Generalizations of the constrained mock-Chebyshev least squares in two variables: tensor product vs total degree interpolation

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Acronymns and settings

c-L: Chebyshev Lobatto;
m-C: mock-Chebyshev;
Cm-CLS: constrained m-C least squares;
m-P: mock-Padua;
$X_n = \{x_i\}$, $x_i = -1 + 2i/n$, $i = 0, \dots, n$;
$m = \lfloor \pi \sqrt{n} / \sqrt{2} \rfloor$;
$X_m^{CL} = \{x_i^{CL}\}$ the C-L nodes of order $m+1$;
$X'_m = \{x'_i\}$ the m-C nodes of order $m+1$: $|x'_i - x_i^{CL}| = \min_{j=0,\dots,n} |x_j - x_i^{CL}|, \quad i = 0, \dots, m;$
Π_r polynomials of degree $\leq r$;
$\hat{\Pi}_r \subset \Pi_r$ polynomials interpolating f on X'_m;
$\mathcal{B}_r = \{u_j(x)\}_{j=0,\dots,r}$ a basis of Π_r;
$X_{n_x} \times Y_{n_y} = \{(x_i, y_j)\}$ a uniform rectangular grid on $[-1,1]^2$ of $(n_x+1)(n_y+1)$ nodes;
$\Pi_{r_x} \otimes \Pi_{r_y}$ the tensor product of the spaces Π_{r_x}, Π_{r_y} of polynomials in x, y, respectively;
$\hat{\Pi}_{(r_x,r_y)} \subset \Pi_{r_x} \otimes \Pi_{r_y}$ polynomials interpolating f on $X'_{m_x} \times Y'_{m_y}$;
Pad_m the set of Padua nodes of degree m;
$Pad'_m = \{(x'_i, y'_j)\}_{\substack{i=0,\dots,m \\ j=1,\dots,\lfloor m/2 \rfloor + 1 + \delta_k}}$, $\delta_k = 0$ if m is even or m is odd but i is even, $\delta_k = 1$ if m is odd and i is odd, the set of m-P nodes of degree m, obtained from the m-C subset X'_m and Y'_{m+1};
$\Pi_r(\mathbb{R}^2)$ bivariate polynomials of degree $\leq r$;
$\hat{\Pi}_r(\mathbb{R}^2) \subset \Pi_r(\mathbb{R}^2)$ polynomials interpolating f on Pad'_m;
$\tilde{\mathcal{B}}_r = \{u_i(x)u_j(y)\}_{\substack{i=0,\dots,r \\ j=0,\dots,r-i}}$ a basis of $\Pi_r(\mathbb{R}^2)$.

Constrained mock-Chebyshev least squares interpolation

Given an analytic function $f(x)$ in $[-1,1]$, $r \in \mathbb{N}$ s.t. $m < r \leq n$, the *Cm-CLS problem* is

$$\textit{find } \hat{P}_{X_n} \in \hat{\Pi}_r \textit{ such that } ||f - \hat{P}_{X_n}||_2^2 = \min_{P \in \hat{\Pi}_r} ||f - P||_2^2. \tag{1}$$

Let be $P_{X'_m}(t) = \sum_{i=0}^m \ell_i(t) f(x'_i) \in \Pi_m$ the interpolation polynomial on the mock-Chebyshev nodes and $\omega_m(t) = \prod_{i=0}^m (t - x'_i)$. In [1] it is proven that problem (1) has a unique solution

$$\hat{P}_{X_n}(t) = P_{X'_m}(t) + \hat{Q}_{X''_{n-m}}(t)\omega_m(t), \quad \text{where } ||\hat{f} - \hat{Q}_{X''_{n-m}}||^2_{2,\omega_m^2} = \min_{Q \in \Pi_{r-m-1}} ||\hat{f} - Q||^2_{2,\omega_m^2},$$

$$\text{with } ||u||_{2,\omega_m^2} = \left(\sum_{k=1}^{n-m} \omega_m^2(x''_k) u^2(x''_k) \right)^{\frac{1}{2}} \quad \text{and} \quad \hat{f}(t) = \frac{f(t) - P_{X'_m}(t)}{\omega_m(t)}.$$

Alternatively, we can use the *Lagrange multipliers method* [2] which requires the choice of a basis $\mathcal{B}_r$. Let be $V = [u_j(x_i)]_{i,j}$ the interpolation matrix at the nodes of X_n relative to $\mathcal{B}_r$, $b = [f(x_0), \dots, f(x_n)]^T$ and we assume both that the first $m+1$ points of X_n are those ones of X'_m and that $\mathcal{B}'_m = \{u_j(x) : j = 0, \dots, m\}$ spans Π_m. Let be $C = [c_i^T]_{i=1,\dots,m+1}$ the matrix formed by the first $m+1$ rows of V and $d = [d_1, \dots, d_{m+1}]^T$ the vector of the first $m+1$ components of b. Then

$$\hat{P}_{X_n}(x) = \sum_{i=0}^r \hat{a}_i u_i(x), \quad \text{where } C\hat{a} = d \quad \text{and} \quad ||V\hat{a} - b||_2^2 = \min_{a \in \mathbb{R}^{r+1}} ||Va - b||_2^2$$

solves problem (1). The coefficients $\hat{a}$ and the Lagrange multipliers $\hat{z}$, satisfying the optimality conditions for Lagrangian function of the problem, form the unique solution of the linear system

$$\begin{bmatrix} 2V^TV & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \hat{a} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} 2V^Tb \\ d \end{bmatrix} \qquad \text{(KKT matrix).}$$

The unisolvence of the interpolation problem on X_n by polynomials of Π_n assures the invertibility of the KKT matrix, since C has linear independent rows and V has linear independent columns.

Tensor product vs total degree interpolation

Tensor product interpolation. Given an analytic function $f(x,y)$ in $[-1,1]^2$, $r_x, r_y \in \mathbb{N}$ s.t. $m_x < r_x \leq n_x$, $m_y < r_y \leq n_y$, the Cm-CLS tensor product interpolation problem is

$$\textit{find } \hat{P}_{X_{n_x} \times Y_{n_y}} \in \hat{\Pi}_{(r_x,r_y)} \textit{ s.t. } ||f - \hat{P}_{X_{n_x} \times Y_{n_y}}||_2^2 = \min_{P \in \hat{\Pi}_{(r_x,r_y)}} ||f - P||_2^2. \tag{2}$$

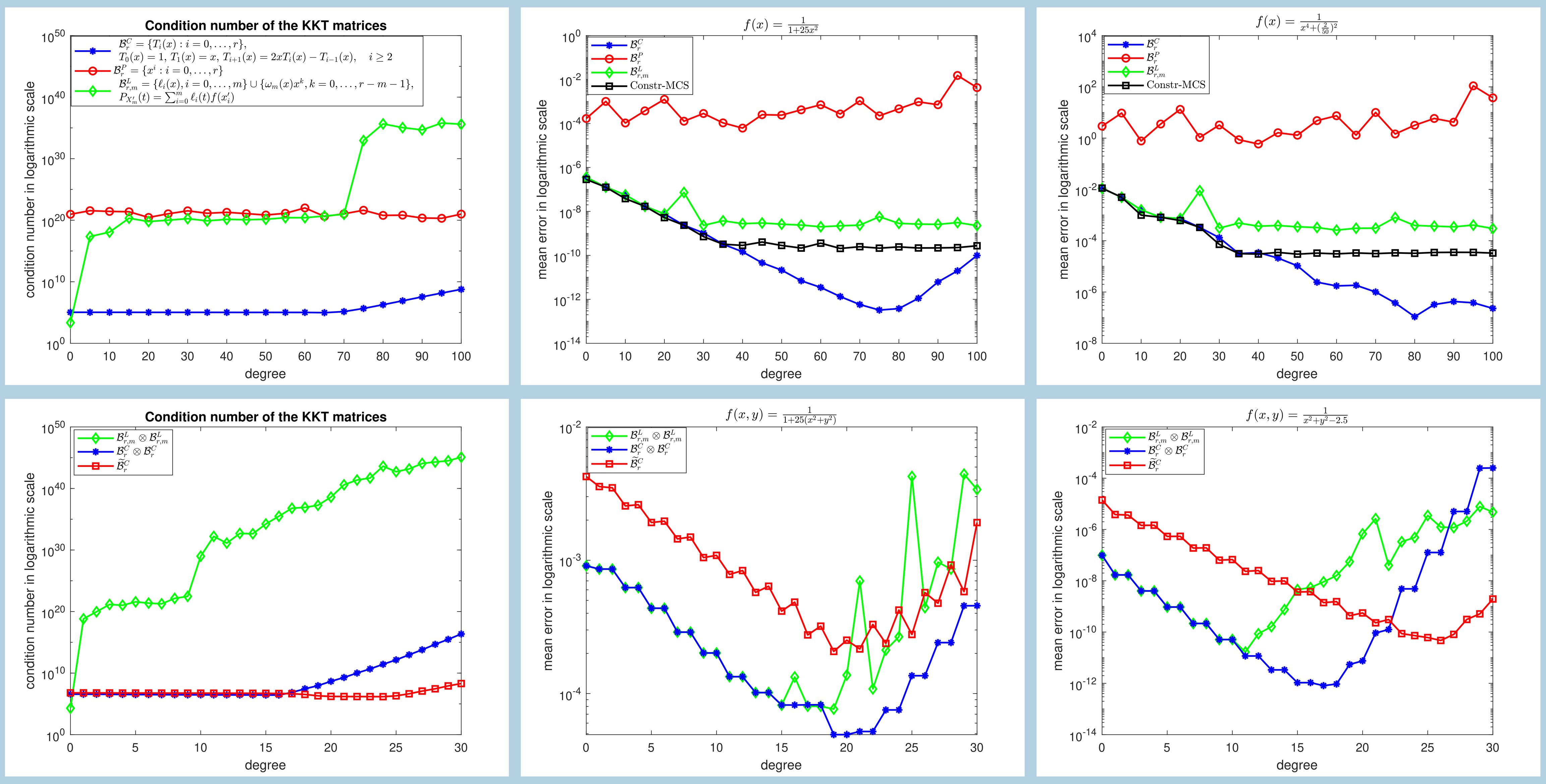
The method proposed in [1] to solve the univariate Cm-CLS problem is not applicable in this case. The Lagrange multipliers method instead can be used in analogy with the univariate case with the settings and requirements there specified. We assume both that the nodes of $X_{n_x} \times Y_{n_y}$ and the elements of $\mathcal{B}_{r_x} \otimes \mathcal{B}_{r_y}$ are reshaped into a sequence and that the first $(m_x+1)(m_y+1)$ nodes of $X_{n_x} \times Y_{n_y}$ are those ones of $X'_{m_x} \times Y'_{m_y}$ and that the first $(m_x+1)(m_y+1)$ elements of $\mathcal{B}_{r_x} \otimes \mathcal{B}_{r_y}$ spans $\Pi_{m_x} \otimes \Pi_{m_y}$. We introduce the matrices V and C in analogy with the univariate case. The unisolvence of tensor product interpolation problem on $X_{n_x} \times Y_{n_y}$ by polynomials of $\Pi_{n_x} \otimes \Pi_{n_y}$ assures the invertibility of the KKT matrix in this case, since C has linear independent rows and V has linear independent columns.

Total degree interpolation. The extension of the Cm-CLS problem in this case is based on the Padua points [3] and on the m-P points which mimic their behavior [5]. Given an analytic function $f(x,y)$ in $[-1,1]^2$, $r \in \mathbb{N}$ s.t. $m < r \leq n$, the constrained m-P least-squares problem is

$$\textit{find } \hat{P}_{X_n \times Y_n} \in \hat{\Pi}_r(\mathbb{R}^2) \textit{ such that } ||f - \hat{P}_{X_n \times Y_n}||_2^2 = \min_{P \in \hat{\Pi}_r(\mathbb{R}^2)} ||f - P||_2^2. \tag{3}$$

The problem (3) has a unique solution since $\tilde{\mathcal{B}}_m$ interpolates on Pad'_m [4] and $\tilde{\mathcal{B}}$ can be completed to the basis $\mathcal{B}_n \otimes \mathcal{B}_n$ interpolating on $X_n \times Y_n$.

Numerical results



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