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Reconstruction of volatility surfaces: a computational study

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Abstract

- Usually, points of options' volatility surfaces are implicitly obtained by Black-Scholes' formula or by stochastic models like the Heston model. Then, given volatility's points, reconstruction of the surface is needed.
- From the point of view of interpolation is interesting to work on a particular data set as the one used. This set is characterized by not scattered or specifically distributed data, as usual, rather they are arranged on lines.
- The computational study is based on radial basis function methods. Initially, reconstruction has been made globally, then the surface obtained has been tested by removing points and evaluating RMS error.
- Local methods as RBF-partition of unity method have been used with variable size of subdomains and shape parameters. To improve the interpolant's accuracy we added points obtained by the least square method.
- One of the aims of the economic world is to forecast options' volatility, so the study is continuing to test these methods to extrapolate volatility surfaces.

1 RBF-Global Interpolant Problem

Let $X_N = \{\mathbf{x}_i \in \Omega \subset \mathbb{R}^d, i = 1, .., N\}$, $d \geq 2$ and $\mathcal{F}_N = \{f_i \in \mathbb{R}, i = 1, .., N\}$. The interpolation problem consists in recovering a continuous function $F : \Omega \longrightarrow \mathbb{R}$, which satisfies the interpolation condition $F(x_i) = f_i$. F is described by radial basis functions $\varphi_\varepsilon(|| \cdot - \mathbf{x_j}||_2)$, i.e.:

$$F(x_i) = \sum_{j=1}^{N} c_j \varphi_\varepsilon(||\mathbf{x_i} - \mathbf{x_j}||_2) = f_i \, .$$

In matricial form system becomes:

$$A\mathbf{c} = \mathbf{f} \, ,$$

where $A_{ij} = \varphi_\varepsilon(||\mathbf{x_i} - \mathbf{x_j}||_2)$, $i, j = 1, .., N$, $\mathbf{c}$ and $\mathbf{f}$ N-dimentional vectors.

2 Partition of unity method

The basic idea of the PUM is to consider a partition of the open and bounded domain $\Omega \subseteq \mathbb{R}^2$ into d subdomains Ω_j such that $\Omega \subseteq \bigcup_{j=1}^{d} \Omega_j$ with some mild overlap among the subdomains.

I) We choose a partition of unity, i.e. a family of compactly supported, non-negative, continuous functions W_j with $\text{supp}(W_j) \subseteq \Omega_j$ such that $\sum_{j=1}^{d} W_j(x, y) = 1$.

II) For each subdomain Ω_j we consider R_j as local approximant obtained with linear combination of radial basis functions.

Global approximant is gained as:

$$\mathcal{I}(x, y) = \sum_{j=1}^{d} R_j(x, y) W_j(x, y), \qquad (x, y) \in \Omega. \qquad (1)$$

Note that if the local approximants satisfy the interpolation conditions at node (x_i, y_i), i.e. $R_j(x_i, y_i) = f_i$, then also the global one interpolates points, i.e. $\mathcal{I}(x_i, y_i) = f_i$, for $i = 1, 2, \ldots, n$.

3 Improvement of interpolation's accuracy by least square method

One of the aims of the study is to understand how to improve interpolant's accuracy, especially for local interpolation obtained by PUM method.

$\Downarrow$

To achieve that result we have used least square method.

Let $f_{\alpha_1, \alpha_2, \cdots, \alpha_k}$, function described by parameters $\alpha_1, \alpha_2, \cdots, \alpha_k$ that better describe points' set. The least square problem is finding these parameters such that the sum S of square of distances is minimized:

$$S = \sum_{i=0}^{n} (\bar{y}_i - f_{\alpha_1, \alpha_2, \cdots, \alpha_k})^2$$

The minimum problem can be written as:

$$\frac{\partial S}{\partial \alpha_j} = 0 \text{ for } j = 1, \ldots, k.$$

Finally found $f_{\alpha_1, \alpha_2, \cdots, \alpha_k}$, we use it to obtain new points that allow us to improve interpolant's accuracy.

4 Implied Volatility

The fundamentals of economic problem are:

- One way to price options is basing on Black-Scholes model described by the following PDE and its boundaries conditions:

$$\frac{\partial V}{\partial t} = rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV \, ,$$

where V is the price of the option, S the value of underlying, r risk free rate and σ volatility, commonly assumed to be constant.

- We can observe volatility, $\sigma(t, S)$, varying with time t and value of the strike price S. The target, removing that assumption, is to forecast future volatility and its surface to carry out financial reasons as hedging strategies for portfolios.

- Numerically derived value of volatility σ from Black-Scholes' PDE is named *implied volatility*. The aim of the study is perform an extensive computational analysis, such as to identify the appropriate method to accurately reconstruct implied volatility surface using radial basis functions.

5 Numerical Test

We test various methods to interpolate a set of $n = 144$ data get from Bloomberg of maturity, strike price and implied volatility. The following figure is obtained with global interpolant method based on radial basis functions. Best surfaces are obtained using thin plate spline adding a first grade polynomial term (conditionally definite positive function) and with Multiquadrics RBF.

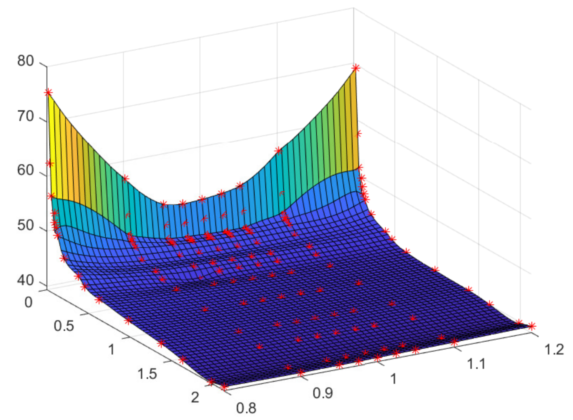

Figure 1. *Plots of volatility surface obtained with TPS.*

Multiquadric's surface is very close to TPS' one, most differences emerge on the boundary of the domain.

Global interpolants are tested removing points and subsequently compare the new surface with the oldest one. The differences between the two interpolants is reported in the following figures.

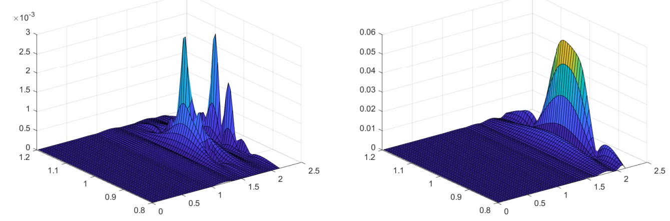

Figure 2. *Error surface obtained removing 4 points (left) and error surface obtained removing 5 points (right).*

The associated root mean square errors, relative to the removed points, are: $rms_{err} = 0.0025$ for the first interpolant and $rms_{err} = 0.0484$ for the second one. These errors can be considered acceptable since most capitalized assets are traded at the second decimal digit.

6 Partition of unity numerical tests

A local approach is taken using the partition of unity method. This method is performed using a variable size of subdomains and shape parameters to find a better fit. First results are shown on the following figures.

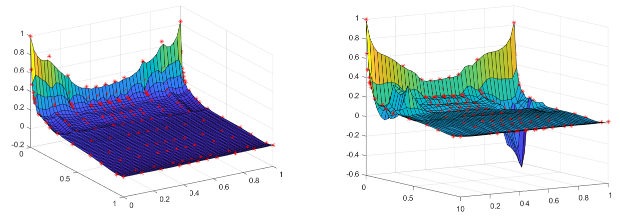

Figure 3. *Plots of volatility surface obtained with Multiquadric (left) and TPS (right).*

Most of the problems emerge on the steepest zone of the boundary also characterized by a low density of data.
To prevent a low level of accuracy as the one shown previously, we add points to the low-density data area. These new points are obtained with the least square method: approximation has been made only on the time axis and has been chosen only four points every row. Accuracy improvements are shown in the following figure.

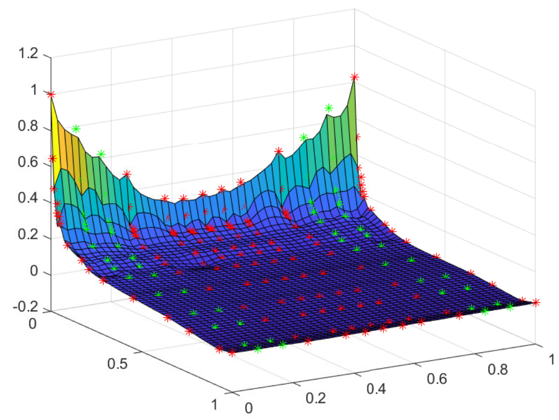

Figure 4. *Plots of volatility surface obtained with Multiquadric, green points are obtained with least square method.*

7 Work in progress

Following studies are focused on forecasting futures values of options' volatility. To do so we try to extrapolate the obtained surfaces. First attempt is shown in the following image.

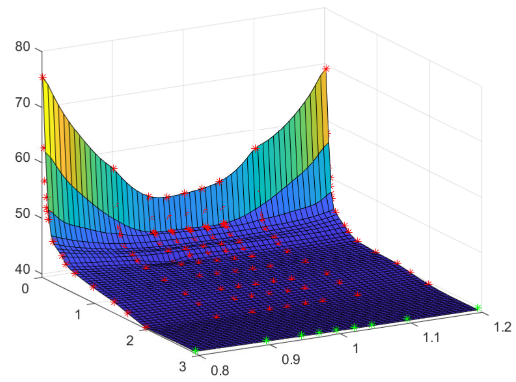

Figure 5. *Extrapolation obtained with Multiquadric.*

We compare the surface obtained using Multiquadric with Bloomberg's surface (the green points in the figure), so the maximum distance between our extrapolation and Bloomberg's data is of the order of 0.8%.

References

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