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Numerical methods to evaluate hypersingular integrals of highly oscillatory functions on the positive semiaxis

Maria Carmela De Bonis and Valeria Sagaria, University of Basilicata

Introduction

We propose two methods to approximate

$$H_p^\omega(f,t) = \int_0^{+\infty} \frac{f(x)e^{i\omega x}}{(x-t)^{p+1}} u(x)\,dx, \qquad (1)$$

where $t > 0$, $p \in \mathbb{N}$, $i^2 = -1$, $\omega \gg 1$ and $u(x) = x^\gamma e^{-\frac{x}{2}}, \gamma \geq 0$, is a generalized Laguerre weight. The integral is understood in the Cauchy principal value sense if $p = 0$ and in the finite part Hadamard sense if $p > 0$.

Integrals (1) frequently appear in boundary element methods (BEMs) and the efficiency of BEMs often depends upon the efficiency of the numerical evaluation of such hypersingular integrals.

Basic definition

Sobolev-type space function

Let $r \in \mathbb{N}, r \geq 1$ and $\varphi(x) = \sqrt{x}$

$$W_r(u) = \left\{ f \in C_u : f^{(r-1)} \in AC(\mathbb{R}^+), \|f^{(r)}\varphi^r u\| < +\infty \right\}$$

equipped with $\|f\|_{W_r(u)} := \|fu\| + \|f^{(r)}\varphi^r u\|$.

Truncated Lagrange polynomial [1]

Let $w_\alpha(x) = x^\alpha e^{-x}$ the Laguerre weight, $\alpha > -1$, and p_m the corresponding $m-$th orthonormal polynomial. The truncated Lagrange polynomial is

$$L_{m+1}(w_\alpha, f) := \sum_{k=1}^{j} f(z_k)\ell_{m+1,k}(x),$$

where z_k, $k = 1, \dots, j$, are the zeros of p_m, $j = \min_{k=1,\dots,m}\{k : z_k \geq 4\theta m\}$, $0 < \theta < 1$ is fixed and

$$\ell_{m+1,k}(x) = \frac{(4m-x)p_m(x)}{p'_m(z_k)(x-z_k)(4m-z_k)}.$$

Dilation rule

We recall the dilation rule introduced in [2]

$$I_b^\omega(f) = \int_0^b f(x)e^{i\omega x}x^\gamma\,dx = \frac{1}{\omega^{\gamma+1}}\int_0^{\omega b} f\left(\frac{x}{\omega}\right)e^{ix}x^\gamma\,dx$$

$$= \frac{1}{\omega^{\gamma+1}}\left(\int_0^d + \sum_{k=1}^{S-1}\int_{kd}^{(k+1)d} + \int_{Sd}^{\omega b}\right) f\left(\frac{x}{\omega}\right)e^{ix}x^\gamma\,dx,$$

with $S = \lfloor \frac{\omega b}{d} \rfloor$. Transforming the intervals $A_k = [kd, (k+1)d]$, $k = 0, \dots, S-1$, and $A_S = [Sd, \omega b]$ into $[-1,1]$, we approximate the first integral by a Gauss-Jacobi quadrature rule w.r.t. the Jacobi weight $v^{0,\gamma}$ and the remaining integrals by a Gauss-Legendre quadrature rule. Then, we obtain

$$I_b^\omega(f) = \Psi_m^\omega(f) + r_m(f).$$

Product rule

We replace f in (1) by $L_{m+1}(w_\alpha, f)$ obtaining

$$H_p^\omega(f,t) = \sum_{k=1}^{j} \frac{f(z_k)\lambda_{m,k}}{4m - z_k}\sum_{i=0}^{m-1} p_i(z_k)A_i(t) + e_{p,m}(f,t)$$

$$=: H_{p,m}^\omega(f,t) + e_{p,m}(f,t),$$

where, using the recurrence relation for Laguerre polynomials,

$$A_i(t) = \int_0^{+\infty} \frac{(4m-x)p_i(x)e^{i\omega x}}{(x-t)^{p+1}}u(x)\,dx$$

$$=: (4m-t)M_i^{(p)}(t) - M_i^{(p-1)}(t),$$

with $M_i^{(p)}(t) = H_p^\omega(p_i, t)$, $i = 0, \dots, m-1$, and

$$\begin{cases} M_1^{(0)}(t) = \dfrac{d_0 + (t-b_0)M_0^{(0)}(t)}{a_1}, \\ M_{i+1}^{(0)}(t) = \dfrac{d_i + (t-b_i)M_i^{(0)} - a_i M_{i-1}^{(0)}}{a_{i+1}}, \\ M_1^{(p)} = \dfrac{M_0^{(p-1)}(t) + (t-b_0)M_0^{(p)}(t)}{a_1}, \\ M_{i+1}^{(p)}(t) = \dfrac{M_i^{(p-1)}(t) + (t-b_i)M_i^{(p)}(t) - a_i M_{i-1}^{(p)}(t)}{a_{i+1}}, \\ d_i = \displaystyle\int_0^{+\infty} p_i(x)e^{i\omega x - \frac{x}{2}}\,dx \end{cases}$$

The starting moments are

$$M_0^{(0)}(t) = p_0 \begin{cases} -e^{bt}E_i(-bt), & \gamma = 0, \\ e^{bt}t^\gamma\left[i\pi + \gamma e^{i\gamma\pi}\Gamma(\gamma)\Gamma(-\gamma, bt)\right], & \gamma \neq 0, \end{cases}$$

$$M_0^{(p)}(t) = \frac{1}{p}\frac{d}{dt}M_0^{(p-1)}(t) = \frac{1}{p!}\frac{d^p}{dt^p}M_0^{(0)}(t),$$

with $p_0 = \frac{1}{\sqrt{\Gamma(\alpha+1)}}$ and $b = i\omega - \frac{1}{2}$.

Stability and convergence

Theorem 1 *Let $p \geq 0$ and $r \in \mathbb{N}$, $r \geq 1$. For any $f \in W_{p+r}(u)$ with γ, α satisfying the following condition*

$$\max\left\{-1, 2\gamma - \frac{5}{2}\right\} < \alpha \leq 2\gamma - \frac{1}{2}$$

and for any $t > 0$ we get

$$t^p|H_{p,m}^\omega(f,t)\| \leq \mathcal{C}\left(\|f\|_{W_{p+r}(u)} + \|f\|_{W_p(u)}\right)\log m,$$

$$t^p|e_{p,m}(f,t)| = \mathcal{O}\left(m^{-\frac{r}{2}}\log m\right),$$

where $\mathcal{C} \neq \mathcal{C}(m,f)$.

Dilation type rule

We suppose that $\exists M > 0$ s.t.

$$\left|\frac{f(x)u(x)}{(x-t)^{p+1}}\right| < \varepsilon \quad \forall x > M. \qquad (2)$$

• If $t < M$, with $T_p(f,x,t) = \sum_{k=0}^{p}\frac{f^{(k)}(t)}{k!}(x-t)^k$, we write

$$H_p^\omega(f,t) = \int_0^M \frac{f(x) - T_p(f,x,t)}{(x-t)^{p+1}}e^{i\omega x}u(x)dx$$

$$+ \int_M^{+\infty}\frac{f(x) - T_p(f,x,t)}{(x-t)^{p+1}}e^{i\omega x}u(x)dx$$

$$+ \sum_{k=0}^{p}\frac{f^{(k)}(t)}{k!}\int_0^{+\infty}\frac{e^{i\omega x}u(x)}{(x-t)^{p+1}}dx$$

$$=: H_{p,M}^\omega(f - T_p(f),t) + H_{p,\infty}^\omega(f - T_p(f),t) + B(t),$$

where $B(t) = \sum_{k=0}^{p}\frac{f^{(k)}(t)}{k!}p_0 M_0^{(p-k)}(t)$.

Under (2) we neglect $H_{p,\infty}^\omega(f - T_p(f))$ and we approximate only $H_{p,M}^\omega(f - T_p(f))$ as follows

$$H_p^\omega(f - T_p(f), t) = \Psi_m^\omega(F) + r_m(F),$$

with $F(x,t) := \frac{f(x) - T_p(f,x,t)}{(x-t)^{p+1}}e^{-\frac{x}{2}}$.

Let $\bar{k} \in \{0, \dots, S\}$ s.t. $t \in A_{\bar{k}}$. To avoid the numerical cancellation occurring when t is close to one of the quadrature knots, we use a control algorithm (see [3]).

• If $t > M$, writing

$$H_p^\omega(f,t) = H_{p,M}^\omega(f,t) + H_{p,\infty}^\omega(f,t)$$

and recalling (2), we approximate directly

$$H_{p,M}^\omega(f,t) = \Psi_m^\omega(\tilde{F}) + r_m(\tilde{F}),$$

with $\tilde{F}(x,t) := \frac{f(x)}{(x-t)^{p+1}}e^{-\frac{x}{2}}$.

Stability and convergence

Theorem 2 *Let $f \in C^{p+1+r}((0,+\infty))$, $p \geq 0$, $r \in \mathbb{N}$, $r \geq 1$. Then*

$$|\Psi_m^\omega(F)| \leq \mathcal{C}\|F\|, \qquad |\Psi_m^\omega(\tilde{F})| \leq \mathcal{C}\|\tilde{F}\|,$$

$$|r_m(F)| = \mathcal{O}\left(m^{-r}\right), \qquad |r_m(\tilde{F})| = \mathcal{O}\left(m^{-r}\right),$$

where $\mathcal{C} \neq \mathcal{C}(m,f)$.

Examples

Example 1

$$H_0^\omega(f,t) = \int_0^{+\infty}\frac{e^{-x}}{x-t}e^{i\omega x}\,dx,$$

$f(x) = e^{-\frac{x}{2}}, u(x) = e^{-\frac{x}{2}}, p = 0, t = 0.02.$

	$H_{0,m}^\omega(f,t)$		
m	$\omega = 10$	$\omega = 320$	$\omega = 5000$
8	$4.45\,e^{-6}$	$6.80\,e^{-5}$	$7.29\,e^{-5}$
16	$7.02\,e^{-9}$	$4.42\,e^{-9}$	$4.97\,e^{-9}$
32	$2.32\,e^{-16}$	$1.70\,e^{-17}$	$9.77\,e^{-18}$

	$\Psi_m^\omega(F)$		
m	$\omega = 10$	$\omega = 320$	$\omega = 5000$
4	$8.42\,e^{-9}$	$2.61\,e^{-10}$	$1.66\,e^{-11}$
8	$1.59\,e^{-17}$	$4.73e^{-19}$	$3.02\,e^{-20}$

Example 2

$$H_1^\omega(f,t) = \int_0^{+\infty}\frac{\cos(x-3)}{(x-t)^2}e^{i\omega x}x^{\frac{3}{5}}e^{-\frac{x}{2}}\,dx,$$

$f(x) = \cos(x-3),\ u(x) = x^{\frac{3}{5}}e^{-\frac{x}{2}},\ p = 1,\ t = 3.01.$

	$H_{1,m}^\omega(f,t)$		
m	$\omega = 20$	$\omega = 200$	$\omega = 2000$
32	$2.46\,e^{-4}$	$2.43\,e^{-2}$	$2.43\,e^{-2}$
64	$1.56\,e^{-9}$	$3.01\,e^{-8}$	$3.01\,e^{-7}$
128	$2.85\,e^{-17}$	$1.20\,e^{-16}$	$7.75\,e^{-16}$

	$\Psi_m^\omega(F)$		
m	$\omega = 10$	$\omega = 320$	$\omega = 5000$
4	$1.14\,e^{-9}$	$2.83\,e^{-11}$	$7.11\,e^{-13}$
8	$1.83\,e^{-17}$	$4.62\,e^{-19}$	$1.16\,e^{-20}$

References

[1] Mastroianni, G., Occorsio, D., Numerical Approximation of weakly singular integrals on the half-line. J. Comp. Appl. Math., 140 (2002), 1-2, 587-598.

[2] De Bonis, M.C., Pastore, P., A quadrature rule for integrals of highly oscillatory functions. Rendiconti del circolo matematico di Palermo, Serie II, Suppl. 82 (2010), 1-25.

[3] Criscuolo, G., Mastroianni, G., Convergenza di formule Gaussiane per il calcolo delle derivate di integrali a valor principale secondo Cauchy, Calcolo 24 (2) (1987), 179-192

mariacarmela.debonis@unibas.it, valeria.sagaria@unibas.it