Cpolymesh: a Matlab/Python suite for complex polynomial approximation on Chebyshev-like polynomial meshes

L. Bialas-Cież, D.J. Kenne, **A. Sommariva**, M. Vianello

DWCAA 2024,
Dedicated to Len Bos in occasion of his retirement
Alba di Canazei (TN) - Italy, 09-13 September 2024

Some years ago we discovered the following phenomena. Marco Vianello and I wrote the following Matlab code:

```
function example1

% 1. define a dense set of equispaced points in [-1,1].
N=1000; x=linspace(-1,1,N); x=x';

% 2. evaluate the Cheb.-Vandermonde matrix of degree "n=10".
n=10; V=chebpolys(n,x);

% 3. magic wand (wow effect)!
w=V'\ones(n+1,1);  ind=find(w ~= 0);  xi=x(ind);

% 4. Lebesgue constant
Vxi=chebpolys(n,xi); leb=norm(Vxi'\V',1);
fprintf('\n \t Lebesgue const. :%1.3e \n',leb);

% 5. plot.
plot(x,zeros(size(x)),'r.'); hold on;
plot(xi,zeros(size(xi)),'bo',...
               'MarkerFaceColor','k','MarkerSize',10);
hold off;
```

Some historical note

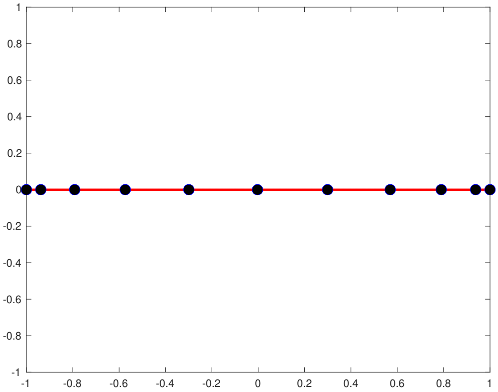


Figure: At degree 10, the extracted pointset looks like Chebyshev points and you ask yourself what is going on (its Lebesgue constant is ≈ 2.26 to be compared with ≈ 2.05 of Extended Chebyshev points and ≈ 2.36 of the Chebyshev points).

Being very surprised, we thought it could be a miracle in the 1D case. So we wrote the following Matlab code:

```
function example2
% 1. define a dense set of "tensorial equispaced" points in the
    square [-1,1]^2.
N=50; x=linspace(-1,1,N); [XM,YM]=meshgrid(x);
x=XM(:); y=YM(:); X=[x y];

% 2. evaluate the 2D Cheb-Vandermonde matrix of degree "n=10"
n=10; V = dCHEBVAND(n,X);

% 3. magic wand
w=V'\ones(size(V,2),1);  ind=find(w ~= 0);  xi=X(ind,:);

% 4. % Lebesgue constant
Vxi=dCHEBVAND(n,xi); leb=norm(Vxi'\V',1);
fprintf('\n \t Lebesgue const. :%1.3e \n',leb);

% 5. plot
plot(x,y,'r.'); hold on;
plot(xi(:,1),xi(:,2),'bo',...
           'MarkerFaceColor','k','MarkerSize',10);
axis square
hold off;
```

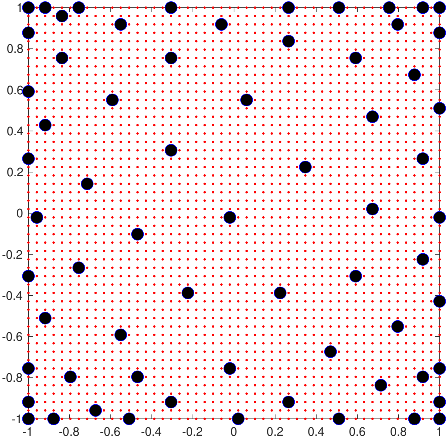
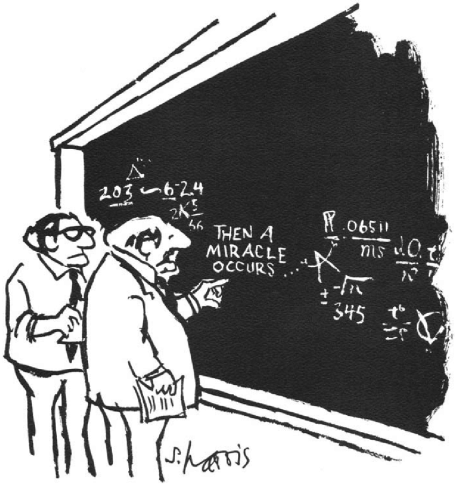


Figure: The extracted pointset looks like Padua points and now you think something big is going on. Notice that the Lebesgue constant is again rather small, i.e. ≈ 12.01, though Padua points give ≈ 6.88.



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

What happened later is that we found this surprising paper

ELSEVIER

Journal of Approximation Theory

Volume 152, Issue 1, May 2008, Pages 82-100

Uniform approximation by discrete least squares polynomials

Jean-Paul Calvi [a], Norman Levenberg [b]

realizing that our meshes were actually *admissible meshes* in the language of the authors.

They wrote in the introduction:

> In the univariate case, the efficiency and accuracy of this classical method generally depends on the location of the interpolation points.
>
> From a theoretical point of view, this has been known for more than 80 years. Very few results are available on multivariate Lagrange interpolation.
>
> The so-called Fekete points work well but are difficult to locate. Indeed, aside from the beautiful Padua points recently discovered by Caliari et al. in a square in $\mathbb{R}^2$ no explicit sets of "good" interpolation points are known on any compact set.
>
> Thus we attack the problem of constructing complex approximation polynomials using point evaluations from a different perspective. The method we study in this paper is commonly used in applied mathematics; it is a classical least squares approximation problem ...

We believed that somehow we were dealing with some strange pointsets sharing properties with Fekete points. We wrote

ELSEVIER
Computers & Mathematics with Applications
Volume 57, Issue 8, April 2009, Pages 1324-1336

Computing approximate Fekete points by QR factorizations of Vandermonde matrices ☆

Alvise Sommariva, Marco Vianello

In its abstract we said explicitly:

> We propose a numerical method (implemented in Matlab) for computing approximate Fekete points on compact multivariate domains.
> It relies on the search of maximum volume submatrices of Vandermonde matrices computed on suitable discretization meshes, and uses a simple greedy algorithm based on QR factorization with column pivoting.

In this paper we provided a theorem about the quality of this pointset, named Approximate Fekete Points, in the 1D case, but what about the multivariate instances? We asked Len what was going on. We wrote all together with J.-P. Calvi and N.Levenberg:

Geometric weakly admissible meshes, discrete least squares approximations and approximate Fekete points

HTML articles powered by AMS MathViewer

by L. Bos, J.-P. Calvi, N. Levenberg, A. Sommariva and M. Vianello PDF

Math. Comp. **80** (2011), 1623-1638 Request permission

Abstract:

Using the concept of Geometric Weakly Admissible Meshes (see §2 below) together with an algorithm based on the classical QR factorization of matrices, we compute efficient points for discrete multivariate least squares approximation and Lagrange interpolation.

By deep results in pluripotential theory, the pointset had some remarkable asymptotic features that makes it very good for interpolation purposes.

Theorem 1. *Suppose that $K \subset \mathbb{C}^d$ is compact, non-pluripolar, polynomially convex and regular (in the sense of Pluripotential theory) and that for $n = 1, 2, \ldots, A_n \subset K$ is a WAM. Let $\{\mathbf{b}_1^{(n)}, \mathbf{b}_2^{(n)}, \ldots, \mathbf{b}_N^{(n)}\}$ be the Approximate Fekete Points selected from A_n by the greedy algorithm of* [20] *described above, using any basis $\mathcal{B}_n$. Then*

(1) $\lim_{n\to\infty} |\mathrm{vdm}(\mathbf{b}_1^{(n)}, \ldots, \mathbf{b}_N^{(n)})|^{1/m_n} = \tau(K)$, *the transfinite diameter of K*

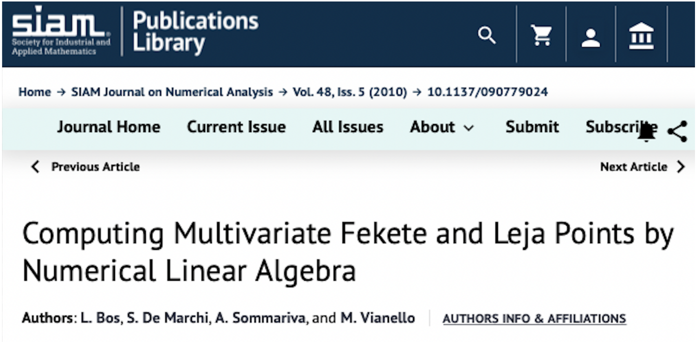
and

(2) *the discrete probability measures $\mu_n := \frac{1}{N}\sum_{j=1}^{N} \delta_{\mathbf{b}_j^{(n)}}$ converge weak-$*$ to the pluripotential-theoretic* equilibrium measure *$d\mu_K$ of K.*

At this stage we had a huge advantage.

- The *true* Fekete points maximize the absolute value of the determinant of the Vandermonde matrix and are known in very few instance.
- Having a (good) polynomial basis and generating a mesh, we could produce good sets for polynomial interpolation in many other multivariate instances!

Later we wrote with S. De Marchi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Computing Multivariate Fekete and Leja Points by Numerical Linear Algebra

Authors: L. Bos, S. De Marchi, A. Sommariva, and M. Vianello | AUTHORS INFO & AFFILIATIONS

- A new pointset, i.e. Discrete Leja Points (Schaback/De Marchi hint).
- Len proved again the remarkable asymptotic properties of DLP that makes it good for interpolation.
- Differently from AFP, DLP are a *sequence*, easily at hand by means of linear algebra routines, specifically LU *vector* factorisation.

As before we consider a code for DLP on the interval $[-1, 1]$.

```
function example3
% define a dense set of equispaced points in [-1,1].
N=1000; x=linspace(-1,1,N); x=x';
% evaluate the Vandermonde matrix of degree "n=10"
n=10; V=chebpolys(n,x);
% magic wand
[L,U,sigma]=lu(V,'vector'); ind=sigma(1:n+1);  xi=x(ind,:);
% Lebesgue constant (DLP degree 10)
Vxi=chebpolys(n,xi); leb=norm(Vxi'\V',1);
fprintf('\n \t n: %2.0f Lebesgue const. :%1.3e \n',n,leb);
% DLP at degree 8
n1=8; ind1=sigma(1:n1+1);  xi1=x(ind1,:);
% Lebesgue constant (DLP degree 8)
V1=V(:,1:n1+1); Vxi1=chebpolys(n1,xi1); leb1=norm(Vxi1'\V1',1);
fprintf('\n \t n1: %2.0f Lebesgue const. :%1.3e \n',n1,leb1);
% plot
plot(x,zeros(size(x)),'k.'.','MarkerSize',1); hold on;
plot(xi,zeros(size(xi)),'bo',...
      'MarkerFaceColor','b','MarkerSize',12);
plot(xi1,zeros(size(xi1)),'mo',...
    'MarkerFaceColor','m','MarkerSize',12);
hold off;
```

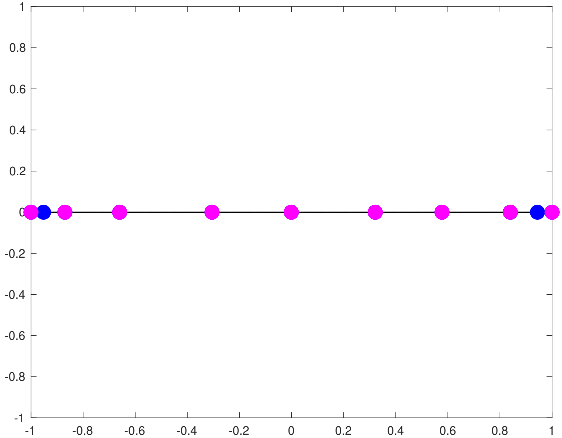


Figure: The DLPs look again like Chebyshev points. Notice that those at degree 8 are nested in those at degree 10 and that their Lebesgue constant is rather small, i.e. ≈ 3.145 at $n = 8$ and 4.333 at $n = 10$.

We can write a code for DLP on the square $[-1,1]x[-1,1]$.

```
function example4

% define a dense set of equispaced points in [-1,1]^2.
N=50; x=linspace(-1,1,N); [XM,YM]=meshgrid(x);
x=XM(:); y=YM(:); X=[x y];

% evaluate the Vandermonde matrix of degree "n=10"
n=10; V = dCHEBVAND(n,X);

% magic wand
[L,U,sigma]=lu(V,'vector'); ind=sigma(1:size(V,2)); xi=X(ind,:);

% Lebesgue constant
Vxi=dCHEBVAND(n,xi); leb=norm(Vxi'\V',1);
fprintf('\n \t Lebesgue const. :%1.3e \n',leb);

% plot
plot(x,y,'r.'); hold on;
plot(xi(:,1),xi(:,2),'bo',...
        'MarkerFaceColor','k','MarkerSize',10);
axis square
hold off;
```

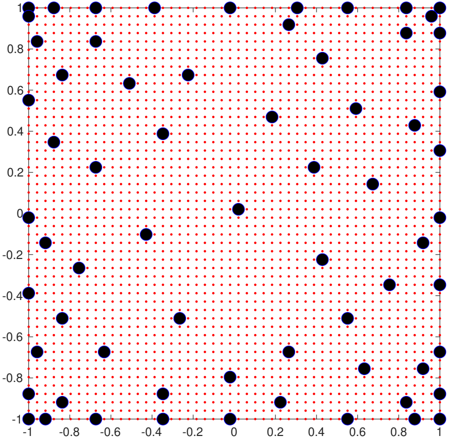


Figure: The DLP pointset looks again like Padua points Notice that the Lebesgue constant is again rather small, i.e. ≈ 27.61, though Padua points give ≈ 6.88 and AFP ≈ 12.01.

What happened next was a blossoming of papers on the topic, in which just to mention some lines of study:

- (weakly)-admissible meshes were found for many multivariate domains (even with low-cardinality, see e.g. papers by A. Kroó, F. Piazzon and M. Vianello);
- application to polynomial optimization problems was investigated (see papers by F. Piazzon and M.Vianello);
- application to PDEs (by P. Zitnan);
- links with statistical theory of optimal designs were described (see, e.g., papers of Len with F. Piazzon and M. Vianello).

If you are interested in this, see e.g.

https://www.math.unipd.it/~marcov/inequalities.html

For a brief summary, see the poster

https://www.math.unipd.it/~marcov/pdf/posterICIAM2011.pdf

We have collected many of these polynomial meshes at the homepage named

WAM: Matlab package for multivariate polynomial fitting and interpolation on Weakly Admissible Meshes

available at

`https://www.math.unipd.it/~marcov/wam.html`

There you can find such sets in the case of

- polygons,
- domains related to circular arcs (sectors, lenses, lunes, ...),
- suitable smooth convex domains,
- cubes,
- cones and pyramids.

Remark

The page must be updated, since in recent works some meshes have been found, e.g. on spherical triangles, on the torus as well as over many complex domains.

If you are interested in many of these connections between approximation theory and pluripotential theory, Len and well-known collaborators (one loves gardening, see the latest DRNA!), wrote

October 17, 2018

POLYNOMIAL INTERPOLATION AND APPROXIMATION IN $\mathbb{C}^d$

T. BLOOM*, L. P. BOS, J.-P. CALVI AND N. LEVENBERG

ABSTRACT. We update the state of the subject approximately 20 years after the publication of [8]. This report is mostly a survey, with a sprinkling of assorted new results throughout.

Some historical note

This paper is a recent update of

T. Bloom, L. P. Bos, J.-P. Calvi, N. Levenberg,

Polynomial Interpolation and Approximation in $\mathbb{C}^d$,

Annales Polonici Mathematici 106 (2012), pp.53–81.

Figure: N. Levenberg and L. Bos (picture by courtesy of J.-P. Berrut, DWCAA 2016)

Recently we have observed that some work could be done for certain subdomains in the complex field and this led us to a

Joint work with

- Dimitri Jordan Kenne (Jagellonian University, Krakow, Poland),
- Leokadia Bialas-Cież (Jagellonian University, Krakow, Poland),
- Marco Vianello (University of Padua).

Work partially supported by

- the DOR funds of the University of Padova,
- INdAM-GNCS 2024 Project "Kernel and polynomial methods for approximation and integration: theory and application software",
- the National Science Center - Poland, grant Preludium Bis 1, N. 2019/35/O/ST1/02245.

Research accomplished within RITA, SIMAI Activity Group ANA&A, UMI Group TAA.

Purpose

The talk is based on the work dedicated to Len Bos:

J. Approx. Softw., Vol. 1, No. 2 (2024) 2

Journal of Approximation Software

CPOLYMESH: Matlab and Python codes for complex polynomial approximation by Chebyshev admissible meshes

D.J. Kenne[1], A. Sommariva[2], and M. Vianello[2,⋆]

[1] *Doctoral School of Exact and Natural Sciences, Jagiellonian University, Krakow, Poland*
[2] *Department of Mathematics, University of Padova, Italy*

- Let $K \subset \mathbb{C}$ be a complex compact set with connected complement;
- polynomial meshes $\{Z_n\}_{n\geq 1}$ are sequences of finite subsets $Z_n \subset K$ such that
$$\|p\|_K \leq c\|p\|_{Z_n}\ ,\ \ \forall p \in \mathbb{P}_n(\mathbb{C})\ , \tag{1}$$
where
  - $\|\cdot\|$ is the *inf*-norm on a continuous/discrete bounded subset;
  - $p \in \mathbb{P}_n$ is any polynomial with complex coefficients with degree not exceeding n;
  - c is usually termed the *constant* of the polynomial mesh;
- it can be easily proven that $card(Z_n) \geq n+1 = dim(\mathbb{P}_n(\mathbb{C}))$;
- the polynomial mesh Z_n is optimal when $card(Z_n) = O(n)$.

Let C_N be the set of

- N Chebyshev points i.e. N Chebyshev zeros in $(-1, 1)$, i.e

$$\cos((2j-1)\pi/(2N)), \quad 1 \leq j \leq N;$$

- or $N+1$ Chebyshev-Lobatto points, i.e..

$$\cos(j\pi/N), \quad 0 \leq j \leq N.$$

Consider the points

$$\mathcal{C}_\nu^m = \tau(C_N) \subset [a, b] \tag{2}$$

where

- in the algebraic case is

$$N = m\nu \ , \ \ \tau(u) = \frac{b-a}{2}\, u + \frac{b+a}{2} \ , \ \ u \in [-1, 1] \ ,$$

- in the trigonometric case is

$$N = 2m\nu \ , \ \ \tau(u) = 2 \arcsin\left(u \sin\left(\frac{b-a}{4}\right)\right) + \frac{b+a}{2} \ , \ \ u \in [-1, 1].$$

Proposition

Let $K \subset \mathbb{C}$ be a complex compact set with connected complement and

$$\partial K \subseteq \bigcup_{j=1}^{s} \Gamma_j \subseteq K$$

alternatively with parametric

- *algebraic arcs* $\Gamma_j = \gamma_j([a_j, b_j])$,
- *trigonometric arcs* $\Gamma_j = \gamma_j([a_j, b_j])$, *with* $b_j - a_j \leq 2\pi$

of degree

$$d_j = \max\{deg\, Re(\gamma_j), deg\, Im(\gamma_j)\}, 1 \leq j \leq s.$$

Then for every $p \in \mathbb{P}_n(\mathbb{C})$, $n \geq 1$, $m > 1$, *the set* $\{Z_n^m\}_{n \geq 1}$ *with*

$$Z_n^m = \bigcup_{j=1}^{s} \gamma_j(\mathcal{C}_{nd_j}^m)$$

is a polynomial mesh for K *with constant* $c_m = \frac{1}{\cos(\pi/(2m))}$.

Numerical example on a cardioid

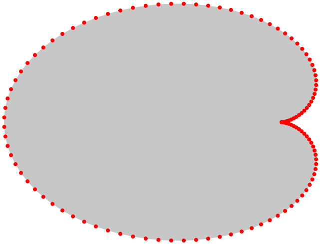


Figure: The red dots represent the polynomial mesh Z_{10}^3 over the complex cardioid.

- The boundary of the cardioid is defined as

$$\gamma_1(t) = \cos(t)(1-\cos(t)) + i(\sin(t)(1-\cos(t))), \quad t \in [0, 2\pi]$$

  i.e. $d_1 = 2$ and the trigonometric case is involved.
- The mesh Z_{10}^3 has degree $n = 10$ and factor $m = 3$.
- Since $Z_n^m = \gamma_1(\mathcal{C}_{nd_1}^m)$, where $\mathcal{C}_{nd_j}^m$ are Chebyshev points suitably scaled in $[0, 2\pi]$.
- The cardinality of the mesh is $N = 2(n \cdot d_1 \cdot m) = 120$.
- It is $c_3 = \frac{1}{\cos(\pi/(2\cdot 3))} \approx 1.15470$.

1. The polynomial mesh $Z_n^m = \bigcup_{j=1}^{s} \gamma_j(\mathcal{C}_{nd_j}^m)$ depends
   - on the polynomial degree n,
   - on the degrees of the boundary d_j,
   - on the fact that γ_j may algebraic or trigonometric (different scaling function τ and possibly a factor 2 in the cardinality),
   - on the coefficient m that increases the cardinality of the polynomial mesh;
2. increasing $m > 1$, the smaller is the constant
$$c_m = \frac{1}{\cos(\pi/(2m))}$$
s.t.
$$\|p\| \leq c_m \|p\|_{Z_n^m}, \quad p \in \mathbb{P}_n(K)$$
and $\lim_m c_m = 1$;
3. the meshes Z_n^m have $O(mn)$ cardinality, improving asympt. the $O(n^2)$ cardinality of previously known meshes on these domains (coming from connected compact set of $\mathbb{C}$ whose boundary is a C^1 parametric curve with bounded tangent vectors).

Remark

The class of domains with connected complement and such boundaries is very wide, including

- *curvilinear polygons with boundary tracked by splines,*
- *curvilinear polygons with boundary tracked by polar arcs like*

$$\gamma_j(t) = z_0 + r_j(t)(\cos(t) + i\sin(t))$$

with $r_j(t)$ a trigonometric polynomial.

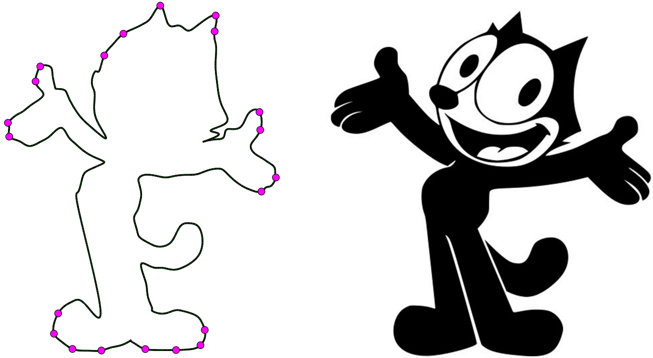


Figure: Felix the cat (1917), written via parametric cubic splines (148 arcs). In magenta, AFP (degree 20).

The previous results allow a computable interval estimate of the Lebesgue constant (uniform operator norm) of any linear projection operator $L_n : C(K) \to \mathbb{P}_n(\mathbb{C})$ of the form

$$L_n f(z) = \sum_{j=1}^{M} f(\xi_j)\, \phi_j(z) \;, \tag{3}$$

where $\{\xi_j\} \subset K$ and $\{\phi_j\}$ is a set of generators of $\mathbb{P}_n(\mathbb{C})$.

Well-known examples are

- polynomial interpolation at $M = n+1$ distinct nodes, where the $\phi_j(z)$ are the corresponding cardinal Lagrange polynomials,
- polynomial least-squares at $M > n+1$ sampling nodes, with

$$\phi_j(z) = K_n(z, \xi_j) = \sum_{k=1}^{n+1} q_k(z)\overline{q_k(\xi_j)} \;, \tag{4}$$

  where K_n is the reproducing kernel of the discrete scalar product with unit weights supported at the sampling nodes $\{\xi_j\}$ and $\{q_k\}$ a discrete orthogonal polynomial basis.

For interpolation in K at degree n, one needs to compute

$$\|L_n\|_K = \sup_{f \in C(K), f \neq 0} \frac{\|L_n f\|_K}{\|f\|_K} = \sup_{z \in K} \sum_{k=0}^{n} |l_k(z)|$$

where l_k, $k = 0, \ldots, n$, are the Lagrange polynomials.

In order to solve this difficult optimization problem, keeping fingers crossed, one usually evaluates the simpler

$$\|L_n\|_{K_d} = \sup_{z \in K_d} \sum_{k=0}^{n} |l_k(z)|$$

where K_d is a discrete and *sufficiently* dense subset of K.

The problem is that there is in general no control on how good is the approximation

$$\|L_n\|_{K_d} \approx \|L_n\|_K.$$

Proposition

Let

- $K \subset \mathbb{C}$ *be one the sets previously defined;*
- $\lambda_n(z) = \sum_{j=1}^{M} |\phi_j(z)|, \quad z \in K$, *be the Lebesgue function of* L_n*;*
- Z_n^m *the polynomial mesh previously defined.*

Then for every $n \geq 1$, $m > 1$, *the following inequalities hold*

$$\|\lambda_n\|_{Z_n^m} \leq \|L_n\| \leq c_m \|\lambda_n\|_{Z_n^m} \ , \tag{5}$$

$$0 \leq \|L_n\| - \|\lambda_n\|_{Z_n^m} \leq (c_m - 1)\|L_n\| \ , \tag{6}$$

for the Lebesgue constant $\|L_n\| = \|\lambda_n\|_K = \|\lambda_n\|_\Gamma$

1. Numerical estimates (5) of the Lebesgue constant $\|L_n\|$ are available from below and above;
2. by (6), we can have tighter estimates of the Lebesgue constant increasing m;
3. remember that $\|f - L_n(f)\|_K \leq (1 + \|L_n\|)E_n(f)$ where $E_n(f)$ is the best-approximation error of f in $\mathbb{P}_n$.

In the paper

D.J. Kenne, A. Sommariva, M. Vianello,

CPOLYMESH: Matlab and Python codes for complex polynomial approximation by Chebyshev admissible meshes,

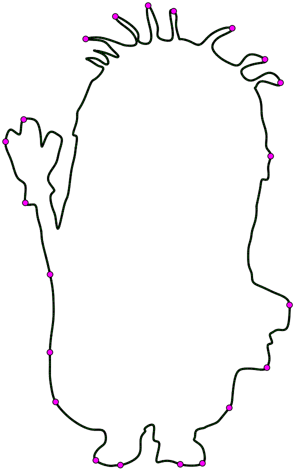
Journal of Approximation Software (volume 1, issue 2).

we have developed codes in Matlab/Octave and Python that implement these ideas, that is

- Polynomial Mesh constructor;
- Stabilized Vandermonde Matrix constructor;
- Discrete Orthogonal Polynomials constructor and evaluator;
- Discrete Extremal sets constructor;
- Polynomial projectors (either interpolation or least-squares);
- Lebesgue constant evaluator.

1. These codes allowed us, for many sets K to define at degree n
   - extremal sets (as Approximate Fekete Points, Discrete Leja Points or Pseudo Leja Points) *good for interpolation*,
   - or alternatively polynomial meshes *good for least-squares*,

   also providing reliable error estimates of their Lebesgue constant either from below that from above.
2. All the routines and demos are available as open-source software on Github and at the authors' homepage, i.e
   - Matlab/Octave version: `https://github.com/alvisesommariva/CPOLYMESH/`;
   - Python version: `https://github.com/DimitriKenne/CPOLYMESH`.
3. The Matlab/Octave and the relative Python codes perform the same experiments and implement the same basic routines.



```
>> tic; demo_cdes_1(20); toc;

        ----------------------------------------
        -> Demo on extremal points
        ----------------------------------------
        * AM degree              : 20
        * AM factor ext.pts.     : 2
        * AM cardinality e.p.    : 23880
        * Ext. points card.      : 21
        * AM factor Leb.const.   : 2
        * AM cardinality l.c.    : 47760
        * AM coefficient         : 1.00863
        * Leb. const. bounds     : [10.095,10.183]
        * Leb. const. approx     : 10.139
        ----------------------------------------
        -> Figure
        ----------------------------------------
        * Green dots   : Admissible mesh
        * Magenta dots: Approximate Fekete Points
        ----------------------------------------
Elapsed time is 0.734138 seconds.
>> |
```

Figure: Minion as parametric cubic spline (197 arcs).

Numerical examples: domains

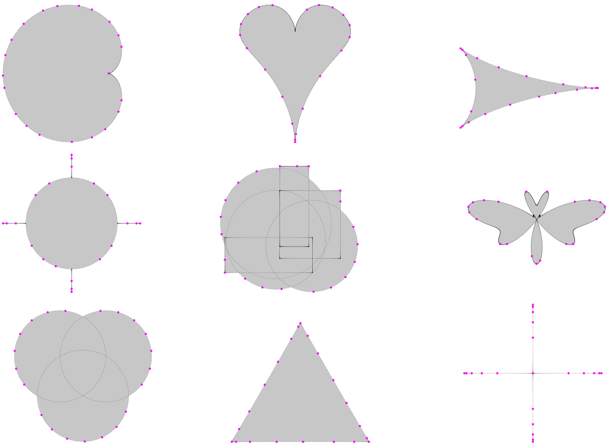


Figure: Exemplifying the variety of feasible (curvi)linear polygons: Approximate Fekete Points (magenta dots) for polynomial interpolation at degree $n = 20$ via Chebyshev admissible meshes (black dots) of the piecewise polynomial or trigonometric boundary, with $m = 2$. Notice that the composite domain in the middle has a boundary difficult to track, but still it is contained in the union of circumferences and segments.

Numerical examples: results

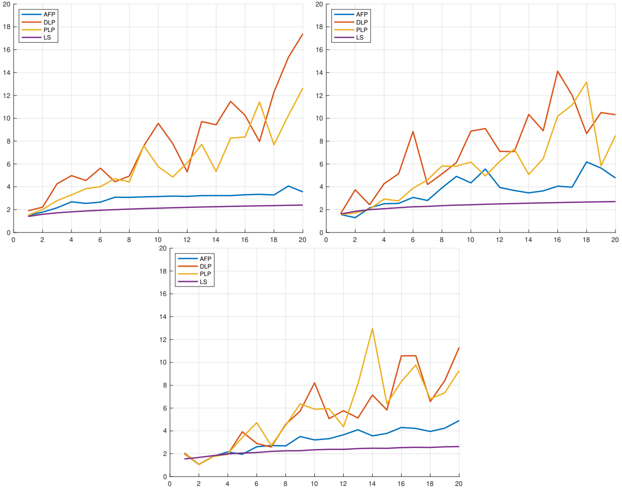


Figure: Lebesgue constants for the extremal interpolation sets AFP, DLP, PLP, and for LS approximation, with degrees $n = 1, 2, \ldots, 20$, on the cardioid, La Porte Heart and deltoid of the previous page. Notice how low is the Lebesgue constant for the Least-Squares and the fact that in these examples they are all less than 18.

Numerical examples: results

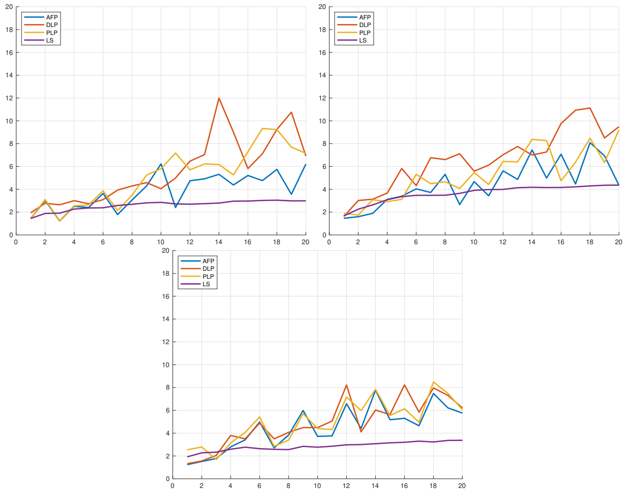


Figure: Lebesgue constants for the extremal interpolation sets AFP, DLP, PLP, and for LS approximation, with degrees $n = 1, 2, \ldots, 20$, on the sun, union of disks and squares and butterfly of the previous page. Notice how low is the Lebesgue constant for the Least-Squares and the fact that in these examples they are all less than 12.

Numerical examples: results

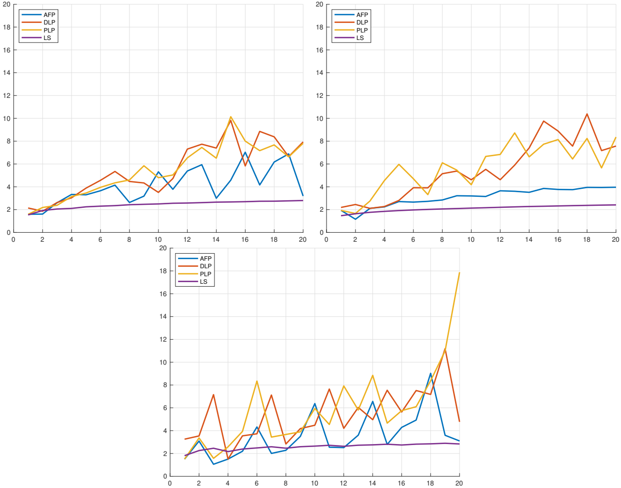


Figure: Lebesgue constants for the extremal interpolation sets AFP, DLP, PLP, and for LS approximation, with degrees $n = 1, 2, \ldots, 20$, on the Borromean domain, simplex and cross of the previous page. Notice how low is the Lebesgue constant for the Least-Squares and the fact that in these examples they are all less than 18.

In the recent work

L. Bialas-Cież, D. Kenne, A. Sommariva, M. Vianello, Evaluating Lebesgue constants by Chebyshev polynomial meshes on cube, simplex and ball, Electron. Trans. Numer. Anal., 60 (2024), pp. 428–445.

we have extended these ideas to multivariate cases.

With techniques similar to the complex case, we have determined product-type Chebyshev grids

- on the cube $[-1,1]^d$,
- on the simplex (via Duffy-like transformation),
- on the disk/ball,

having at hand a tool for computing rigorously the Lebesgue constant on any set useful for interpolation/approximation on these domains.

In particular, in its numerical section we compare the computed Lebesgue constants of some well-known families of interpolation points

- on the square as the Padua, the Morrow-Patterson and the Halton points;
- on the simplex as the Waldron points, the Approximate Lebesgue and the Symmetric Approximate Lebesgue points;
- on the disk as AFP, DLP, the Approximate Lebesgue, the Carnicer–Godes and Halton points.

We finally performed some experiments with some least-square type operators, getting their numerical rigorous bound of the Lebesgue constant on the square and on the disk.

A brief note on multivariate real domains

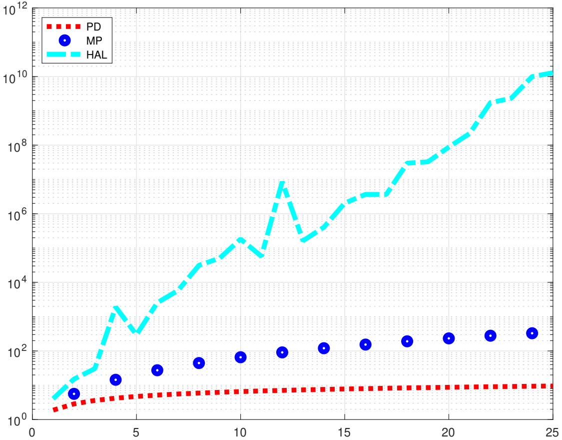


Figure: Certified Lebesgue constants for the interpolation of some pointsets on square.

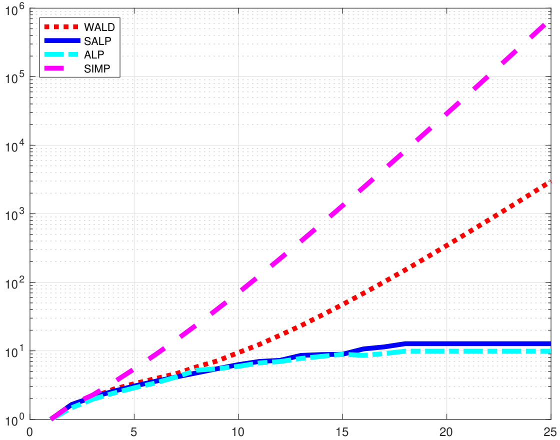


Figure: Certified Lebesgue constants for the interpolation of some pointsets on the simplex.

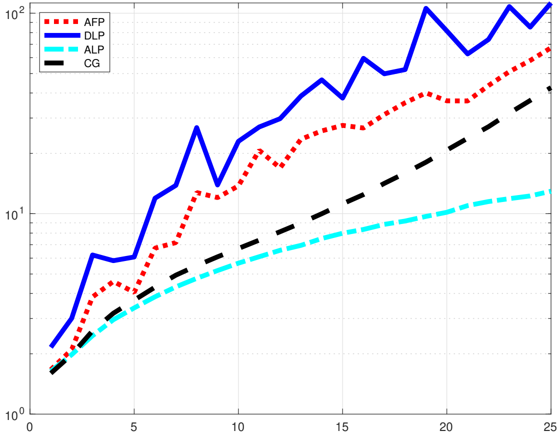


Figure: Certified Lebesgue constants for the interpolation of some pointsets on the unit-disk.

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