_1, x_2, x_3)$ (or F for simplicity) of degree 4. We will use the fact that such a polynomial has 35 coefficients.

Now let us consider the set

$$S = \{K \subset \mathbb{P}^3_{\mathbb{C}} \mid K \text{ a quartic surface}\}.$$

Since a quartic surface is defined by a homogeneous polynomial, S can be identified with the set of homogeneous polynomials of degree 4 modulo the equivalence relation $\sim$ such that for a polynomial $F_1 \sim F_2$ if and only if $F_2 \cong \lambda F_1$ for all non zero $\lambda \in \mathbb{C}$.

A polynomial F can be given by its coefficients $(a_0, a_1, \ldots, a_{34})$, therefore we can identify S with the set

$$\{(a_0, a_1, \ldots, a_{34}) \mid a_i \in \mathbb{C},\ i = 0, \ldots, 34\} / \sim,$$

where $(a_0, a_1, \ldots, a_{34}) \sim (a'_0, a'_1, \ldots, a'_{34})$ if and only if $a'_i = \lambda a_i$ for $i = 1, \ldots, 34$ and for all non zero $\lambda \in \mathbb{C}$. But the latter set is non other than $\mathbb{P}^{34}_{\mathbb{C}}$, therefore $S \cong \mathbb{P}^{34}_{\mathbb{C}}$.

This last result means that the set S possesses the structure and properties of $\mathbb{P}^{34}_{\mathbb{C}}$. But we shall not stop here, since our aim is to study the set of equivalence classes of elements of S, which leads us to define the equivalence relation $\sim_S$ as follows :

For $K_1, K_2 \in S$, $K_1 \sim_S K_2$ if and only if K_1 and K_2 are isomorphic, which is equivalent to say that $F_1(x_0, \ldots, x_3)$ and $F_2(y_0, \ldots, y_3)$ are related by a coordinate transformation, where F_i is the polynomial defining the surface K_i, $i = 1, 2$, i.e., there exists an invertible matrix $G \in GL(4, \mathbb{C})$ such that

$$\begin{pmatrix} x_0 \\ \vdots \\ x_3 \end{pmatrix} = G \begin{pmatrix} y_0 \\ \vdots \\ y_3 \end{pmatrix}$$

By defining a group action of $GL(4, \mathbb{C})$ on $\mathbb{P}^{34}_{\mathbb{C}}$, we get that

$$S / \sim_S \cong \mathbb{P}^{34}_{\mathbb{C}} / GL(4, \mathbb{C}).$$

The topological space $\mathbb{P}^{34}_{\mathbb{C}} / GL(4, \mathbb{C})$ is the moduli space of this classification problem, since $S / \sim_S$ inherits all of its properties.

Now in order to understand further the construction of the moduli space/solving the classification problem in a more general setting, we shall give an example where the moduli space cannot be determined explicitly. For instance, let us consider S to be the set of all curves, and $\sim_S$ is defined such that for $X_1, X_2 \in S$, $X_1 \sim_S X_2$ if and only if $X_1 \cong X_2$. At this point $S / \cong_S$ is just a set.

As in the previous example about projective quartic surfaces, we would like $S / \sim_S$ to be isomorphic to some space with nice properties. To do that, we shall first define the notion of a family of curves parametrized by a complex manifold, afterwards we define a certain functor from the category of complex manifolds to the category of sets, and would like it to be representable by a complex manifold, which would be exactly the moduli space for this case.

We define the notion of family of curves as such: consider a complex analytic family $(\mathscr{M}, B, \varpi)$, such that $\dim(B) = m$ and $\dim(\mathscr{M}) = m+1$. Therefore by the implicit function theorem, for all $t \in B$ we have that $\varpi^{-1}(t)$ is a submanifold of $\mathscr{M}$ of dimension 1, i.e., a curve. We also know that $\varpi^{-1}(t)$ depends holomorphically on t.

Let $\mathscr{C}$ be the category of complex manifolds and $Sets$ be the category of sets, consider the contravariant functor

$$F : \mathscr{C} \longrightarrow Sets$$

such that $F(B)$ is the set of equivalence classes of families of _curves in S parametrized by B. Here equivalence of families are defined as in (definition 1.1.9).

At this point, we ask whether this functor is representable or not. If the answer is positive, it means that there exists a complex manifold $M \in ob(\mathscr{C})$ such that

$$F(B) \cong \mathrm{Hom}_{\mathscr{C}}(B, M)$$

for all $B \in ob(\mathscr{C})$. Now when we consider the complex manifold whose underlying topological space is just a point $\{pt\} \in ob(\mathscr{C})$, we have that $F(\{pt\})$ is the set of equivalence classes of families parametrized by $\{pt\}$ (which are actually curves in S), therefore $F(\{pt\}) = S/\sim_S$. Furthermore, since $\mathrm{Hom}_{\mathscr{C}}(\{pt\}, M) = M$, we get that $S/\sim_S \cong M$ as in the previous example.

Remark 3.1.1 *The choice of the category $\mathscr{C}$ is not restricted to that of complex manifolds. However, depending on the objects that one seeks to classify, one has to choose the right category $\mathscr{C}$ otherwise the functor F will fail to be representable.*

At this point, we shall give a more general definition of a classification problem along with the construction of the moduli space, based on the two previous examples.

Let S be a given set of mathematical objects (curves, matrices, sheaves...), and $\sim_S$ an equivalence relation on S, so that we can construct the set $S/\sim_S$. In order to define a notion of family, we need to define the "classification" functor from the category $\mathscr{C}$ comprised of the objects which are sets with a certain mathematical structure and morphisms which preserve the said structure.

Next, we require this structure to be naturally related to the structure on the elements of S. It is made clearer by the notion of a family of elements of S parametrized by an object of $\mathscr{C}$. In the good case, we want the structure on $S/\sim_S$ to reflect properties of families of objects of S parametrized by objects of $\mathscr{C}$.

We want the category $\mathscr{C}$ whose objects parametrize the families in $S/\sim_S$ to verify the following conditions:

- There is a forgetful functor For from $\mathscr{C}$ to the category of sets such that for every objects $M, N \in ob(\mathscr{C})$, the mapping of sets induced by For
$$\mathrm{Hom}_{\mathscr{C}}(M,N) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(For(M), For(N))$$
is injective.
- There exists a base point object in $\mathscr{C}$ denoted by $\{pt\}$ such that its underlying set consists of one element, furthermore there exists a canonical identification of the set $\mathrm{Hom}_{\mathscr{C}}(\{pt\}, M)$ with $For(M)$, for every $M \in ob(\mathscr{C})$.

From now on, $\mathscr{C}$ will denote a category satisfying the above conditions. Now we can give the following definition:

Definition 3.1.2 *Let S be a set of geometric objects and $\sim_S$ be an equivalence relation on S. Let $\mathscr{C}$ be a category. A functor of families is a contravariant functor*
$$FAM : \mathscr{C} \longrightarrow \textbf{\textit{Sets}}$$
such that :

1. *$FAM(\{pt\}) = S$.*
2. *For every $T \in ob(\mathscr{C})$, there exists an equivalence relation $\sim_T$ on the set $FAM(T)$ such that if $T = \{pt\}$, $\sim_T$ reduces to the already given $\sim_S$.*
3. *for any morphism $\phi : T_1 \longrightarrow T_2$ in $\mathscr{C}$, we have*
$$FAM : FAM(T_2) \longrightarrow FAM(T_1)$$
which takes an equivalence relation $\sim_{T_2}$ to $\sim_{T_1}$.

Representable functors reflect the intuition that the structure of an object M in a good category $\mathscr{C}$ (in the sense that it verifies the conditions above) must be described by the knowledge of all morphisms inside $\mathrm{Hom}_{\mathscr{C}}(X, M)$, for any object $X \in ob(\mathscr{C})$.

Example 3.1.3 *Let M be a complex manifold. We denote by M_{C^∞} the underlying differentiable manifold of M. We want to study the structures of complex manifolds that can be defined on M_{C^∞} and occur as deformation of the complex structure M on M_{C^∞}. Let $S = \{(M_{C^\infty}, \{U_j, z_j\}),\ j \in \mathbb{N}\}$, where $\{U_j, z_j\}$ is a system of local coordinates on M_{C^∞} which gives it the structure of M (or of a complex manifold which occurs as the deformation of M). We define $\sim_S$ such that two elements of S are equivalent if and only if they give rise to biholmorphic complex structures.*

Let $\mathscr{C}$ be the category whose objects are pairs (B, t_0) where B is a connected manifold and $t_0 \in B$ (called the base point), and whose morphisms are maps of manifolds that preserve base points.

First, a **moduli problem** of a collection of objects S consists of the following notions:

- An object X in a category $\mathscr{C}$,
- a collection of objects S and an equivalence relation $\sim_S$ on it,
- a functor *FAM* of families of objects of S parametrized by objects of $\mathscr{C}$
- a pullback of families compatible with the equivalence class.

The problem of moduli of S is given by the following steps :

1. To find $M \in ob(\mathscr{C})$ such that the elements of the underlying set of M are in a canonical one to one correspondence with the elements of the set of equivalence classes $S/\sim_S$.
2. To investigate the ways in which the properties of families influence the structure of M.

Thanks to Yoneda's Lemma, we know that the structure of M is uniquely determined by the representable functor $\mathrm{Hom}_{\mathscr{C}}(\_\_, M)$.

The problem is to check whether *FAM* is representable by an object M, i.e., $FAM \simeq \mathrm{Hom}(\_\_, M)$.

Definition 3.1.4 *Given a moduli problem, the object $M \in ob(\mathscr{C})$ which represents the moduli functor FAM is called a* ***fine moduli space****.*

In fact, if *FAM* is representable by M, then for any object $B \in ob(\mathscr{C})$ elements of *FAM*(B) are families of the type $\phi : \mathscr{D} \longrightarrow B$, which are identified with a morphism χ in $\mathrm{Hom}(B, M)$ by the natural transformation isomorphism. χ sends an element $b \in B$ to the moduli point in M determined by the fiber $\phi^{-1}(b)$. By going the other way around, if we consider the identity morphism $id_M \in \mathrm{Hom}(M, M)$, then it corresponds to a family $\mathbb{1} : C \longrightarrow M$ called **the universal family**. This special family has the property that any other family over B is the pullback of C via a unique map of B to M. We can now rephrase the definition of a fine moduli space into:

Definition 3.1.5 *A fine moduli space is an object $M \in ob(\mathscr{C})$ which is the base space of a universal family.*

Fine moduli spaces are very useful in the sense that they allow us to get data about the families of objects that we desire to study by looking at only one family. Unfortunately, it is very rare to find a fine moduli space which represents a given moduli problem. So we use a weaker version called the coarse moduli space.

Definition 3.1.6 *Given a moduli problem, a* ***coarse moduli space*** *is a pair (M, Φ) where $M \in ob(\mathscr{C})$ and*

$$\Phi : FAM \longrightarrow \mathrm{Hom}(\_\_, M)$$

is a natural transformation such that:

1. *for $\{pt\} \in ob(\mathscr{C})$, the mapping $\Phi(\{pt\})$ is bijective.*
2. *for all objects $N \in ob(\mathscr{C})$ and any natural transformation*

   $$\Psi : FAM \longrightarrow \mathrm{Hom}_{\mathscr{C}}(\_\_, N)$$

   there exists a unique natural transformation

   $$\Omega : \mathrm{Hom}_{\mathscr{C}}(\_\_, M) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(\_\_, N)$$

   such that $\Psi = \Omega \circ \Phi$.

It turns out that it is also hard to find coarse moduli spaces representing a certain moduli functor, therefore we take some measures as restricting the class of objects that we want to parametrise and take only suitable objects. For instance if we take our collection to be of equivalence classes of families of vector bundles on a smooth algebraic variety X, by considering only families of stable vector bundles we get a coarse moduli space representing the moduli functor in this case. This space can also be constructed, using geometric invariant theory, as the quotient of a certain scheme by a natural group action.

Once the existence of the moduli space is proven, the question that we may ask ourselves here is what structure does the moduli space have? In other words, is it connected, irreducible...

3.2 Local properties of moduli spaces

Infinitesimal deformation theory gives information on the local structure of the moduli space. Kodaira-Spencer's existence theorem, or more generally Kuranishi's theorem, allow us to construct a local moduli space. In fact, given a compact complex manifold M, the complete and effectively parametrized complex analytic family $(\mathscr{M}, B, \varpi)$ given by the theorem such that $M = \varpi^{-1}(t_0)$ for some $t_0 \in B$ is locally universal over the germ of neighborhoods of $t_0 \in B$. This germ is precisely the local moduli space. We can get information on its structure using various techniques.

In the following, we shall use many results without proving them, but we shall give a reference. For the notion of the degree of a vector bundle V (denoted by $\deg(V)$), see [[9], chapter 4, section 4.6.3].

Theorem 3.2.1 (Riemann-Roch for vector bundles) *Let V be a holomorphic vector bundle of rank n over a compact Riemann surface X of genus g. Let $\mathscr{F}$ denote the sheaf of germs of holomorphic sections of V. Then the degree of V is given by the formula*

$$\deg(V) = \dim_{\mathbb{C}}(H^0(X, \mathscr{F})) - \dim_{\mathbb{C}}(H^1(X, \mathscr{F})) + n(g-1).$$

Theorem 3.2.2 (Serre duality) *Let V be a holomorphic vector bundle over a compact Riemann surface X, V^* its dual bundle, $\mathscr{F}$ and $\mathscr{F}^*$ the corresponding sheaves of germs of holomorphic sections and $\mathscr{K} = \Theta^*$ the sheaf of holomorphic sections associated to $K = T^*X$. Then we have the canonical isomorphism of $\mathbb{C}$-vector spaces*

$$H^1(X, \mathscr{F})^* \cong H^0(X, \mathscr{K} \otimes \mathscr{F}^*).$$

Proposition 3.2.3 ([7],(2.7)) *Let L be a line bundle over a Riemann surface X of genus $g \geq 2$, and let the degree of the line bundle L be greater than $2g-1$. Then we have $H^1(X,\mathscr{L}) = 0$, where $\mathscr{L}$ is the sheaf of holomorphic sections associated to L.*

Example 3.2.4 *Let M be a compact Riemann surface of genus $g \geq 2$. Then Riemann's formula states that the number of moduli which is the number of effective independent complex parameters on which the deformation of complex structures of M depend is given by $3g-3$.*

Since M is of dimension 1, then $H^2(M,\Theta) = 0$ and thus we can use Kodaira-Spencer's existence theorem to find a complete and effectively parametrized complex analytic family $(\mathscr{M}, B, \varpi)$ such that $M = \varpi^{-1}(t_0)$, thus the local moduli space X of M exists and is smooth. Moreover, we have that the dimension of X is equal to

$$\dim(H^1(M,\Theta)) = \dim(H^0(M,\mathscr{K} \otimes \mathscr{K}))$$

using Serre duality, where $\mathscr{K} = \Theta^*$ described above. Using proposition 3.2.3 and the Riemann-Roch theorem, we have that

$$\begin{aligned}
\deg(K) &= \dim(H^0(M,\mathscr{K})) - \dim(H^1(M,\mathscr{K})) + (\mathrm{rank}(K)).(g-1) \\
&= g - \dim(H^0(M,\mathscr{O}_M)) + (g-1) \\
&= 2g-2,
\end{aligned}$$

where we have used the definition of the genus and Serre duality, and $\mathscr{O}_M$) denotes the sheaf of germs of holomorphic functions of M. Since we have $deg(\mathscr{K}\otimes\mathscr{K}) = 2.deg(K) = 2(2g-2) \geq (2g-1)$, we get that $\dim(H^1(M,\mathscr{K})) = 0$. Again using Riemann-Roch theorem, we have that the number of moduli of M is

$$\begin{aligned}
\dim(H^0(M,\mathscr{K} \otimes \mathscr{K})) &= \dim(H^1(M,\mathscr{K} \otimes \mathscr{K})) + \deg(K \otimes K) - g + 1 \\
&= 0 + (4g-4) - g + 1 \\
&= 3g-3.
\end{aligned}$$

Let us define the notion of families for vector bundles. Let X be a compact complex manifold and $V = (V, X, \pi)$ a holomorphic vector bundle over X of rank n. A complex analytic family of holomorhic vector bundles of rank n over X parametrized by a complex manifold B is a holomorphic vector bundle $\mathbb{V}$ on the product complex manifold $X \times B$ such that for all $t \in B$, the pullback $\mathbb{V}_t$ of $\mathbb{V}$ via the canonical inclusion

$$X \cong X \times \{t\} \hookrightarrow X \times B$$

is a holomorphic vector bundle of rank n over X. If there exists $t' \in B$ such that $\mathbb{V}_{t'}$ is isomorphic to V, then $\mathbb{V}_{t'}$ is called a deformation of V over B, for all $t \in B$.

Let $(\mathbb{V}, X \times B, \varpi)$ be an analytic family of holomorphic vector bundles of rank v over a compact Riemann surface X and parametrized by the complex manifold B, V a holomorphic vector bundle such that $V = \varpi^{-1}(X \times \{0\})$. Let $\mathscr{U} = \{U_i\}_{i \in I}$ be a finite covering of open subsets of X, and let $\Delta \subset B$ be a neighborhood of 0. Consider $\mathbb{V}_\Delta = (\mathbb{V}_\Delta, X \times \Delta, \varpi)$ be the holomorphic vector bundle over $X \times \Delta$, then by the definition of a vector bundle (1.3.1), if U_i is small enough, we have that $\pi^{-1}(U_i) \cong U_i \times \mathbb{C}^v \times \Delta$. Furthermore, for any $(x, t) \in U_i \times \Delta \cap U_j \times \Delta$, we can define the transition matrices $g_{ij}(x, t) \in GL(v, \mathbb{C})$ which we will denote by $G_{ij}(t)$. We recall that these transition matrices are endomorphisms and satisfy the cocycle condition

$$G_{ij}(t).G_{jk}(t) = G_{ik}(t) \tag{3.1}$$

on each $U_i \times \Delta \cap U_j \times \Delta \cap U_k \times \Delta \neq \emptyset$.

As we did in chapter 2 section (2.1), we want to differentiate these transition matrices with respect to $t \in B$ and then construct infinitesimal deformations of the vector bundle V which will lie in a certain cohomology group. In fact, by taking the condition (3.1) and differentiating each side of the equality with respect to t, we have

$$\begin{aligned} \frac{\partial G_{ik}(t)}{\partial t} &= \frac{\partial \big(G_{ij}(t).G_{jk}(t)\big)}{\partial t} \\ &= \frac{\partial G_{ij}(t)}{\partial t}.G_{jk}(t) + \frac{\partial G_{jk}(t)}{\partial t}.G_{ij}(t) \end{aligned}$$

Now to get our cocycles, we consider the two isomorphisms $h_j : \varpi^{-1}(U_i \cap U_j, \Delta) \longrightarrow (U_i \cap U_j) \times \mathbb{C}^v \times \Delta$ and $h_i^{-1} : (U_i \cap U_j) \times \mathbb{C}^v \times \Delta \longrightarrow \varpi^{-1}(U_i \cap U_j, \Delta)$. We know that $\dfrac{\partial G_{ij}(t)}{\partial t} : (U_i \cap U_j) \times \mathbb{C}^v \times \Delta \longrightarrow (U_i \cap U_j) \times \mathbb{C}^v \times \Delta$, therefore we put $\zeta_{ij}(t) = h_i^{-1} \dfrac{\partial G_{ik}(t)}{\partial t} h_j$, and it is easily verified that

$$\zeta_{ik}(t) = \zeta_{ij}(t) + \zeta_{jk}(t)$$

on each $U_i \times \Delta \cap U_j \times \Delta \cap U_k \times \Delta \neq \emptyset$, and that $\zeta_{ii}(t) = 0$.

Hence $\{\zeta_{ij}(t)\} \in Z^1(\mathscr{U}, END(V))$, where $END(V)$ is the sheaf of endomorphisms of the vector bundle V. Now we define the infinitesimal deformation of V to be the cohomology class $\zeta(t) \in H^1(X, END(V))$ of the 1-cocycle $\{\zeta_{ij}(t)\}$.

As in the case of complex manifolds, we can show that this infinitesimal deformation doesn't depend on the choice of the open covering $\mathscr{U}$ of X. We define the Kodaira-Spencer map for this case as follows

$$\rho_t : T_t B \longrightarrow H^1(X, END(V))$$

which takes a tangent vector $\dfrac{\partial}{\partial t}$ and associates it to the infinitesimal deformation $\zeta(t)$ through the construction shown above.

Now for this example we suppose that V has no non-trivial endomorphisms, i.e., the only endomorphisms of V are multiplication by a complex number. Such a vector bundle is called a **simple vector bundle**. Since X is a Riemann surface, it is of dimension 1, therefore $H^2(X, END(V)) = 0$ (where the obstructions lie). So we can apply the analogue of Kodaira-Spencer's theorem of existence for vector bundles and thus we get a complex holomorphic family $(\mathbb{V}, X \times B, \varpi)$ such that $V = \varpi^{-1}(X \times \{t_0\})$ for some $t_0 \in B$. We then have the existence of a smooth local moduli space of V. Since $T_t B \cong H^1(X, END(V))$, we have that the dimension of the local moduli space is equal to the dimension of $H^1(X, END(V))$. We will compute this dimension:

First, we use without proving the fact that the degree of the vector bundle $END(V)$ (whose sheaf of germs of endomorphisms of V is precisely $END(V)$) is equal to zero. Next we notice that since V is supposed to be a simple bundle, $H^0(X, END(V)) \cong \mathbb{C}$. Now we use Riemann-Roch theorem for vector bundles so we get

$$\begin{aligned}
\dim(H^1(X, END(V))) &= \dim(H^0(X, END(V))) - \deg(END(V)) \\
&\quad + (\operatorname{rank}(END(V))).(g-1) \\
&= 1 - 0 + v^2(g-1) \\
&= v^2(g-1) + 1
\end{aligned}$$

where g is the genus of X. So the number of independent parameters on which the structure of simple holomorphic vector bundle on V depend is $v^2(g-1)+1$.

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