On unisolvence of unsymmetric random Kansa collocation

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Purpose

In this talk we will briefly discuss

- unisolvence of unsymmetric random Kansa collocation.
- introduce some basics on the topic;
- show the results that we have proved.

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In this short talk we will briefly

- discuss unisolvence of unsymmetric random RBF-Kansa collocation.
- introduce some basics on the topic;
- mention the results that we have proved.

Examples of RBF that will be considered are:

- Thin Plate-Splines: $\phi(r) = r^{2\nu}\log(r)$, $\nu \in \mathbb{N}$,
- Multiquadrics: $\phi(r) = \sqrt{1 + r^2}$,
- Generalized Inverse MultiQuadrics: $\phi(r) = (1 + r^2)^{\beta}$, $\beta < 0$
- Gaussians: $\phi(r) = e^{-r^2}$,
- Matérn: $\phi(r) = \frac{2^{1-\nu}}{\Gamma(\nu)} r^{\nu} K_{\nu}(r)$.

For each of them one can defined the scaled version $\phi_{\varepsilon}(r) = \phi(\varepsilon r)$.

We consider the Poisson equation with Dirichlet boundary conditions

$$\begin{cases} \Delta u(P) = f(P)\ ,\ P \in \Omega\ , \\ u(P) = g(P)\ ,\ P \in \partial\Omega\ , \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain (connected open set), $P = (x_1, \ldots, x_d)$ and $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_d^2$ is the Laplacian.

Depending on the cases in analysis we will consider additional assumptions.

Unsymmetric Kansa collocation consists in seeking a function

$$u_N(P) = \sum_{j=1}^{n} c_j\,\phi_j(P) + \sum_{k=1}^{m} d_k\,\psi_k(P)\ ,\quad N = n + m\ , \tag{2}$$

where

$$\phi_j(P) = \phi_\varepsilon(\|P - P_j\|_2)\ ,\quad \{P_1, \dots, P_n\} \subset \Omega\ , \tag{3}$$

$$\psi_k(P) = \phi_\varepsilon(\|P - Q_k\|_2)\ ,\quad \{Q_1, \dots, Q_m\} \subset \partial\Omega\ , \tag{4}$$

such that

$$\begin{cases} \Delta u_N(P_i) = f(P_i)\ ,\ i = 1, \dots, n \\ u_N(Q_h) = g(Q_h)\ ,\ h = 1, \dots, m\ . \end{cases} \tag{5}$$

Kansa collocation can be rewritten in matrix form as

$$\begin{pmatrix} \Delta\Phi & \Delta\Psi \\ \\ \Phi & \Psi \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \\ \mathbf{g} \end{pmatrix} \tag{6}$$

where the $N \times N$ block matrix is

$$K_N = K_N(\{P_i\}, \{Q_h\}) = \begin{pmatrix} \Delta\Phi & \Delta\Psi \\ \\ \Phi & \Psi \end{pmatrix} = \begin{pmatrix} (\Delta\phi_j(P_i)) & (\Delta\psi_k(P_i)) \\ \\ (\phi_j(Q_h)) & (\psi_k(Q_h)) \end{pmatrix}$$

and

- $\mathbf{f} = \{f(P_i)\}$,
- $\mathbf{g} = \{g(Q_h)\}, 1 \leq i, j \leq n, 1 \leq h, k \leq m$.

A main issue is to show if the Kansa collocation matrix

$$K_N = K_N(\{P_i\}, \{Q_h\}) = \begin{pmatrix} \Delta\Phi & \Delta\Psi \\ & \\ \Phi & \Psi \end{pmatrix} \tag{7}$$

is non-singular.

- Existence of sufficient conditions ensuring invertibility of unsymmetric Kansa collocation matrices is still a substantially open problem (Hon and Schaback in 2001 has showed that there exist point configurations leading to singularity of the collocation matrices).
- The lack of well-posedness conditions has been considered one of the main drawbacks of unsymmetric Kansa collocation, despite its manifest effectiveness in many applications.

For Thin-Plates splines $\phi(r) = r^{2\nu}\log(r)$, $\nu \in \mathbb{N}$ we proved the following.

Theorem (F.Dell'Accio, A.Sommariva, M.Vianello, 2024)

- *Assume that $\Omega \subset \mathbb{R}^2$ is a domain whose boundary curve has an analytic parametrization (namely a curve $\gamma : [a, b] \to \mathbb{R}^2$, $\gamma(a) = \gamma(b)$, that is* analytic *and* regular*, i.e. $\gamma'(t) \neq (0, 0)$ for every $t \in (a, b)$).*
- *Let K_N be the TPS-Kansa collocation matrix defined above, with $N = N_I + N_B \geq 2$, where*
  - (a) *$\{P_j\}$ is a sequence of independent uniformly distributed random points in Ω,*
  - (b) *$\{Q_h\}$ a sequence of independent uniformly distributed points on $\partial\Omega$*

Then for every $N \geq 2$ the matrix K_N is a.s. (almost surely) nonsingular.

For MultiQuadrics (MQ) $\phi_\varepsilon(r) = \sqrt{1 + (\varepsilon r)^2}$ and Inverse MultiQuadrics (IMQ) $\phi_\varepsilon(r) = \frac{1}{\sqrt{1+(\varepsilon r)^2}}$ the following result holds.

Theorem (R.Cavoretto, F.Dell'Accio, A.De Rossi, A.Sommariva, M.Vianello, 2024)

Let K_n be the MQ or IMQ Kansa collocation matrix where

(a) *$\{P_j\}$ is a sequence of i.i.d. (independent and identically distributed) random points in Ω with respect to any probability density $\sigma \in L^1_+(\Omega)$,*

(b) *$\{Q_h\}$ is any fixed set of m distinct points on $\partial\Omega$*

Then for every $m \geq 1$ and for every $n \geq 0$ the matrix K_n is a.s. (almost surely) nonsingular.

Remark

There is no restrictive assumption on $\partial\Omega$, except for the usual ones that guarantee well-posedness and regularity of the solution (like e.g. that the boundary is Lipschitz).

Theorem (A.Sommariva, M.Vianello, 2024)

Let $\phi : [0, +\infty) \to \mathbb{R}$ *be a radial function such that:*

(*i*) $\phi \in C^2([0, +\infty)) \cap H((0, +\infty))$, $\lim_{r \to \infty} \phi(r) = 0$;

(*ii*) $\ell(r) = \phi''(r) + \phi'(r)/r$ *is continuous at* $r = 0$, $\ell(0) \neq 0$, $\lim_{r \to \infty} \ell(r) = 0$;

(*iii*) *the RBF interpolation matrix* $V_m = \phi_\varepsilon(\|Q_h - Q_k\|)$, $1 \leq h, k \leq m$, *is nonsingular for every set of distinct points* $\{Q_1, \ldots, Q_m\} \subset \mathbb{R}^d$.

Let

(a) $\{P_i\}$ *is a sequence of i.i.d. (independent and identically distributed) random points in* Ω *with respect to any probability density* $\sigma \in L^1_+(\Omega)$;

(b) $\{Q_h\}$ *is any fixed set of* m *distinct points on* $\partial\Omega$.

Then for every $m \geq 1$ *and for every* $n \geq 0$ *the collocation matrix* K_n *is a.s. (almost surely) nonsingular.*

RBFs that fulfil the assumptions of this theorem are:

- Gaussians: $\phi(r) = e^{-r^2}$;
- Generalized Inverse MultiQuadrics: $\phi(r) = (1 + r^2)^{\beta}$, $\beta < 0$;
- Matérn: $\phi(r) = \frac{2^{1-\nu}}{\Gamma(\nu)} r^{\nu} K_{\nu}(r)$, where K_ν is the modified Bessel function of the second kind or Macdonald function of order $\nu > 0$.

Of course the theorem is valid also for their scaled version ϕ_ε.

We are considering the investigation of similar results for

- other classes of RBF (e.g. RBF with compact support);
- other differential equations;
- other boundary conditions (e.g. Neumann, Robin, mixed-type);
- other different pointsets (e.g. QMC).

At present we believe that all these topics are not trivial and will make us sweat a lot (any help is welcome)!

- On unsymmetric collocation by radial basis functions:
  C.Hon, R.Schaback, On unsymmetric collocation by radial basis functions, Applied Mathematics and Computation, 119 2–3 (2001), Pages 177–186.
- TPS-Kansa (random points):
  F.Dell'Accio, A.Sommariva, M.Vianello, Unisolvence of random Kansa collocation by Thin-Plate Splines for the Poisson equation, Engineering Analysis with Boundary Elements, (2024), to appear.
- MQ/IMQ-Kansa (random points):
  R.Cavoretto, F.Dell'Accio, A.De Rossi, A.Sommariva, M.Vianello, Nonsingularity of unsymmetric Kansa matrices: random collocation by MultiQuadrics and Inverse MultiQuadrics, submitted.
- Smooth RBF-Kansa (random points):
  A.Sommariva, M.Vianello, Unisolvence of unsymmetric random Kansa collocation by Gaussians and other analytic RBF vanishing at infinity, submitted.
- RBF interpolation at random points:
  You can find our works at `https://www.math.unipd.it/~alvise/papers.html`