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A new product integration rule for the finite Hilbert transform

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Topic: Approximation of the Hilbert transform of f

$$\mathcal{H}^u f(t) := \int_{-1}^{1} \frac{f(x)}{x-t} u(x)dx = \lim_{\epsilon\to 0} \int_{|x-t|\geq\epsilon} \frac{f(x)}{x-t} u(x)dx,$$

$u = v^{a,b}$, $a, b > -1$ Jacobi weight. The rule is based on the de la Vallée Poussin (VP) polynomial approximation of f. Comparison with the product rule based on the Lagrange interpolating polynomial.

The product rules

Given $w = v^{\alpha,\beta}, \alpha, \beta > -1$, let $\{x_k\}_{k=1}^n$ the zeros of $p_n(w)$. The VP rule

$$\mathcal{H}^u f(t) \approx \int_{-1}^{1} \frac{V_n^m(w,f,x)}{x-t} u(x)dx = \sum_{k=1}^{n} f(x_k) \int_{-1}^{1} \frac{\phi_{n,k}^m(x)}{x-t} u(x)dx$$

$$\Phi_{n,k}^m(x) = \lambda_{n,k}(w) \sum_{j=0}^{n+m-1} \mu_n^m(j) p_j(w,x_k) p_j(w,x),$$

$\{\lambda_{n,k}(w)\}$ Christoffel numbers and

$$\mu_n^m(j) := 1,\ j = 0: n-m, \quad \mu_n^m(j) = \frac{n+m-j}{2m},\ n-m+1 : n+m-1.$$

The main characteristic of $V_n^m(w,f)$ is the dependence on the additional degree–parameter $0 < m < n$. The error of the VP rule is

$$e_n^{VP}(f,t) = \mathcal{H}^u f(t) - \sum_{k=1}^{n} f(x_k) \int_{-1}^{1} \frac{\phi_{n,k}^m(x)}{x-t} u(x)dx.$$

Similarly, the L-rule on the same nodes is

$$\mathcal{H}^u f(t) \approx \int_{-1}^{1} \frac{L_n(w,f,x)}{x-t} u(x)dx = \sum_{k=1}^{n} f(x_k) \int_{-1}^{1} \frac{\ell_{n,k}(x)}{x-t} u(x)dx,$$

$$\ell_{n,k}(x) = \lambda_{n,k}(w) \sum_{j=0}^{n-1} p_j(w,x_k) p_j(w,x),$$

and the error of the L-rule

$$e_n^L(f,t) = \mathcal{H}^u f(t) - \sum_{k=1}^{n} f(x_k) \int_{-1}^{1} \frac{\ell_{n,k}(x)}{x-t} u(x)dx.$$

Error estimates: case $u = v^{\gamma,\delta},\ \gamma, \delta \neq 0$

Let $u = \frac{u^+}{u^-}$, $u_1 := u_+(x) := v^{a_+,b_+}(x)$, $u_2 := u_-(x) := v^{a_-,b_-}(x)$, $c_+ := \max\{0, c\}$, $c_- := \max\{0, -c\}$. For $r > 0$, $f \in B_r(u_1) \subset C_{u_1}$,

$$B_r(u_1) = \left\{ f \in C_{u_1} : \sum_{n=1}^{\infty} (n+1)^{r-1} E_n(f)_{u_1} < \infty \right\},$$

$$\|f\|_{B_r(u_1)} := \|f u_1\|_\infty + \sum_{n=1}^{\infty} (n+1)^{r-1} E_n(f)_{u_1},$$

for suitable choices of w, and for $m \sim n$,

$$|e_n^{VP}(f,t)| u_2(t) \leq \mathcal{C} \left(|[f(t) - V_n^m(w,f,t)] u_1(t)| + \frac{\log n}{n^r} \|f\|_{B_r(u_1)} \right),$$

$$|e_n^{L}(f,t)| u_2(t) \leq \mathcal{C} \left(|[f(t) - L_n(w,f,t)] u_1(t)| + \frac{\log^2 n}{n^r} \|f\|_{B_r(u_1)} \right),$$

Hence the convergence rate of both the rules depends on two components: the pointwise approximation that VP and Lagrange polynomials of f provide at the specific $t \in (-1,1)$, and the degree of smoothness of f.

Example 1

$$\mathcal{H}^u f(t) = \int_{-1}^{1} \frac{f(x)}{x-t} \sqrt{1-x^2} dx, \ (f \text{ in Fig.})$$

$u_1 = v^{\frac{1}{2},0}$, $u_2 = v^{0,\frac{1}{2}}$ and $w = v^{-\frac{1}{2},-\frac{1}{2}}$.

t = 0.5				t = 0.8			
n	m	e_n^{VP},	e_n^L	n	m	e_n^{VP}	e_n^L
30	5	**3.10e-02**	**3.10e-02**	20	2	**1.11e-02**	1.27e-02
60	23	**3.92e-03**	**3.92e-03**	50	20	**1.82e-04**	9.12e-03
80	7	4.86e-02	**4.31e-02**	70	13	**5.37e-05**	6.01e-03
150	105	**6.20e-06**	**6.20e-06**	100	60	**1.40e-04**	1.42e-03
200	13	5.06e-04	**4.19e-04**	150	11	**3.41e-06**	3.82e-05
250	16	**5.54e-05**	6.67e-05	250	15	**8.01e-09**	3.11e-07
300	150	**2.11e-10**	**2.11e-10**	300	28	**6.11e-10**	1.94e-07

Indeed, for suitable choices of m, $V_n^m(w,f)$ strongly reduces the Gibbs phenomenon affecting $L_n(w,f)$ and provides a better pointwise approximation, as shown in Fig.1 (right)

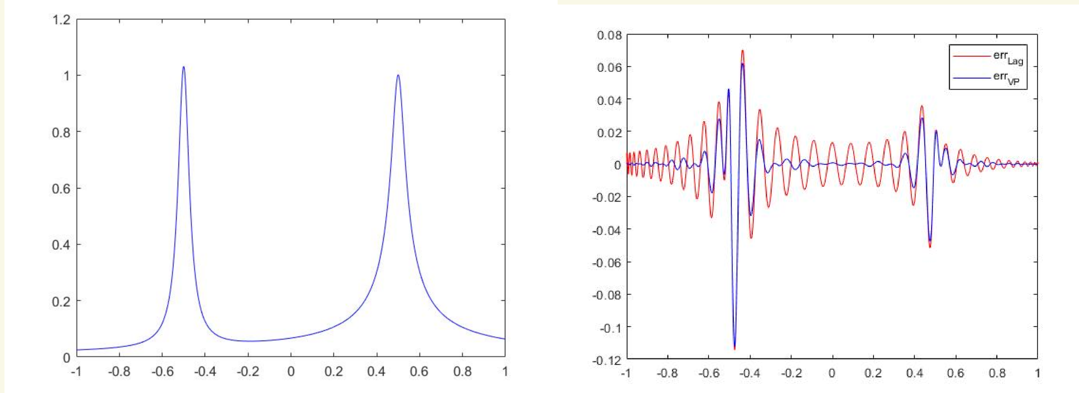
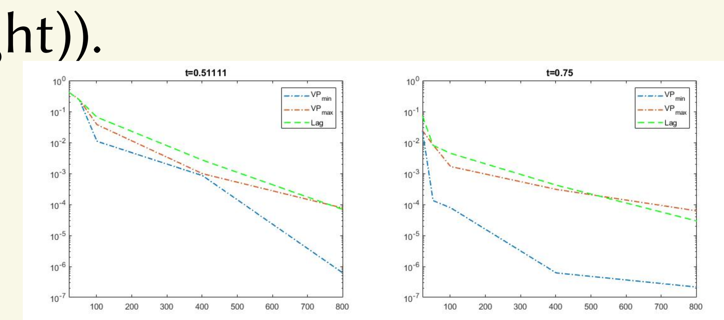


Figure : 1: f (left), the errors $f(x) - L_n(w,f,x)$ and $f(x) - V_n^m(w,f,x)$, $n = 70$, $m = 13$ (right)

We show for increasing n the absolute errors (in log-scale) of VP-rules, for the "optimal" m (inducing the smallest error), for the "worst" m (inducing the largest error), for the L-rule ($t = 0.51111$ (left), $t = 0.75$ (right)).



The results by the VP-rule are globally better than those achieved by the L-rule, when the function f presents a quick variation in a localized range. As matter of fact, the parameter m can be used to reduce the quadrature error.

Example 2

$$\mathcal{H}^u(f,t) = \int_{-1}^{1} \frac{|x|^{\frac{1}{7}}}{x-t} \sqrt[4]{\frac{1-x}{1+x}} dx.$$

$$u = v^{\frac{1}{4},-\frac{1}{4}},\ w = u,\ u_1 = v^{\frac{1}{4},0}.$$

In this case, the function is very smooth, except a small interval around 0. In Fig.2 we show for a fixed t, the graphics of the errors in log-scale, for increasing values of n and for $m = \lfloor n\theta \rfloor$, $\theta \in \{0.3,\ 0.6,\ 0.9\}$. As we can see, all the sequences of the VP rule converge faster than the L-rule.

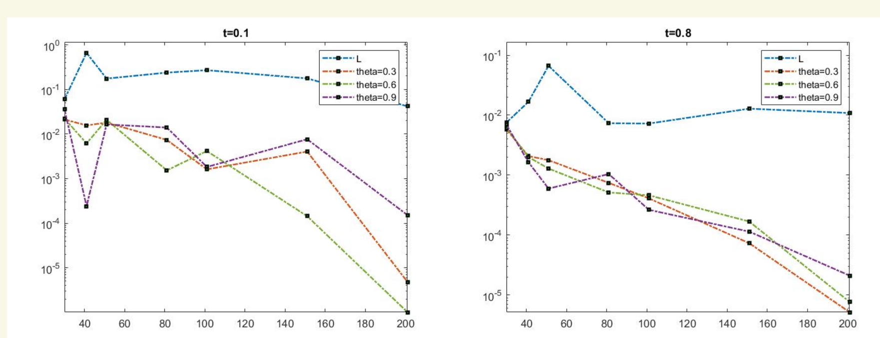


Figure : 2: Absolute errors for $t = 0.1$ (left), $t = 0.8$ (right)

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