Numerical Quadrature and Hyperinterpolation over Spherical Triangles/Polygons by the dCATCH Package.

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The main purpose of the talk is to show a novel strategy for determining by Caratheodory-Tchakaloff compression implemented in the d-CATCH package

- the computation of nodes and weights of a low-cardinality positive quadrature formula, nearly exact for polynomials of a given degree,
- hyperinterpolation of mild degree,

over spherical polygons.

The Matlab software used in this talk is available at the authors' homepage.

A great circle is the intersection of the unit-sphere  $\mathbb{S}^2$  with a plane passing through the origin.

Let  $P_1, \ldots, P_L$  be distinct points of  $\mathbb{S}^2$  and set  $P_0 = P_L$ .

A spherical polygon is the region  $\mathcal{P}\subset\mathbb{S}^2$  whose boundary  $\delta\mathcal{P}$ 

- is determined by the geodesic arcs {γ<sub>k</sub>}<sub>k=0,...,L</sub>, where each γ<sub>k</sub> is the portion of the great circles joining P<sub>k</sub> with P<sub>k+1</sub>.
- is oriented counterclockwise.
- In this talk we suppose that  ${\mathcal P}$ 
  - is contained in a cap Ω whose *polar angle* is strictly inferior than π;
  - $\blacksquare$   $\mathcal{P}$  is simple, i.e. it has no self intersections.

## Spherical polygon triangulation

Under these assumptions,

- if C is the center of such a cap Ω containing P then determine the stereographic projection P' of P on the plane π<sub>C</sub> tangent in C to the unit-sphere;
- compute a triangulation over the planar polygon  $\mathcal{P}'$  (e.g. by Matlab built-in environment polyshape), i.e.  $\mathcal{P}' = \bigcup_{i=1}^{m} T'_i$ where  $T'_i \subset \pi_C$  are planar triangles whose interior do not overlap, i.e. if  $j \neq k$  then  $\operatorname{int}(T'_i) \cap \operatorname{int}(T'_k) = \emptyset$ ;
- map back to the sphere, by means of the inverse of the stereographic projection, each planar triangle  $T'_k$  into a spherical triangle  $T_k$ .

The spherical triangles  $\{T_k\}_{k=1,...,M}$  determine a spherical triangulation of  $\mathcal{P}$ .

The purpose of this section is to determine a cubature rule over the spherical polygon  $\mathcal{P}$ , that has internal nodes  $\{Q_k\}_{k=1,...,N}$  and positive weights  $\{w_k\}_{k=1,...,N}$  so that if f is a continuous function then

$$\int_{\mathcal{P}} f(x, y, z) d\sigma \approx \sum_{k=1}^{N} w_k f(x_k, y_k, z_k).$$

Since  $\{T_k\}_{k=1,...,M}$  is a spherical triangulation of  $\mathcal{P}$ , if we determine such a cubature rule over each spherical triangle  $T_k$  then by the addivity of integration we have such a rule on  $\mathcal{P}$  either.

## Cubature on spherical polygons: spherical triangles



Figura: What we will get: quadrature nodes (and weights) on a spherical triangle lifted from the projected elliptical triangle.

With no loss of generality, up to a suitable rotation, we concentrate on spherical triangles  $\mathcal{T} = ABC$  with centroid  $(A + B + C)/||A + B + C||_2$  at the north pole, that are completely contained in the northern-hemisphere, and do not touch the equator.

Then, if 
$$f \in C(\mathcal{T})$$
,  $g(x, y) = \sqrt{1 - x^2 - y^2}$ ,  
$$I_{\mathcal{T}} := \int_{\mathcal{T}} f(x, y, z) d\sigma = \int_{\mathcal{T}^{\perp}} f(x, y, g(x, y)) \frac{1}{g(x, y)} dx dy,$$

where  $\mathcal{T}^{\perp}$  is the projection of  $\mathcal{T}$  onto the xy-plane, that is the curvilinear triangle whose vertices, say  $\hat{A}$ ,  $\hat{B}$ ,  $\hat{C}$ , are the xy-coordinates of A, B, C, respectively.

The sides of  $\mathcal{T}^{\perp}$  are arcs of ellipses centered at the origin, being the projections of great circle arcs. Then we can split the planar integral into the sum of the integrals on three elliptical sectors  $S_i$  with i = 1, 2, 3, obtained by joining the origin with the vertices  $\hat{A}$ ,  $\hat{B}$ ,  $\hat{C}$ , namely

$$I_{\mathcal{T}} = \int_{\mathcal{T}^{\perp}} f(x, y, g(x, y)) \frac{1}{g(x, y)} dx dy = \sum_{i=1}^{3} \int_{S_{i}} f(x, y, g(x, y)) \frac{1}{g(x, y)} dx dy$$

If the purpose is to compute an algebraic rule over T, with degree of precision n, positive weights and internal nodes (i.e. rules of PI-type) we face the problem that while f is a polynomial of total degree n then being  $g(x,y) = \sqrt{1 - x^2 - y^2}$ , we have that in general  $f(x, y, g(x, y)) \frac{1}{g(x, y)}$  may not be a polynomial.

#### Quadrature on spherical polygons: spherical triangles

To see this properly let

$$f(x, y, z) = x^{\alpha} y^{\beta} z^{\gamma}, \quad 0 \le \alpha + \beta + \gamma \le n, \ \alpha, \beta, \gamma \in \mathbb{N};$$
$$g(x, y) = \sqrt{1 - x^2 - y^2}.$$

Thus

$$f(x, y, g(x, y))\frac{1}{g(x, y)} = x^{\alpha}y^{\beta}(1 - x^2 - y^2)^{(\gamma - 1)/2}$$

Thus if  $\gamma$  is

- odd then f(x, y, g(x, y))<sup>1</sup>/<sub>g(x,y)</sub> is a polynomial of degree at most n,
- even then  $f(x, y, g(x, y)) \frac{1}{g(x, y)}$  is 1/g multiplied for a polynomial of degree at most n.

#### Quadrature on spherical polygons: spherical triangles

Key points:

- Approximate 1/g by a polynomial  $p_{\epsilon}$  of degree  $m = m_{\epsilon}$  such that  $|p_{\epsilon} 1/g| \le \epsilon \cdot 1/|g|$ .
- Since  $f/g \approx f \cdot p_{\epsilon} \in \mathbb{P}_{n+m}$ , we integrate  $f \cdot p_{\epsilon}$  instead of f/g on the elliptical sectors  $S_i$ , i = 1, 2, 3.

By determining a rule of algebraic degree of precision n + m over each elliptical sector  $S_i$ , i = 1, 2, 3, with internal nodes, and positive weights then we have a rule on  $\mathcal{T}^{\perp} := \bigcup_{i=1}^{3} S_i$  with nodes  $(x_k, y_k)_{k=1,...,N_{n+m}}$ , weights  $w_{k=1,...,N_{n+m}}$  of PI-type.

Mapping back the nodes on the sphere, we have a rule over the spherical triangle that is near algebraic with ADE n since

$$\int_{\mathcal{T}} f(x, y, z) d\sigma \approx \sum_{j=1}^{N_{n+m}} \frac{w_i}{\sqrt{1 - x_j^2 - y_j^2}} f(x_j, y_j, \sqrt{1 - x_j^2 - y_j^2}).$$
(1)

## Quadrature on spherical polygons: spherical triangles

Some details:

- Rule over elliptical sector S<sub>i</sub>: since each S<sub>i</sub> is an affine transformation of a circular sector of the unit-disk S<sup>\*</sup><sub>i</sub>, we determine a formula on S<sup>\*</sup><sub>i</sub> and map it to S<sub>i</sub> (some care on the weights that must be multiplied by absolute value of the transformation matrix determinant);
- Computation of  $m = m_{\epsilon}$ : it is sufficient to find the degree of a (near) optimal univariate polynomial approximation (up to  $\epsilon$ ) to  $1/\sqrt{1-t}$  where  $t \in [0, \max\{\|\hat{A}\|_2^2, \|\hat{B}\|_2^2, \|\hat{C}\|_2^2\}]$ .

Thus  $m = m_{\epsilon}$  can be estimated by Chebfun (even stored in tables!).

• Caratheodory-Tchakaloff rule compression: from the nodes  $\{P_k\}$  and weights  $\{w_k\}$  of the PI-type rule on  $\mathcal{T}$ , we extract one with cardinality at most  $(n + 1)^2$ , by Lawson-Hanson algorithm (see dCATCH suite implementation).

#### Quadrature on spherical polygons: example



Figura: Quadrature nodes on a spherical polygon of a rule of PI-type, with ADE=8, 380544 points, before Caratheodory-Tchakaloff compression. Cputime: about 10 seconds.

#### Quadrature on spherical polygons: example



Figura: Triangulation of a spherical polygon (967 spherical triangles) and quadrature rule of PI-type with ADE=8, 81 points, after Caratheodory-Tchakaloff compression (magenta). Cputime: 3.5 seconds.

As introduced by I.H. Sloan in 1995, given

- an orthonormal basis of  $\mathbb{P}_n^d(\Omega)$  (the subspace of *d*-variate polynomials of total-degree not exceeding *n*, restricted to a compact set or manifold  $\Omega \subset \mathbb{R}^d$ ) w.r.t. a given measure  $d\mu$  on  $\Omega$ ), say  $\{p_j\}$ ,  $1 \leq j \leq N_n = \dim(\mathbb{P}_d^n(\Omega))$ ,
- a quadrature formula exact for  $\mathbb{P}^{2n}_{d}(\Omega)$  with nodes  $X = \{x_i\} \subset \Omega$  and positive weights  $\mathbf{w} = \{w_i\}, 1 \le i \le M$  with  $M \ge N_n$ ,

the discretized orthogonal projection (hyperinterpolation) of  $f \in C(\Omega)$  is

$$(\mathcal{L}_n f)(x) = \sum_{j=1}^M \langle f, p_j \rangle_{l_{2,w}(X)} p_j(x) = \sum_{i=1}^M w_i f(x_i) \sum_{j=1}^{N_n} p_j(x_i) p_j(x).$$

Letting  $\Omega$  be the spherical polygon,

- by means of the routines in our Matlab package dCATCH we determine the required orthonormal basis {p<sub>j</sub>},
- we compute the quadrature rule of degree 2n on  $\Omega$ ,
- we finally get the hyperinterpolant of degree *n*.

As previously mentioned, the Matlab software implementing this approach is available at the authors' homepage.

As example, we take into account a course map of Australia (without taking into account its smaller islands).



Figura: Quadrature nodes of PI-type on a coarse approximation of Australia, useful for hyperinterpolation. ADE is 10, there are 167 sph. triangles, the full rule has 81711 points, the compressed one 121, with moments error of  $\approx 5 \cdot 10^{-15}$ . The whole process took about 3 seconds.

Setting  $(x_0, y_0, z_0) \approx (-6.325e - 01, 6.668e - 01, -3.908e - 01)$ as an approximation of australian centroid, and

$$h(x, y, z) = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$$

we define

- 1  $f_1(x, y, z) = 1 + x + y^2 + x^2 \cdot y + x^4 + y^5 + x^2 \cdot y^2 \cdot z^2$ (polynomial, total degree 6);
- 2  $f_2(x, y, z) = \cos(10 \cdot (x + y + z))$  (oscillations);
- 3  $f_3(x, y, z) = \sin(-h(x, y, z))$  (regular);
- 4  $f_4(x, y, z) = \exp(-h(x, y, z))$  (regular);
- 5  $f_5(x, y, z) = ((x x_0)^2 + (y y_0)^2 + (z z_0)^2)^{3/2}$  (hard);
- 6  $f_6(x, y, z) = ((x x_0)^2 + (y y_0)^2 + (z z_0)^2)^{5/2}$  (medium);



Figura: Inf-Norm hyperinterpolation error.



Figura: Hyperinterpolation operator Inf-Norm.

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