A Nyström-type method based on anti-Gauss quadrature rules

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1. The problem

Let us approximate the solution f of the following second-kind Fredholm integral equation

$$f(y) - \int_{-1}^{1} k(x,y) f(x) w(x) dx = g(y), \quad y \in [-1,1],$$

where k and g are two given functions, and $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$ is the Jacobi weight with parameters $\alpha, \beta > -1$.

One of the most popular approach is the Nyström method based on the well-known n-point Gauss quadrature rule

$$I(f) = \int_{-1}^{1} f(x) w(x) dx = \sum_{k=1}^{n} \lambda_k f(x_k) + e_n(f) =: \mathcal{G}_n(f) + e_n(f).$$

Basically, the equations $(I - K_n) f_n = g$ are considered where f_n is the unknown and $(K_n f_n)(y) = \sum_{k=1}^{n} \lambda_k k(x_k, y) f_n(x_k)$. They are required to hold at the nodes $\{x_i\}_{i=1}^{n}$ and this yields a linear system whose unknowns are $a_k = f_n(x_k)$. Once computed, the Nyström interpolant

$$f_n(y) = \sum_{k=1}^{n} \lambda_k k(x_k, y) a_k + g(y)$$

provides an approximated solution for our equation.

2. Anti-Gauss quadrature formula

In 1996, Laurie [Math. Comp. 65] introduced an interpolatory $(n+1)$-point quadrature rule, named anti-Gauss rule,

$$I(f) = \int_{-1}^{1} f(x) w(x) dx = \sum_{k=1}^{n+1} \tilde{\lambda}_k f(\tilde{x}_k) + \tilde{e}_{n+1}(f) =: \tilde{\mathcal{G}}_{n+1}(f) + \tilde{e}_{n+1}(f).,$$

such that it has an error precisely opposite to the error $e_n(f)$ of a n-point Gauss rule that is

$$\tilde{e}_{n+1}(f) = -e_n(f), \qquad \text{for all} \qquad f \in \mathbb{P}_{2n+1}.$$

The coefficients $\{\tilde{\lambda}_k\}_{k=1}^{n+1}$ are all positive and the nodes $\{\tilde{x}_k\}_{k=1}^{n+1}$ are all real and interlace with the Gauss nodes $\{x_k\}_{k=1}^{n}$ that is $\tilde{x}_1 < x_1 < \tilde{x}_2 < \ldots < x_n < \tilde{x}_{n+1}$.
The formula can be easily constructed by solving a suitable eigenvalue problem for a modified Jacobi matrix.

Spaces of functions Let us introduce the Jacobi weight $u(x) = (1-x)^{\gamma}(1+x)^{\delta}$ with $\gamma, \delta \geq 0$ and define

$$L_u^{\infty} = \left\{ f \in C^0((-1,1)) : \lim_{x \to \pm 1} (fu)(x) = 0 \right\},$$

equipped with the weighted uniform norm

$$\|fu\|_{\infty} = \max_{x \in [-1,1]} |(fu)(x)|.$$

For smoother functions, we introduce the weighted Sobolev space

$$\mathcal{W}_r^{\infty}(u) = \left\{ f \in L_u^{\infty} : \|f\|_{\mathcal{W}_r^{\infty}(u)} = \|fu\|_{\infty} + \|f^{(r)} \varphi^r u\|_{\infty} < \infty \right\},$$

where $r = 1, 2, \ldots,$ and $\varphi(x) = \sqrt{1-x^2}$.

3. A Nyström-type method

From now on, let us assume that the given equation has a unique solution $f^* \in L_u^{\infty}$ for a given right-hand side $g \in L_u^{\infty}$ with $0 \leq \gamma < \alpha + 1$ and $0 \leq \delta < \beta + 1$.
In order to approximate it, let us consider the equations

$$\tilde{f}_{n+1}(y) - \sum_{j=1}^{n+1} \tilde{\lambda}_j k(\tilde{x}_j, y) \tilde{f}_{n+1}(\tilde{x}_j) = g(y), \qquad j = 1, \ldots, n+1$$

where $\tilde{f}_{n+1}$ are the unknowns. By evaluating the equation at the anti-Gauss nodes we get

$$\sum_{k=1}^{n+1} \left[\delta_{ik} - \tilde{\lambda}_k k(\tilde{x}_k, \tilde{x}_i) \right] \tilde{a}_k = g(\tilde{x}_i), \qquad i = 1, \ldots, n+1, \qquad (1)$$

where $\tilde{a}_k = \tilde{f}_{n+1}(\tilde{x}_k)$. Then, the Nyström interpolant is given by

$$\tilde{f}_{n+1}(y) = \sum_{k=1}^{n+1} \tilde{\lambda}_k k(\tilde{x}_k, y) \tilde{a}_k + g(y).$$

4. Analysis of the method

Theorem Let us assume that, for an integer r,

$$g \in \mathcal{W}_r^{\infty}(u), \qquad \sup_{|x| \leq 1} \|k(x,\cdot)\|_{\mathcal{W}_r^{\infty}(u)} < \infty, \qquad \sup_{|y| \leq 1} u(y) \|k(\cdot, y)\|_{\mathcal{W}_r^{\infty}} < \infty.$$

Then, for n sufficiently large, system (1) is uniquely solvable and well-conditioned in the ∞-norm. Finally, one has

$$\|[f^* - \tilde{f}_{n+1}]u\|_{\infty} = \mathcal{O}\left(n^{-r}\right),$$

where the constants in $\mathcal{O}$ are independent of n and f^*.

5. Averaged Nyström interpolant

Theorem Let the assumptions of the previous theorem be satisfied and let us assume that, for any $y \in [-1,1]$, the terms $\{\alpha_i(y)\}$ of the series

$$k(x,y) f^*(y) = \sum_{i=0}^{\infty} \alpha_i(y) \pi_i(x)$$

converge to zero sufficiently rapidly where $\{\pi_i\}$ are the Jacobi polynomials orthonormal with respect to the weight w. Moreover, let us hypothises that

$$\max \left\{ \left| \sum_{i=2n+2}^{\infty} \alpha_i(y) G_n(\pi_i) \right|, \left| \sum_{i=2n+2}^{\infty} \alpha_i(y) \widetilde{G}_{n+1}(\pi_i) \right| \right\} < \left| \sum_{i=2n}^{2n+1} \alpha_i(y) G_n(\pi_i) \right|,$$

for n large enough. Then, either

$$f_n(y) \leq f^*(y) \leq \tilde{f}_{n+1}(y), \qquad \text{or} \qquad \tilde{f}_{n+1}(y) \leq f^*(y) \leq f_n(y).$$

The previous theorem allows us to obtain a better approximation of the solution by the averaged Nyström interpolant

$$\mathfrak{f}_n(y) = \frac{1}{2}[f_n(y) + \tilde{f}_{n+1}(y)], \qquad y \in [-1,1].$$

6. Numerical Results

Example 1 Let us consider the following equation in L^{∞}

$$f(y) - \int_{-1}^{1} \frac{e^{(x+y)}}{1 + x^2 + 3y^2} f(x) \frac{dx}{\sqrt{1-x^2}} = \sqrt{|y|^9}.$$

Next graphs display our numerical results.

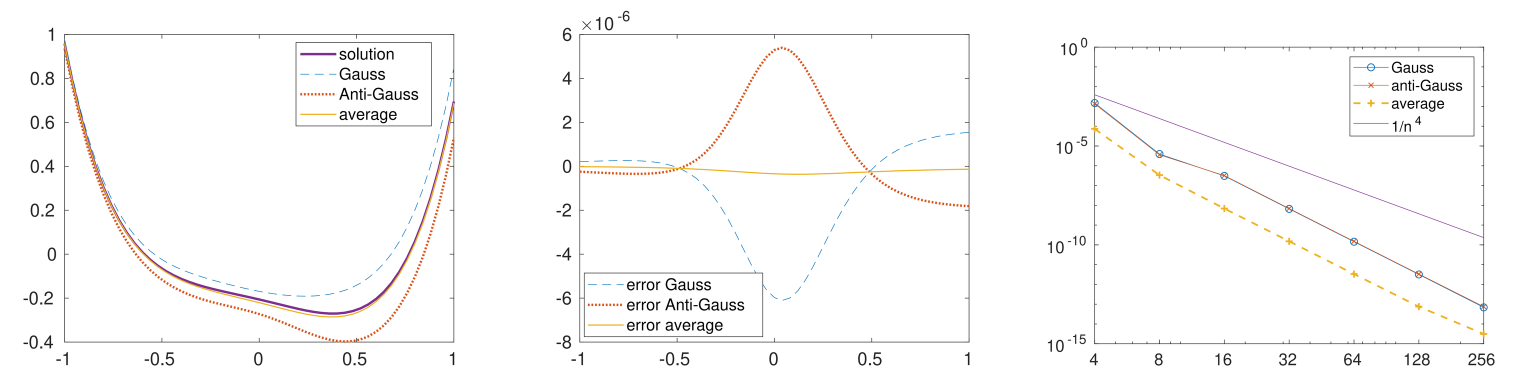

Fig. 1: Form left to right: comparison of the exact weighted solution with the approximations produced by the Gauss, anti-Gauss, and averaged rules, for $n = 2$; errors corresponding to the three quadrature formulae when $n = 8$; weighted ∞-norm errors

Example 2 We apply our approach to approximate the unique solution $f^* \in L_u^{\infty}$, $u(x) = \sqrt[4]{1-x^2}$ of the equation

$$f(y) - \int_{-1}^{1} (y+3) \sqrt{|\cos(1+x)|^5} f(x) \sqrt{1-x^2} dx = \ln(1+y^2).$$

Next table contains the numerical errors at the point $y = -0.3$

n	$(f_n - f_{512})u$	$(\tilde{f}_{n+1} - f_{512})u$	$(\mathfrak{f}_n - f_{512})u$
4	1.87e-03	-1.90e-03	-1.87e-05
8	1.66e-05	-1.38e-05	1.39e-06
16	-1.25e-06	1.43e-06	9.09e-08
32	-1.39e-07	1.35e-07	-2.08e-09
64	-1.12e-08	9.97e-09	-5.96e-10
128	1.03e-09	-1.20e-09	-8.23e-11
256	-2.48e-11	3.20e-11	3.58e-12

Perspectives of research In collaboration with L. Reichel and M.M. Spalević, we are exploring the application of other averaged Gauss quadrature formulae [M.M. Spalević, Math Comp 76, 2007; L. Reichel and M.M. Spalević Appl. Num. Math. 165, 2021] to integral equations.

Reference

[1] P. Díaz de Alba, L. Fermo and G. Rodriguez. Solution of second kind Fredholm integral equations by means of Gauss and anti-Gauss quadrature rules. Numerische Mathematik, 146: 699-728, 2020.