On Kosloff Tal-Ezer least-squares quadrature formulas

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Problem

In this work we propose a method for the quadrature of **analytic functions** on compact intervals based on function values on **arbitrary grids**. In practice it is not always possible to sample functions on optimal nodes with a low-order **Lebesgue constant**. Therefore, we extend interpolatory quadrature formulas via the so-called **Fake Nodes** [2, 3]. More precisely, we analyse the **Kosloff and Tal-Ezer** (KT) map as stabilizing component of interpolatory and **least-squares** quadrature formulas (referred to as KTI and KTL formulas).

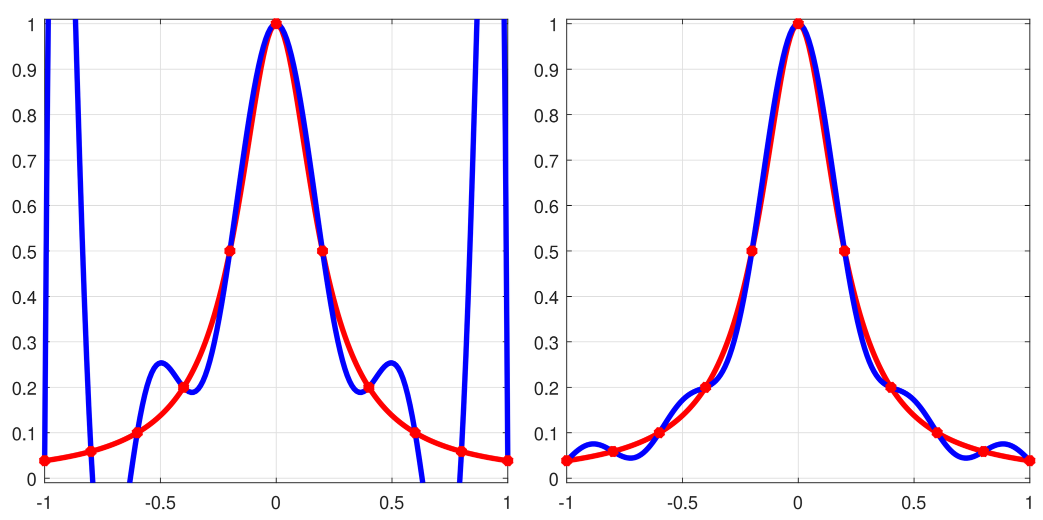

Fig 1. Polynomial and KT interpolation of Runge function on 11 equispaced nodes.

Preliminaries on KTL

Let $\mathcal{X} = \{x_0, \dots, x_m\}$ be a set of quadrature nodes in the interval $I = [-1, 1]$ and f a continuous function on I. The Kosloff Tal-Ezer map $M_\alpha : I \to I$ is given by

$$M_\alpha(x) := \frac{\sin(\alpha\pi x/2)}{\sin(\alpha\pi/2)},\ x \in I,\ \text{for } 0 < \alpha \leq 1,$$

and $M_0(x) := \lim_{\alpha \to 0^+} M_\alpha(x)$ for $\alpha = 0$. While M_0 is the identity map on I, the KT function M_α with $\alpha = 1$ maps the open and closed equidistant Newton-Cotes quadrature nodes to the Chebyshev and Chebyshev-Lobatto nodes. If $\mathbb{P}_n$ denotes the space of polynomials of degree at most n, we can associate to M_α the approximation space

$$\mathbb{P}_n^\alpha = \{P \circ M_\alpha \ : \ P \in \mathbb{P}_n\}.$$

If $\alpha < 1$, it is shown in [1] that the polynomial interpolant on $M_\alpha(\mathcal{X})$ displays **Runge type artifacts** if $\mathcal{X}$ is a set of equidistant nodes in I. To overcome this issue, the node set $\mathcal{X}$ was chosen larger such that $m > n$, and the following **weighted least-squares approximant** of the function f was introduced:

$$F_{n,\mathcal{X}}^\alpha(f) := \min_{P^\alpha \in \mathbb{P}_n^\alpha} \sum_{i=0}^{m} \mu_i |f(x_i) - P^\alpha(x_i)|^2,$$

with the weights μ_i given by

$$\mu_i = \frac{1}{2}\int_{M_\alpha(x_{i-1})}^{M_\alpha(x_{i+1})} \frac{1}{\sqrt{1-x^2}}\mathrm{d}x,\ i = 0, \dots, m,$$

where $x_{-1} = -1$ and $x_{m+1} = 1$.

References

[1] B. Adcock, R.B. Platte, *A mapped polynomial method for high-accuracy approximations on arbitrary grids*, SIAM J. Numer. Anal. **54** (2016).

[2] S. De Marchi, F. Marchetti, E. Perracchione, D. Poggiali, *Polynomial interpolation via mapped bases without resampling*, J. Comput. Appl. Math., **364** (2020), 112347.

[3] S. De Marchi, F. Marchetti, E. Perracchione, D. Poggiali, *Multivariate approximation at fake nodes*, Appl. Math. Comput., **391** (2021), 125628.

[4] S. De Marchi, G. Elefante, E. Perracchione, D. Poggiali, *Quadrature at fake nodes*, Dolomites Res. Notes Approx. **14** (2021), pp. 27–32.

KTL quadrature: development and analysis

For the calculation of the quadrature formula, we choose a basis $\Phi^\alpha = \{\phi_i^\alpha : i = 0, \dots, n\}$ for the space $\mathbb{P}_n^\alpha$. Then, we can write the least-squares approximant as

$$F_{n,\mathcal{X}}^\alpha(f) = \sum_{i=0}^{n} \gamma_i \phi_i^\alpha,$$

where the coefficient vector $\boldsymbol{\gamma}$ is determined by the least-squares solution of the linear system

$$\mathbf{W}\mathbf{A}^\alpha \boldsymbol{\gamma} = \mathbf{W}\boldsymbol{f}.$$

In this system $\mathbf{W} = \mathrm{diag}(\sqrt{\mu_0}, \dots, \sqrt{\mu_m})$ denotes the matrix with the least-squares weights, the matrix $\mathbf{A}^\alpha$ is defined by the entries $\mathbf{A}_{ij}^\alpha = \phi_j^\alpha(x_i)$ and $\boldsymbol{f}$ is the vector with all samples of f on $\mathcal{X}$. Based on this decomposition we have the **KTL** formula

$$\mathcal{I}_{n,\mathcal{X}}^\alpha(f, I) = \boldsymbol{\gamma}^\top \boldsymbol{\tau}^\alpha,$$

where $\boldsymbol{\tau}^\alpha \in \mathbb{R}^{n+1}$ is a moment vector with the entries $\tau_i^\alpha = \int_I \phi_i^\alpha(x)\,\mathrm{d}x$. If $m = n$ the formula $\mathcal{I}_{n,\mathcal{X}}^\alpha(f, I)$ is interpolatory and will be referred to as **KTI** quadrature formula.

The usage of the Chebyshev polynomials $\{T_i(x) : i = 0, \dots, n\}$ leads to the basis $\{T_i(M_\alpha(x)) : i = 0, \dots, n\}$ and allows to calculate the KTL and KTI quadrature weights $\boldsymbol{w}^\alpha$ in terms of a cosine and a non-equidistant fast Fourier transform.

Th. 1 (Computation of the moments)

For $0 < \alpha \leq 1$ and $i \in \mathbb{N}_0$, the moment

$$\tau_i^\alpha = \int_{-1}^{1} T_i(M_\alpha(x))\mathrm{d}x$$

corresponds to the i-th coefficient $\mathcal{F}_{\cos}(g_\alpha)(i)$ of the continuous cosine transform of the function

$$g_\alpha(t) = \frac{\sin(t)}{\sqrt{\frac{1}{\sin^2(\alpha\pi/2)} - \cos^2(t)}}\frac{1}{\alpha}, \qquad t \in [0, \pi].$$

We continue by analyzing the computation of the quadrature formula using the standard monomial basis. Let $C = \sin\left(\alpha\frac{\pi}{2}\right)$ and suppose that S_i is a sequence of numbers satisfying the recursion

$$S_i = -\frac{1}{i}\left[\sin(Cx)^{i-1}\frac{\cos(Cx)}{C}\right]_{-1}^{1} + \frac{(i-1)}{i}S_{i-2}$$

and in which the initial value S_0 is a the exact moment value $\int_{-1}^{1} 1\mathrm{d}x = 2$, then $\tau_i^\alpha = S_i / C^i$.

Th. 2 (Error divergence of the moments)

If the initial value S_0 is a slight perturbation of the exact moment value, then the error $\mathcal{E}_i$ between S_i and $\int_{-1}^{1}\sin(Cx)^i\mathrm{d}x$ satisfies the recurrence relation

$$\mathcal{E}_i = S_i - \int_{-1}^{1}\sin(Cx)^i\mathrm{d}x = \frac{(i-1)}{i}\mathcal{E}_{i-2}.$$

Moreover, $\mathcal{E}_{2k}/\sin(\alpha\frac{\pi}{2})^{2k} \to \infty$ for $k \to \infty$.

This implies that the calculation of the moments τ_i^α via the monomial basis is not stable.
The next theorem allows us to have a better understanding of the numerical results.

Th. 3 (Limit relations)

For the interpolatory KTI quadrature formula we have the limit relations

$$w_i^\alpha \xrightarrow{\alpha \longrightarrow 0^+} \int_{-1}^{1} \ell_i^0(y)\mathrm{d}y$$

and

$$w_i^\alpha \xrightarrow{\alpha \longrightarrow 1^-} \frac{2}{\pi}\int_{-1}^{1} \ell_i^1(y)\frac{1}{\sqrt{1-y^2}}\mathrm{d}y,$$

where ℓ_i^α denotes the i-th Lagrange polynomial defined by the nodes $M_\alpha(x_i)$, $i = 0, \dots, m$.

Numerical Test: KTI, KTL, Perturbation of the nodes

To test the scheme, we use analytic functions in an open neighborhood of $[-1, 1]$: $f_1(x) = \frac{1}{1+100x^2}$ (first row) and $f_2(x) = \frac{1}{1+16\sin^2(7x)}$ (second row). We sample them at equispaced nodes on $[-1, 1]$.

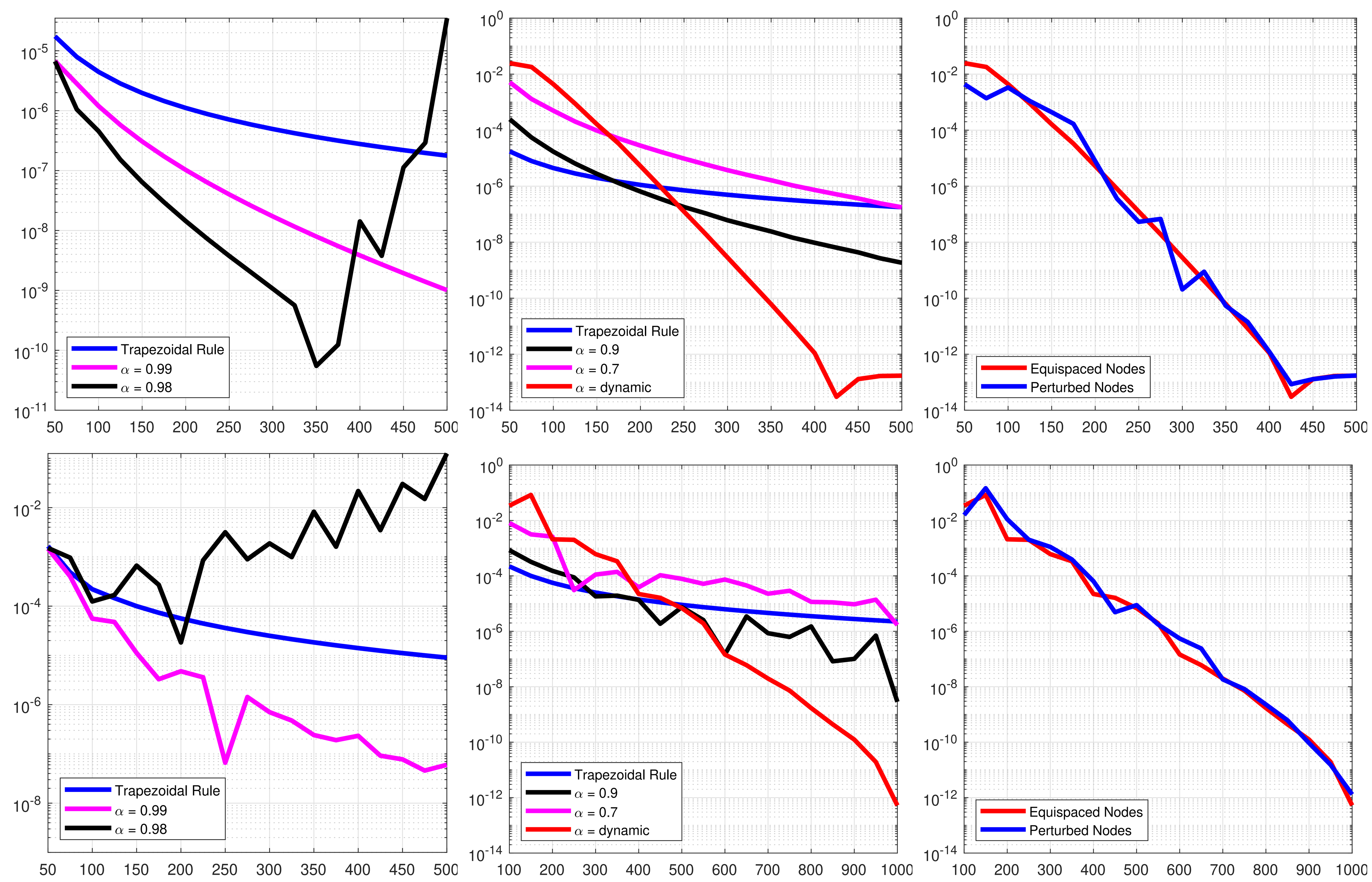

Fig 2. Relative Error for KTI quadrature formula (left), KTL quadrature formula (center), KTL with perturbed nodes (left). The x-axis describes the number of nodes used by the scheme.