On some theorems by Tchakaloff, Davis and Wilhelmsen and their applications

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Purpose

In this talk,

- we start introducing the well-known Tchakaloff theorem (and one of its variants)

  existence theorem of certain algebraic and *low* cardinality cubature rules with positive weights on a multivariate compact domain $\Omega \subset \mathbb{R}^d$,;

- we compress cubature rules,

  we show how, from cubature rules on Ω with positive weights and interior nodes (i.e. of PI-type), whose algebraic degree of precision *ADE* is equal to m and the number of nodes higher than the dimension "r" of the polynomial space $\mathbb{P}_m(\Omega)$ of total degree m, we can extract rules of PI-type but with at most "r" nodes, by means of Lawson-Hanson algorithm;

- we recall some results by Davis and Wilhelmsen

  show how they imply that we can determine rules of PI-type, with ADE equal to m, if the moments w.r.t. to a basis of $\mathbb{P}_m(\Omega)$ are available as well as if we can evaluate numerically the characteristic function $\mathcal{X}_\Omega$;

- examples of this novel meshless approach

  time permitting

Important: all the Matlab routines used in this talk are available at the author's homepage.

Paper

Product Gauss cubature over polygons based on Green's integration formula (2007)

- purpose: cubature formula over convex, nonconvex or even multiply connected polygons Ω.
- ADE: the formula with nodes $\{\mathbf{x}_k\}_k \subset \mathbb{R}^2$ and weights $\{w_k\}_k \subset \mathbb{R}$ has algebraic degree of exactness ADE=$2n-1$, i.e. denoting by $\mathbb{P}_m$ the space of polynomials of total degree m, as well as $\mathbf{x}_k = (x_k, y_k)$,

$$\int_{\Omega} p(x,y)dxdy = \sum_{k=1}^{N} w_k p(x_k, y_k), \text{ for all } p \in \mathbb{P}_{2n-1},$$

  with $N \approx m\,n^2$, where "m" is the number of sides that are not orthogonal to a given line, and not lying on it.
- preprocessing: it does not need any preprocessing like triangulation of the domain, but relies directly on univariate Gauss-Legendre quadrature via Green's integral formula.

Example: quadrature over polygons (meshfree approach)

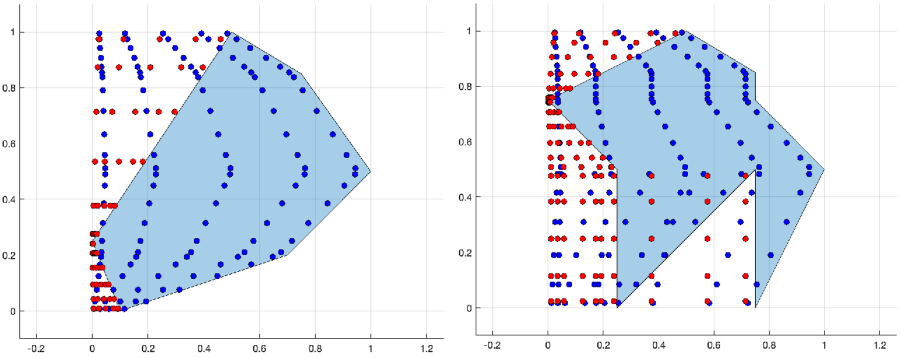

Figure: Examples of polygonal domains (ADE=9).

Left: a convex domain with 6 sides (180 nodes), Right: a non-convex domain with 9 sides (255 nodes).

Red dots: nodes with negative weights. Blue dots: nodes with positive weights.

- **Pros**: meshfree approach.
- **Cons**: In general these rules may have external nodes as well as negative weights.
- **Pros**: With some trick based on roto-translations we can compute rules over convex domains of PI-type (positive weights and internal nodes).
- **Cons**: With some trick based on roto-translations we can compute rules over some non-convex domains but in general this is not always possible.
- **Cons**: Rule may have high cardinality if the polygon has many sides.

We prefer rules with

- internal nodes (integrand may not be defined outside the domain),
- positive weights (more numerical stability and application to hyperinterpolation),

i.e. of PI-type.

Furthermore, we look for rules with low cardinality (few nodes, hence few samples of the function).

Paper

Compressed cubature over polygons with applications to optical design (2020)

- purpose: cubature formula over convex, nonconvex or even multiply connected polygons Ω.
- strategy: once a minimal triangulation is available (see Matlab `polyshape` toolbox), we obtain the rule by applying an almost-minimal rule of PI-type on each triangle with the wanted ADE, summing the contributions.

Some remarks

- minimal triangulation: one can triangulate a general M-sides polygon via $M-2$ triangles (easy task in a convex polygon, not trivial for a general one),
- almost-minimal rule the number of its nodes is almost minimal between those having a certain degree of precision and requirement on the nodes (e.g. internal) and weights (e.g. positive).

Example: quadrature over polygons (triang. based approach)

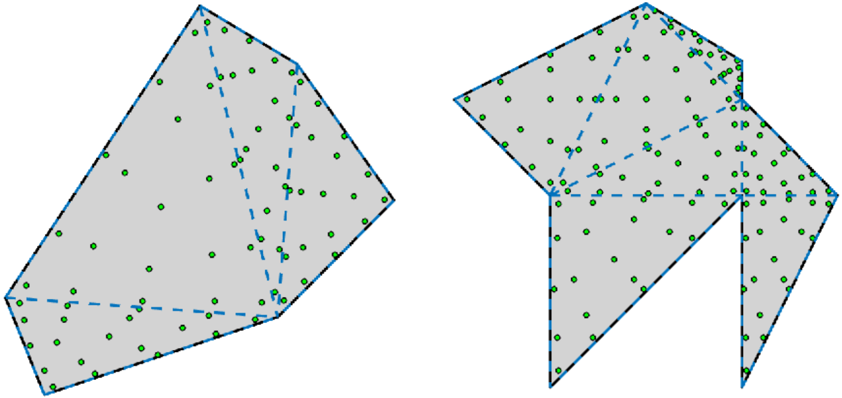

Figure: Examples of polygonal domains (ADE=9).

Left: a convex domain with 6 sides (77 nodes, the previous rule had 180 nodes), Right: a non-convex domain with 9 sides (133 nodes, the previous rule had 255 nodes).

Note: all the weights are positive.

- **Cons**: requires triangulation.
- **Pros**: In general these rules always have internal nodes as well as positive weights.
- **Cons**: Rule still may have high cardinality if the polygon has many sides.

Observe that in the examples above for ADE$=9$

- convex domain: the rule has 77 nodes,
- not convex domain: the rule has 133 nodes.

Remark

In both cases the number of nodes is higher than the dimension of the polynomial space $\mathbb{P}_9$ that is equal to $(9+1)(9+2)/2 = 55$.

Our project is to quickly extract from the previous one, another rule of PI-type with the same degree of precision but with a number of nodes at most equal to dim$(\mathbb{P}_9)$ (i.e. a rule compression).

Tchakaloff theorem, a cornerstone of quadrature theory, substantially asserts that:

For every compactly supported measure there exists a positive algebraic quadrature formula of ADE=m with cardinality not exceeding the dimension of $\mathbb{P}_m$ (restricted to the measure support)

i.e. the goal of the previous section can be achieved.

- Originally proved by V. Tchakaloff in 1957 for absolutely continuous measures, it has then be extended to any measure with finite polynomial moments, and to arbitrary finite dimensional spaces of integrable functions.
- We begin by stating a discrete version of Tchakaloff theorem whose proof is based on a theorem by Carathéodory. See also

Paper

M. Putinar, A Note on Tchakaloff's Theorem, Proceedings of the American Mathematical Society Vol. 125, No. 8 (Aug., 1997), pp. 2409–2414.

Theorem (Carathéodory-Tchakaloff, see more general Putinar theorem)

Let

1. μ *be a multivariate discrete measure supported at a finite set* $X = \{\mathbf{x}_k\}_{k=1,\dots,N} \subset \mathbb{R}^d$, *with correspondent positive weights* $\{w_k\}_{k=1,\dots,N}$,
2. $\Phi = span(\phi_1, \dots, \phi_r)$ *a finite dimensional space of d-variate functions defined on* $\Omega \supseteq X$, *with* $dim(\Phi|_X) \leq r$.

Then there exist a quadrature formula with nodes $T = \{\mathbf{t}_k\}_{k=1,\dots,N_c} \subseteq X$ *and positive weights* $\{u_k\}_{k=1,\dots,N_c}$, *such that* $N_c \leq dim(\Phi|_X)$ *and*

$$\int_\Omega f(\mathbf{x}) d\mu := \sum_{k=1}^{N} w_k f(\mathbf{x}_k) = \sum_{i=1}^{N_c} u_i f(\mathbf{t}_i), \text{ for all } f \in \Phi|_X.$$

Roughly speaking, adapting the theorem to our needs,

If we have a rule of PI-type with ADE=m and cardinality N higher than $r = dim(\mathbb{P}_m(\Omega))$ *then we can extract one of PI-type with ADE=m and cardinality* $N_c \leq r$.

Given

- a formula of PI-type with ADE=m, nodes $X = \{\mathbf{x}_k\}_{k=1,\ldots,N} \subset \mathbb{R}^d$ and positive weights $\{w_k\}_{k=1,\ldots,N}$,
- a basis $\{\phi_1, \ldots, \phi_r\}$ of $\mathbb{P}_m(\Omega)$,

let

- $V_{i,j} = (\phi_j(\mathbf{x}_i))$ the Vandermonde matrix at the nodes,
- $\mathbf{b} = (b_j)_{j=1,\ldots,r}$ where $b_j = \int_\Omega \phi_j d\mu = \sum_{i=1}^N w_i \phi_j(\mathbf{x}_i)$, the vector of the μ moments.

The problem mentioned above resorts into computing a nonnegative solution with at most "r" nonvanishing components to the underdetermined linear system

$$V^T \mathbf{u} = \mathbf{b}.$$

The computation of a nonnegative solution with at most $r = \dim(\mathbb{P}_m(\Omega))$ nonvanishing components to the underdetermined linear system $V^T\mathbf{u} = \mathbf{b}$ can be perfomed finding a sparse solution to the quadratic minimum problem

$$\text{NNLS:} \begin{cases} \min_u \|V^T\mathbf{u} - \mathbf{b}\|_2 \\ \mathbf{u} \geq 0 \end{cases}$$

via Lawson-Hanson active set method for NonNegative Least Squares (NNLS).

In Matlab this can be done by means of the Matlab built-in routine `lsqnonneg` as well as by the more recent `LHDM` by Dessole, Marcuzzi and Vianello.

Remark

- *The approach mentioned above is effective for mild ADE, say on the order of ADE=20 for bivariate domains and ADE=10 for trivariate domains.*
- *There are also other approaches, e.g. by based on linear programming or by a different combinatorial algorithm (recursive Halving Forest), based on SVD.*

As example, we can consider the application of the technique mentioned above to extract a rule of PI-type, for computing a similar one on the polygonal domains treated above.

Algorithm

`input`: *the nodes* $\{\mathbf{x}_k\}_{k=1,\dots,N}$, *the weights* $\{w_k\}_{k=1,\dots,N}$ *of a PI-type rule with* $N > r$ *(r is the dimension of the polynomial space* $\mathbb{P}_m$ *when ADE=m) and a polynomial basis* $\{\psi_j\}_{j=1,\dots,r}$;

1. *Vandermonde matrix: compute* $U = (\phi_k(\mathbf{x}_i))$;
2. *fight ill-conditioning: compute the QR factorization with column pivoting* $\sqrt{W}U(:,\pi) = QR$, *where* $\sqrt{W} = diag(\{w_k\})$ *and* π *is a permutation vector; this corresponds to a change of basis* $(\phi_1, \dots, \phi_r) = (\psi_1, \dots, \psi_r)R^{-1}$, *so obtaining an orthonormal basis w.r.t. the discrete measure defined by the nodes* $\{\mathbf{x}_k\}_{k=1,\dots,N}$ *and the weights* $\{w_k\}_{k=1,\dots,N}$;
3. *moments: evaluate the vector* $\mathbf{b} = Q^T\mathbf{w}$ *where* $\mathbf{w}_k = w_k$;
4. *compute a positive sparse solution: solve* $Q^T\mathbf{u} = \mathbf{b}$ *by Lawson-Hanson algorithm (or its alternatives).*

By this algorithm,

- adopting as basis $\{\psi_j\}$ the total-degree product Chebyshev basis of the smallest Cartesian rectangle $[a_1, b_1] \times [a_2, b_2]$ containing Ω, with the graded lexicographical ordering,
- from the PI-rules with ADE=9 obtained via triangulation,

we get the PI-rules below with cardinality $55 = \dim(\mathbb{P}_9)$.

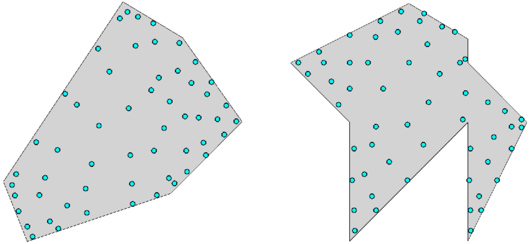

Figure: Examples of polygonal domains (ADE=9).

Left: a convex domain with 6 sides (55 nodes, the previous rule had 77 nodes), Right: a non-convex domain with 9 sides (55 nodes, the previous rule had 133 nodes).

Remark (When do not apply this technique)

We observe that this approach is useful only when the initial rule of PI-type with ADE=m has cardinality higher the dimension L of $\mathbb{P}_m(\Omega)$.

Thus it is worthless in the case of classical domains as the interval, the disk, simplex, cube, sphere, where there are explicit rules of PI-type with cardinality inferior to L.

Remark (Cputimes on the previous domains)

For mild ADE the computation of these compressed rules is fast. Running Matlab R2022a, on a computer with an Apple M1 processor and 16GB of RAM, we had average cputimes as in the table below:

Domain	tri. rule	compress.
Convex domain	*1.1-3s*	*1.7-2s*
Non-convex domain	*5.2-3s*	*1.1-2s*

Table: *Average cputime for computing rules with ADE=9 in the polygonal domains used in the tests.*

In the paper

D. R. Wilhelmsen, A Nearest Point Algorithm for Convex Polyhedral Cones and Applications to Positive Linear approximation, Math. Comp., (30) 1976, pp. 48–57,

the author extended a result from

P. J. Davis, A construction of nonnegative approximate quadratures, Math. Comp., (21) 1967, pp. 578–582.

In his own words

A constructive proof of the Tchakaloff theorem was given by Davis. Although his paper deals only with the integration functional, his results are easily adapted to more general functionals.

In view of both the contributions we will refer to the next result as Davis-Wilhelmsen theorem and show its impact to the numerical construction of algebraic cubature rules.

Definition (Tchakaloff set)

1. Consider a finite-dimensional function space

$$\Phi = span\{\phi_1, \ldots, \phi_r\}, \quad r = \dim(\Phi),$$

on a compact domain $\Omega \subset R^d$, satisfying the Krein condition, i.e. there exists a function in Φ not vanishing on Ω (e.g., a polynomial space).

2. Let $\mathcal{L}$ be a strictly positive linear functional on Φ, i.e. $\mathcal{L}(\phi) > 0$ for every $\phi \in \Phi$, $\phi \geq 0$ not identically vanishing on Ω.

A Tchakaloff set for $\mathcal{L}$ on Φ is a subset, say $T \subset \Omega$, that contains the support of a Tchakaloff-like representation for $\mathcal{L}$, i.e.

$$\mathcal{L}(\phi) = \sum_{j=1}^{N_c} w_j \, \phi(t_j), \quad \forall \phi \in \Phi,$$

where $N_c \leq r = \dim(\Phi)$, $\{t_j\} \subset T$ and $w_j > 0, \ j = 1, \ldots, \nu$.

Example (Wilhelmsen, p.49)

Set $\Phi = \mathbb{P}_L$ and $\mathcal{L}(\phi) = \int_\Omega \phi(\mathbf{x}) w(\mathbf{x}) \, d\mathbf{x}$, $w \geq 0$ weight function.

Theorem

Let

1. Φ *be the linear span of continuous, real-valued, linearly independent functions* $\{\phi_k\}_{k=1,\dots,r}$ *defined on a compact set* $\Omega \subset \mathbb{R}^d$.
2. *Assume that* Φ *satisfies the Krein condition (i.e. there is at least one* $f \in \Phi$ *which does not vanish on* Ω*) and that* $\mathcal{L}$ *is a strictly positive linear functional on* Φ*, i.e.* $\mathcal{L}(\phi) > 0$ *for every* $\phi \in \Phi$*,* $\phi \geq 0$ *not identically vanishing on* Ω.
3. $\{P_i\}_{i=1}^{+\infty}$ *is an everywhere dense subset of* Ω

Then for sufficiently large $\mathcal{I}$*, the set* $X = \{P_i\}_{i=1}^{\mathcal{I}}$ *is a Tchakaloff set, i.e.*

$$\mathcal{L}(f) = \sum_{j=1}^{N_c} w_j f(Q_j), \ \forall f \in \Phi \tag{1}$$

where $w_j > 0$*,* $j = 1, \dots, N_c$ *and* $\{Q_j\}_{j=1}^{N_c} \subset X \subset \Omega$*, with* $N_c = card(\{Q_j\}) \leq r$.

In our numerical framework this theorem is important because it says that:

> a sufficiently dense set of the compact domain Ω contains the nodes of a PI-type rule with algebraic degree of precision ADE=m

Algorithm (sketch)

- *Moment computation: determine $\{\gamma_k\}_{k=1,\ldots,r}$ of a certain polynomial basis $\{\phi_k\}_{k=1,\ldots,r}$ of $\mathbb{P}_m(\Omega)$;*
- *Pointset: using an* `in-domain` *routine on a mesh in a domain $\mathcal{R}$ containing Ω, determine a sufficient number of points $\{\tilde{P}_l\}_{l=1,\ldots,N}$ inside Ω so that the overdetermined linear system $V^T w = \gamma$, with $V_{l,k} = (\phi_k(\tilde{P}_l))$, has a nonnegative solution w with at most $N_c \leq r$ positive components.*
- *Rule extraction: solve $V^T w = \gamma$ via fast Lawson-Hanson algorithm (or alternatives).*

In spite of the simplicity of this approach there are many aspects that deserve explanations, on the implementation side as well as on the theoretical one.

The choice of the polynomial basis $\{\phi_k\}_{k=1,\dots,r}$ of $\mathbb{P}_m$ is not trivial. It must be

- not too badly conditioned,
- allow a fast computation of the moments

$$\gamma_k = \int_\Omega \phi_k(\mathbf{x})\, d\mu, \quad k = 1, \dots, r.$$

Remark (Choice of the polynomial basis)

For many multivariate regions, it is enough to use a specific basis based on total-degree product Chebyshev basis of the smallest Cartesian hyper-rectangle $\prod_{k=1}^{d}[a_k, b_k]$ containing Ω, with the graded lexicographical ordering.

However the choice may be much more difficult for regions over manifolds.

Example. If Ω is compact domain on the unit-sphere $\mathbb{S}_2 \subset \mathbb{R}^3$, e.g. a spherical polygon,

- *the classical spherical harmonics restricted on Ω may be severely ill-conditioned,*
- *alternatively, using the tensorial basis of $\mathbb{R}^3$, restricted to the sphere, some elements of the trivariate basis mentioned above must be somehow discarded, since the dimension of the polynomials $\mathbb{P}_m(\mathbb{S}_2)$ is $(m+1)^2$ while $\mathbb{P}_m(\mathbb{R}^3)$ is $(m+1)(m+2)(m+3)/6$ (i.e. for $m = 9$ we have $\dim(\mathbb{P}_9(\mathbb{S}_2)) = 100$ while $\dim(\mathbb{P}_9(\mathbb{R}^3)) = 220$).*

A second issue is the availability of the in-domain routine.

- Verification of certain inequalities: for example, the unit-ball $B(\mathbf{0},1)$ is defined as the set

$$B(\mathbf{0},1) := \{\mathbf{x} = (x,y,z) \in \mathbb{R}^3 \text{ such that } x^2+y^2+z^2 \leq 1\}.$$

- Specific codes: Matlab built-in `inpolygon` (polygonal domains), `inpolyhedra` (polyhedral domains).

At the same time, it is important the choice of a sufficient dense set.

- Too low cardinality: many iterations of the algorithm.
- Too high cardinality: expensive cost of the indomain routine.

Thus, to determine quickly a sufficiently dense set T,

- its cardinality and distribution must be well-thought a priori
- the in-domain routine must be fast.

A third issue is the moment computation.

This can be done in various ways, for instance resorting to one of the following:

1. they are explicitly known;
2. a Gauss-Green approach in bivariate domains, or a divergence theorem based in trivariate domains;
3. specific rules even with negative weights or external nodes,
4. adaptive rules.

As first example we consider a bivariate domain Ω whose boundary $\partial\Omega$ is parametrically defined by piecewise rational functions i.e.

$$\partial\Omega = \{\mathbf{x} = (x, y), x = x(t), y = y(t), t \in [a, b]\}$$

where x, y are certain rational functions defined on $[a, b]$.

Example

This class of bivariate domains Ω includes those such that $\partial\Omega$ is defined by

- piecewise by NURBS curves,
- composite Bezier curves,
- parametric splines.

Applications:

- NEFEM with NURBS-shaped curvilinear elements,
- VEM with NURBS-shaped curvilinear elements.

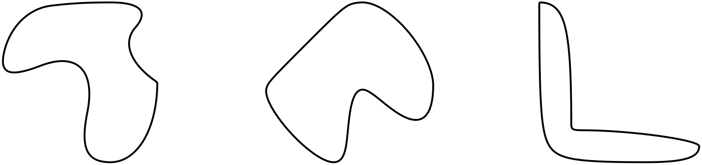

Figure: Examples of integration domains.

The pointset in Ω is extracted from a fine mesh on the minimal rectangle $\mathcal{R}^* \supseteq \Omega$. The `in-domain` routine can be based on Jordan curve theorem:

> a point P belongs to a Jordan domain Ω if and only if, having taken a point $P^* \notin \Omega$ then the segment $\overline{P^* P}$ crosses $\partial\Omega$ an odd number $c(P)$ of times.

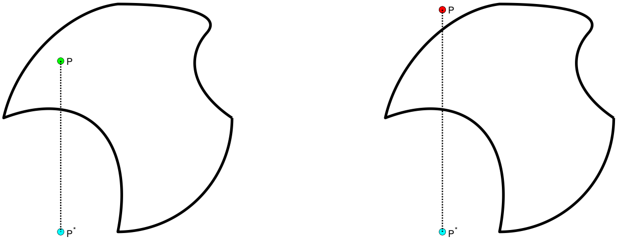

Figure: Points and boundary intersections. On the left $c(P) = 1$ and the point P is in the domain. On the right $c(P) = 2$ and the point P is outside the domain.

For details see: inRS: implementing the indicator function for NURBS-shaped planar domains, Applied Mathematics Letters, Volume 130, August 2022.

Having in mind to compute a rule with algebraic degree of precision $ADE = m$ by moments equations, we

- define a suitable basis $\{\phi_j\}$ of the polynomial space $\mathbb{P}_m$ (*tensorial Chebyshev* on the bounding box $\mathcal{R}^*$ of Ω),
- compute the moments $\gamma_1, \ldots, \gamma_r$, where

$$\gamma_j := \int_{\mathcal{S}} \phi_j(x, y) dxdy.$$

To this purpose:

1. By applying the Gauss-Green theorem, each γ_j is the sum of some line integrals, that after some computation are shown to require the integration in $[-1, 1]$ of continuous rational functions.
2. We compute these integrals in $[-1, 1]$ by high-order Gauss-Legendre rule (other techniques may be used).

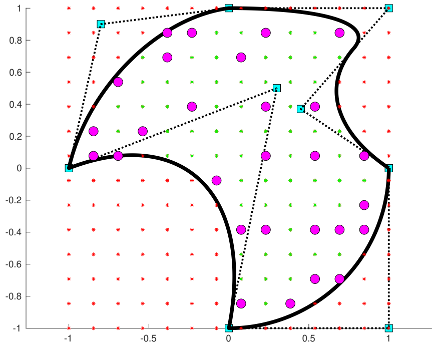

Figure: A curvilinear domain Ω, the mesh points P outside the domain or on its boundary (in red), those inside the domain (in green) and the nodes of a cubature formula of PI-type for $n = 6$ (28 magenta dots). The control points of the NURBS curve are represented as cyan squares, joined to represent the so called *control points polygon*.

ADE	#	# trial pts	cond	moment res	cpu
2	6	50 (121)	1	$5e{-}16$	$5.3e{-}3$
4	15	50 (121)	1	$7e{-}16$	$6.1e{-}3$
6	28	89 (196)	1	$1e{-}15$	$7.6e{-}3$
8	45	239 (484)	1	$2e{-}15$	$1.1e{-}2$
10	66	491 (961)	1	$4e{-}15$	$1.6e{-}2$

Table: Degree of precision $ADE = 2, 4, 6, 8, 10$ of the rule, cardinality $\#$ of the extracted nodes, cubature conditioning and moment residual of the rule on curvilinear domain Ω, number of trial points used in the extraction, cubature condition number cond, moment residual of the rule and median of the cputime over 50 tests.

For details see: Low cardinality Positive Interior cubature on NURBS-shaped domains (to appear on BIT Numer. Math.).

The same approach can be used to compute meshless cubature rules over polyhedra:

- polynomial basis: tensorial-type Chebyshev basis on the bounding box $\mathcal{R}$ of the domain Ω;
- mesh points: *sufficiently dense* Halton points in the bounding box $\mathcal{R}$;
- in-domain routine: `inpolyhedron.m`;
- moment computation: via divergence theorem on the facets (routine requested: cubature formula over a polygon).

Furthermore it can be used to compress Quasi-Montecarlo cubature rules on $\Omega \subset \mathbb{R}^d$ obtained by set operations of compact domains $\Omega_1, \ldots, \Omega_\nu \subset \mathbb{R}^d$:

- polynomial basis: tensorial-type Chebyshev basis $\mathbb{P}_m$ on the bounding box $\mathcal{R}$ of the domain Ω;
- mesh points: *sufficiently dense* Halton points in the bounding box $\mathcal{R}$;
- in-domain routine: in domain routine on each domain Ω_k, $k = 1, \ldots, \nu$ followed by suitable set operations;
- moment computation: via Quasi-Montecarlo cubature.

This allows to achieve a rule with few nodes that equals the results of the QMC rule applied to polynomials in $\mathbb{P}_m$.

Purpose: retaining the approximation power of the original QMC formula (up to a quantity proportional to the best polynomial approximation error of degree m to f, in the uniform norm on Ω), using much fewer nodes.

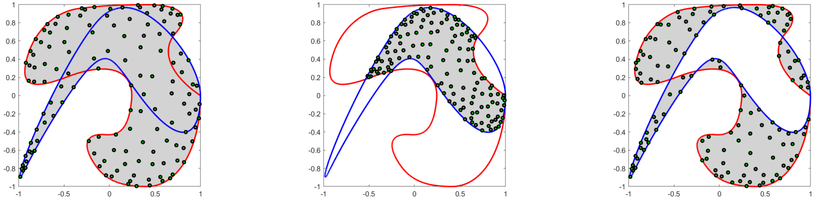

Figure: 231 compressed QMC nodes with exactness degree $n = 20$, on complex shapes arising from union (top-left), intersection (top-right) and symmetric difference (bottom) of two NURBS-shaped domains (extraction from a million Halton points of domain bounding boxes, basis Φ obtained by orthonormalization of a tensorial type basis in the bounding box $\mathcal{R}$ of the domain Ω).

deg	5	10	15	20
card. Q_c	21	66	136	231
compr. ratio	1.2e+04	3.9e+03	1.9e+03	1.1e+03
cpu Q_c	4.0e-02	1.2e-01	2.8e-01	5.8e+00
speed-up	6.0	9.0	11.0	1.4
mom. resid. Q_c^{new}	5.8e-16	1.4e-15	2.4e-15	7.0e-15

Table: Compression parameters of QMC cubature with $N = 255923$ Halton points on the intersection of two NURBS-shaped domains as in Figure above top-right.

deg	5	10	15	20
$E(f_1)$	2.7e-04	1.4e-08	3.0e-13	4.5e-16
$E(f_2)$	2.3e-04	2.4e-05	1.1e-05	5.6e-06

Table: Relative QMC compression errors $E(f_k)$, $k = 1, 2$ for the two test functions $f_1(P) = \exp(-|P - P_0|^2)$, $f_2(P) = |P - P_0|^5$ on the intersection of Fig. 1 top-right.

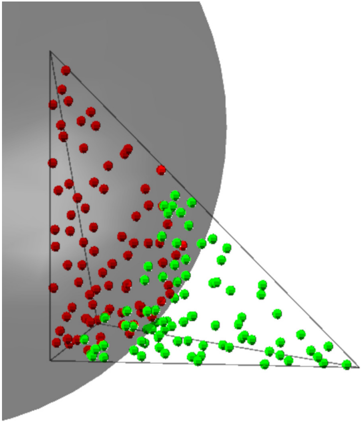

Figure: 84 compressed QMC nodes with exactness degree n = 6, on intersection (red bullets) and difference (green bullets) of a tetrahedral element with a ball (extraction from a million Halton points of domain bounding boxes, cputime: $\approx 5 \cdot 10^{-2}s$, basis Φ obtained by orthonormalization of a tensorial type basis in the bounding box $\mathcal{R}$ of the domain Ω).

deg	2	4	6
card. Q_c	10	35	84
compr. ratio	2.2e+04	6.2e+03	2.6e+03
cpu Q_c	4.1e-02	4.1e-02	1.8e-01
mom. resid. Q_c	1.7e-16	6.0e-16	1.2e-15

Table: Compression parameters of QMC cubature with $N = 216217$ Halton points on the intersection of a tetrahedral element with a ball as in the last figure.

deg	2	4	6
card. Q_c	10	35	84
compr. ratio	5.9e+03	1.7e+03	7.0e+02
cpu Q_c	3.3e-02	2.1e-02	5.5e-02
speed-up	3.4	6.9	8.8
mom. resid. Q_c	5.0e-16	6.1e-16	1.2e-15

Table: Compression parameters of QMC cubature with $N = 58561$ Halton points on the difference of a tetrahedral element with a ball as in the last figure.

1. M. Dessole, F. Marcuzzi, M. Vianello, Accelerating the Lawson-Hanson NNLS solver for large-scale Tchakaloff regression designs, DRNA 13, 20–29 (2020).
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